Chapter 7

PULSATIONS, WAVES, AND DISCONTINUITIES IN STELLAR WINDS

Wave after wave
Will flow with the tide
And bury the world as it does
Tide after tide
Will flow and recede
Leaving life to go on
As it was.

Neil Peart, *Natural Science*

Hot luminous stars (spectral types O, B, Wolf-Rayet) are observed to have strong stellar winds which exhibit variability on time scales ranging from hours to years. Many classes of these stars are also seen, via photospheric line-profile or photometric variability, to pulsate radially or nonradially. It has been suspected for some time that these oscillations can induce periodic modulations in the surrounding stellar wind and produce observational signatures in, e.g., ultraviolet P Cygni line profiles. This Chapter outlines a dynamical foundation for understanding the propagation of stellar pulsations into an accelerating wind, and presents initial observational predictions and constraints for various types of stars.

Section 7.1 contains an overview of the theory of nonradial pulsations (NRPs) of stars, with a description of how the *discrete* spectrum of small-amplitude oscillation modes ("standing waves") may be observable from the photosphere. Section 7.2 presents a parallel derivation of linear wave theory, with emphasis on the *continuous* spectrum of possible oscillations ("traveling waves") in the photosphere and accelerating wind. This work represents a preliminary attempt to bridge the gap between interior pulsations and propagating wind structure. Finally, Section 7.3 examines the nonlinear steepening of finite-amplitude waves and the resulting discontinuities and "kinks" that may develop in stellar winds.

7.1 Global Stellar Pulsation

There exist several classes of early-type stars which are inferred to pulsate strongly enough to be detected either photometrically or via line profile variations. The β Cep variables (spectral types $\sim$ B0 to B3) and the "slowly pulsating B stars" (SPBs) or 53 Per variables (spectral types $\sim$ B3 to B9) have recently been explained in terms of the standard Cepheid opacity instability mechanism (Dziembowski 1994), which leads to strong spontaneous pulsation. Many classical Be stars have been observed to pulsate, and Kambe et al. (1993) has found a correlation between circumstellar emission episodes and increased NRP amplitudes. O and B supergiants exhibit complex variability on many time scales, and it is difficult to isolate clear signatures of pulsation, rotational modulation, or intrinsic wind activity (Fullerton 1990; Kaper 1993; Fullerton, Gies, & Bolton 1996). Let us begin to disentangle these effects by examining the theory of NRP in stellar interiors, and hopefully the mass motions of the underlying star can act as a "seed" for photospheric and wind variability.

7.1.1 Linearized Hydrodynamic Equations

For the equilibrium state, let us assume a spherical non-rotating star, with density ρ_o, pressure P_o, and temperature T_o functions of radius only. Also, let us take the equilibrium fluid velocity $\mathbf{v}_o$ to be identically zero everywhere inside the star. Following Cox (1980) and Unno et al. (1979), we can define the Lagrangian displacement vector

$$\boldsymbol{\xi} \equiv \delta\mathbf{r} = \mathbf{r}(t) - \mathbf{r}_o = (\delta r)\hat{\mathbf{e}}_r + (r\,\delta\theta)\hat{\mathbf{e}}_\theta + (r\sin\theta\,\delta\phi)\hat{\mathbf{e}}_\phi \ , \tag{7.1}$$

as the instantaneous spatial departure from equilibrium. Let us denote equilibrium quantities as f_o, their Eulerian (fixed in space) variations as f', and their Lagrangian (moving with the fluid) variations as δf. To first order accuracy, the Lagrangian and Eulerian perturbations in *velocity* are equivalent, and

$$\mathbf{v}' = \delta\mathbf{v} = \mathbf{v}(t) - \mathbf{v}_o = \mathbf{v}(t) = \frac{d\boldsymbol{\xi}}{dt} = \frac{\partial\boldsymbol{\xi}}{\partial t} \ . \tag{7.2}$$

For other fluid quantities, however, the Lagrangian and Eulerian variations are not equivalent. To first order, for a scalar quantity f,

$$\delta f \;=\; f' + \boldsymbol{\xi} \cdot \nabla f_o \ . \tag{7.3}$$

The linearly perturbed equations of hydrodynamics can be written as follows. (See Section 7.2 for a more in-depth explanation of the linearization of the fluid equations.) In Eulerian form, the linearized mass continuity equation (noting that $\mathbf{v}' = \delta \mathbf{v} = \mathbf{v}$) is

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_o \mathbf{v}) \;=\; 0 \ . \tag{7.4}$$

However, using equations (7.2) and (7.3) above, integrating with respect to time, and expressing the result in Lagrangian form, we obtain the more useful form

$$\delta \rho + \rho_o \nabla \cdot \boldsymbol{\xi} \;=\; 0 \ . \tag{7.5}$$

The vector momentum equation is given in Eulerian form as

$$\frac{\partial \mathbf{v}}{\partial t} \;=\; \frac{\rho'}{\rho_o^2} \nabla P_o - \frac{1}{\rho_o} \nabla P' + \mathbf{g}' \ , \tag{7.6}$$

and in Lagrangian form as

$$\frac{d^2 \boldsymbol{\xi}}{dt^2} \;=\; -\delta \left(\frac{\nabla P}{\rho} \right) + \delta \mathbf{g} \ , \tag{7.7}$$

where $\mathbf{g}$ is the general external acceleration (often assumed due to gravity only). Both forms of the momentum equation will be useful later.

Because we will eventually be concerned with only *adiabatic* (isentropic) variations, the equation of energy conservation can be solved by two particularly simple solutions which express, e.g., the pressure and temperature perturbations in terms of the density perturbation. Defining the adiabatic exponents,

$$\Gamma_1 \;\equiv\; \left(\frac{d \ln P}{d \ln \rho} \right)_{ad} \ , \qquad \Gamma_3 - 1 \;\equiv\; \left(\frac{d \ln T}{d \ln \rho} \right)_{ad} \ , \tag{7.8}$$

$$\frac{\Gamma_2 - 1}{\Gamma_2} \;\equiv\; \left(\frac{d \ln T}{d \ln P} \right)_{ad} \;=\; \frac{\Gamma_3 - 1}{\Gamma_1} \;=\; \nabla_{ad} \ , \tag{7.9}$$

the adiabatic energy relations can be expressed in terms of the thermodynamic identities,

$$\frac{\delta P}{P_o} \;=\; \Gamma_1 \frac{\delta \rho}{\rho_o} \ , \qquad \frac{\delta T}{T_o} \;=\; (\Gamma_3 - 1) \frac{\delta \rho}{\rho_o} \ , \tag{7.10}$$

where the adiabatic exponents Γ_1 and Γ_3 are assumed unperturbed. Isothermal variations can be modeled by setting $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$, but this is not applicable in the stellar interior (see Section 7.2 for the differences between an isothermal *mean state* and isothermal *variations*).

Finally, the Eulerian momentum equation can be re-written in terms of a smaller number of perturbed variables (which will be useful when finding solutions). Cox (1980) derives the following form, using equations (7.5) and (7.10):

$$\frac{d^2\boldsymbol{\xi}}{dt^2} = -\nabla\chi + \mathbf{A}\frac{\Gamma_1 P_o}{\rho_o}(\nabla\cdot\boldsymbol{\xi}) \ , \tag{7.11}$$

where

$$\chi \equiv \frac{P'}{\rho_o} + \psi' \ , \qquad \mathbf{A} \equiv \frac{\nabla\rho_o}{\rho_o} - \frac{\nabla P_o}{\Gamma_1 P_o} \ , \tag{7.12}$$

and ψ' is the perturbed gravitational potential ($\mathbf{g} = -\nabla\psi$). Cox mentions that the above momentum equation is true for a spherical equilibrium condition, but it is also valid for any equilibrium configuration which obeys

$$(\boldsymbol{\xi}\cdot\nabla\rho_o)\,\nabla P_o = (\boldsymbol{\xi}\cdot\nabla P_o)\,\nabla\rho_o \ . \tag{7.13}$$

In some cases of rotating *barotropic* stars, where $\mathbf{g}$ is derivable from a potential and the pressure and density surfaces coincide, this relation can hold as well.

The vector $\mathbf{A}$, sometimes called the "Schwarzschild discriminant," has only a radial component $A_r\hat{\mathbf{e}}_r$ for a spherical equilibrium model. This variable represents a convective stability criterion, because the effective buoyant force (per unit volume) of a small density parcel in a radially stratified fluid is

$$f_B = -g\Delta\rho(r) = -\rho_o g A_r \delta r \ , \tag{7.14}$$

where $\Delta\rho$ is the difference between the parcel density and the density of its surroundings. Thus, if $A > 0$, the fluid is convectively *unstable*, and if $A < 0$, the fluid is convectively *stable*, and oscillates with the Brunt-Väisälä frequency ω_{BV} (see Section 7.2.5), where

$$\omega_{BV}^2 = -A_r g \ . \tag{7.15}$$

7.1.2 Simple Oscillatory Solutions

If we make the standard assumption that all perturbation variables vary in time only sinusoidally, as $e^{i\omega t}$, the three components of the momentum equation can be expressed as

$$\omega^2\xi_r = \frac{\partial\chi}{\partial r} + A_r\frac{\Gamma_1 P_o}{\rho_o}\left(\frac{\delta\rho}{\rho_o}\right) \tag{7.16}$$

$$\omega^2 \xi_\theta = \frac{1}{r}\frac{\partial \chi}{\partial \theta} \tag{7.17}$$

$$\omega^2 \xi_\phi = \frac{1}{r \sin\theta}\frac{\partial \chi}{\partial \phi} \ . \tag{7.18}$$

Using the expressions for ξ_θ and ξ_ϕ, the divergence of $\boldsymbol{\xi}$ can be easily represented as

$$\nabla \cdot \boldsymbol{\xi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \xi_r\right) - \frac{1}{\omega^2 r^2}\mathcal{L}^2 \chi \ , \tag{7.19}$$

where the operator $\mathcal{L}^2$ is defined by Cox (1980) to be

$$\mathcal{L}^2 \equiv -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \ . \tag{7.20}$$

This operator is a familiar one: its eigenfunctions are the spherical harmonics $Y_{\ell m}$, and its eigenvalue equation is

$$\mathcal{L}^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1) Y_{\ell m}(\theta,\phi) \ , \tag{7.21}$$

where $\ell = 0, 1, 2, \ldots;\ m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. The azimuthal mode number, m, denotes the number of pulsational minima or maxima around a chosen equator, while the meridional degree, ℓ, describes the latitudinal variation of the pulsational amplitude. Specifically, there are $(\ell - |m|)$ nodes between the north and south poles of a star, and the poles are always nodes. We will assume that all perturbation variables can be *separated* into a product of a radial eigenfunction and a spherical harmonic function. The frequency ω thus becomes an eigenvalue of the problem, with each solution for ω corresponding to a solution for the radial eigenfunction.

In general, there are two classes of oscillatory solutions to the linearly perturbed momentum equation. These can be seen by examining the r-component of the curl of the momentum equation (i.e., the "vorticity" equation)

$$\omega^2 \left(\nabla \times \boldsymbol{\xi}\right)_r = 0 \tag{7.22}$$

(the $\mathbf{g}'$ term vanished because it can be written as a gradient of a scalar potential). Thus, if $\omega^2 = 0$, we can solve for the so-called *"toroidal"* modes, and if $\left(\nabla \times \boldsymbol{\xi}\right)_r = 0$, we can solve for the *"spheroidal"* modes. Both types of modes, taken together, make up the complete set of solutions to the momentum equation. For spherical, non-rotating stars, the toroidal modes are obviously non-oscillatory, but in rotating stars they have non-vanishing frequencies (see Section 7.1.5, below). For **spheroidal** modes,

$$\left(\nabla \times \boldsymbol{\xi}\right)_r = \frac{\partial}{\partial\theta}\left(r \sin\theta\, \xi_\phi\right) - \frac{\partial}{\partial\phi}\left(r \xi_\theta\right) = 0 \ , \tag{7.23}$$

and this will provide a constraint on the angular variation of ξ_θ and ξ_ϕ. The Lagrangian displacement $\boldsymbol{\xi}$ can be separated into "vertical" (radial) and "horizontal" (tangential) components, each separable into radial and spherical harmonic eigenfunctions:

$$\boldsymbol{\xi} \;=\; \xi_r \hat{\mathbf{e}}_r + \xi_h \hat{\mathbf{e}}_h \;, \tag{7.24}$$

where

$$\xi_r(r,\theta,\phi) \;=\; \frac{u_\ell(r)}{r^2} Y_{\ell m}(\theta,\phi) \tag{7.25}$$

$$\xi_h(r,\theta,\phi) \;=\; \frac{v_\ell(r)}{r} Y_{\ell m}(\theta,\phi) \;, \tag{7.26}$$

and the radial eigenfunctions u_ℓ and v_ℓ are also dependent on the eigenvalue ω. In order to separate ξ_h into its horizontal components in the θ and ϕ directions, we note that $\boldsymbol{\xi}$ is the sum of a gradient of a scalar (χ) and a purely radial vector. In this case, ξ_θ and ξ_ϕ are given simply by angular derivatives of ξ_h, and Cox gives

$$\xi_\theta \;=\; \frac{\partial \xi_h}{\partial \theta} \;, \qquad \xi_\phi \;=\; \frac{1}{\sin\theta}\frac{\partial \xi_h}{\partial \phi} \;, \tag{7.27}$$

and the tangential (non-unit) direction vector is given by

$$\hat{\mathbf{e}}_h \;=\; \left(\frac{1}{Y_{\ell m}}\frac{\partial Y_{\ell m}}{\partial \theta}\right)\hat{\mathbf{e}}_\theta + \left(\frac{1}{Y_{\ell m}\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\right)\hat{\mathbf{e}}_\phi \;. \tag{7.28}$$

Note that ξ_θ and ξ_ϕ uniquely satisfy the spheroidal mode constraint given above in equation (7.23).

In addition, we can also now write

$$\nabla \cdot \boldsymbol{\xi} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \xi_r\right) - \frac{\ell(\ell+1)}{r}\xi_h \;, \tag{7.29}$$

and comparison with equation (7.19) provides the useful relation

$$\xi_h \;=\; \frac{\chi}{\omega^2 r} \;, \tag{7.30}$$

which will be utilized below when we attempt to constrain $u_\ell(r)$, $v_\ell(r)$, and the pressure perturbation δP, all in terms of one (unfortunately arbitrary) surface amplitude.

Correspondingly, the **toroidal** set of solutions, with $\omega^2 = 0$ and $\xi_r = 0$ for non-rotating systems, are given by the alternate set of angular derivatives of a horizontal displacement,

$$\xi_\theta \;=\; \frac{T_{\ell m}(r)}{r\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi} \tag{7.31}$$

$$\xi_\phi \;=\; -\frac{T_{\ell m}(r)}{r}\frac{\partial Y_{\ell m}}{\partial \theta} \;, \tag{7.32}$$

but $(\nabla \times \boldsymbol{\xi})_r \neq 0$ for these modes, as allowed by the vorticity equation in this case. The radial eigenfunctions $T_{\ell m}$ ideally depend on the frequency ω, but in the present case ($\omega^2 = 0$), all toroidal solutions are degenerate and uninteresting.

7.1.3 Surface Constraints

Being primarily interested only in the photospheric and wind manifestation of nonradial pulsations (NRPs), we now focus on the surface ($r = R_*$) boundary condition in the solution for the radial eigenfunctions. The assumption that the Lagrangian pressure variation δP vanishes at the surface,

$$\delta P \,=\, P' + \xi_r \frac{\partial P_o}{\partial r} \,=\, 0 \tag{7.33}$$

(a "mechanical" boundary condition) is useful in deriving a relation between the radial and tangential displacements. Using the equilibrium condition of hydrostatic equilibrium ($\partial P_o / \partial r = -\rho_o g$), we can solve equation (7.33) for

$$\frac{\xi_r}{H_p} \,=\, \frac{P'}{P_o} \ , \tag{7.34}$$

where H_p is the pressure scale height, $P_o/\rho_o g$. Thus, using equation (7.30), we can define the ratio of horizontal to vertical motions,

$$K \,\equiv\, \frac{\xi_h(R_*)}{\xi_r(R_*)} \,=\, \frac{g(R_*)}{\omega^2 R_*}\left(1 + \frac{\psi'}{P'/\rho_o}\right) \tag{7.35}$$

$$\approx\, \frac{GM_*}{\omega^2 R_*^3} \ , \tag{7.36}$$

using Cowling's (1941) approximation, which neglects the perturbation of the gravitational potential ψ. As is often done when considering observations of the surfaces of pulsating stars, we can define a "radial velocity amplitude,"

$$V_p \,\equiv\, \frac{u_\ell(R_*)}{R_*^2}\omega \ , \tag{7.37}$$

and, given V_p and ω (or V_p and K), together with the angular eigenvalues ℓ and m, the value of the displacement vector $\boldsymbol{\xi}$ at the surface of the star is uniquely specified.

Although the simple analysis above provides an adequate surface constraint on the Lagrangian displacement, it does not allow for the full computation of the pressure and density variation on the stellar surface. In the theory of stellar interiors, it is often assumed that the pressure P_o and the density ρ_o vanish at the surface, and this implies

$$\delta P \,=\, P' \,=\, 0 \ . \tag{7.38}$$

The quantity $(\delta P/P_o)$, however, should remain finite at all radii, and the adiabatic energy conditions (eq. [7.10]) allow the corresponding density and temperature ratios $(\delta\rho/\rho_o)$ and $(\delta T/T_o)$ to be computed. Dziembowski (1971) and Cox (1980) derive a surface relation for $(\delta P/P_o)$ by solving the radial component of the Lagrangian momentum equation (7.7) for

$$\frac{\partial}{\partial r}\left(\frac{\delta P}{P_o}\right) = \frac{1}{H_p}\left[\frac{\delta P}{P_o} + \frac{\omega^2}{g}\xi_r + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) - \frac{\ell(\ell+1)}{r}\xi_h - \frac{1}{g}\frac{\partial\psi'}{\partial r}\right] \quad . \tag{7.39}$$

Under the assumptions that: (i) $H_p \ll R_*$ near the surface, and (ii) the quantities $(\delta P/P_o)$ and (ξ_r/R_*) do not vary appreciably over the uppermost scale height H_p of the star, the quantity in square brackets above can be set to zero because $(\delta P/P_o)$ must remain finite. This quantity can then be used to solve for $(\delta P/P_o)$. This can be combined with the surface boundary condition for the perturbed gravitational potential – where ψ' and g are assumed continuous across the perturbed stellar surface – to obtain the final surface relation

$$\frac{\delta P}{P_o} = \left[\ell(\ell+1)K - 4 - \frac{1}{K}\right]\frac{\xi_r}{R_*} + \left[\ell(\ell+1)K - (\ell+1)\right]\frac{\psi'}{gR_*} \quad . \tag{7.40}$$

In the Cowling (1941) approximation, only the first term in square brackets, proportional to (ξ_r/R_*), remains, and it can be clearly seen that the Lagrangian pressure variations can either be in phase or 180° out of phase with the perturbation displacements themselves, depending on the values of ℓ and K.

Buta & Smith (1979) derive in detail the light variations from a linear and adiabatic nonradial pulsator. Note that, for a black body, the frequency-integrated intensity variation from the star can be given by

$$\frac{\delta I}{I_o} = 4\frac{\delta T}{T_o} \quad . \tag{7.41}$$

They find three competing factors which produce variability: (i) the purely thermodynamic change $(\delta T/T_o)$ derivable from equation (7.40), (ii) a "surface normal" (limb darkening) effect due to the changing direction of the local normal to points on the star, and (iii) a "surface area" effect due to the changing projected area of surface elements. More sophisticated treatments have included non-adiabatic effects (see, e.g., Stamford & Watson 1981), rapid rotation (Lee & Saio 1990; Aerts 1993; Townsend 1996), and extended atmospheres (Gouttebroze & Toutain 1994). For stellar wind dynamics, however, we need to derive the light (i.e., incident flux) variations for an "observer" at a *finite* distance from the star – not an infinite distance as is often assumed in the works cited above.

7.1.4 Discrete Frequency Eigenvalues

Because stars are bound systems with the majority of their constituent gas within a finite volume, only a *discrete* set of eigenfrequencies ω arises. This is analogous to simpler physical systems, such as a string fixed at both ends exhibiting a discrete set of vibrational overtone frequencies. Clement (1994), however, discusses the case of *local* instabilities inside stars (e.g., in convective zones) which excite a continuous spectrum of frequencies. These arise because no firm "boundaries" exist to limit the solutions in these regions. Here we will examine stars that are *globally* stable, and have only discrete eigenfrequencies. For realistic stellar models, these modes separate into several natural groupings, and it will be useful to analyze a simple model of a stellar interior in order to classify and understand these different solutions.

The classical illustrative model in stellar pulsation theory is the "homogeneous compressible" sphere, which has a constant equilibrium density throughout its volume, but allows first-order density fluctuations to exist. This model is equivalent to a so-called *polytrope* of index $n_p = 0$; a polytrope is an idealized stellar configuration with a quasi-adiabatic equation of state, $P \propto \rho^{1+(1/n_p)}$. For $n_p = 0$, of course, this implies that the zero-order density is completely insensitive to changes in pressure, and the equation of hydrostatic equilibrium can be integrated in this case to obtain

$$\rho(r) = \rho_o \ , \qquad P(r) = \tfrac{2}{3}\pi G \rho_o^2 (R_*^2 - r^2) \ . \tag{7.42}$$

Although unrealistic, the discrete eigenfrequencies which arise in this model have exact counterparts in more centrally-condensed (and realistic) fluids. Pekeris (1938) and Ledoux & Walraven (1958) derived a second-order differential equation for the parameter $\alpha \equiv \nabla \cdot \boldsymbol{\xi} = -\delta\rho/\rho_o$, using Cowling's (1941) approximation ($\psi' = 0$) and equations (7.10), (7.16), and (7.19). This equation for α becomes decoupled from the equation of motion (for ξ_r), and can be expressed, defining $x \equiv r/R_*$, as

$$\left(1 - x^2\right) \frac{\partial^2 \alpha}{\partial x^2} + \left(\frac{2 - 6x^2}{x}\right) \frac{\partial \alpha}{\partial x} + f(x, \ell, K, ,_1)\, \alpha \ = \ 0 \ , \tag{7.43}$$

where

$$f(x, \ell, K, ,_1) \ \equiv \ \frac{2}{,_1} \left[\frac{1}{K} + 4 - \ell(\ell+1) K \right] - \ell(\ell+1) \left(\frac{1 - x^2}{x^2} \right) - 6 \ . \tag{7.44}$$

Note that the stellar surface ($x = 1$) is a regular singular point of this equation, and a power series solution of the form

$$\alpha_{n\ell} \ = \ x^{\ell} \sum_{i=0}^{n} C_{2i}\, x^{2i} \tag{7.45}$$

provides a recursion relation for the coefficients C_{2i} when substituted back into equation (7.43). In order to have finite solutions in the domain $0 \leq x \leq 1$, the series must terminate at a finite n, and this constraint (setting the numerator in the recursion relation to zero for a given n) provides the discrete spectrum of solutions which satisfy the equation

$$\frac{1}{K} - \ell(\ell+1)K \;=\; ,_1 [n(2n+2\ell+5)+2\ell+3] - 4 \;\equiv\; 2D_{n\ell} \;, \tag{7.46}$$

where $n = 0, 1, 2, \ldots$ is the radial *order* of the solution. In most models, the radial eigenfunctions $u_\ell(r)$ and $v_\ell(r)$ exhibit n nodes or roots between the center and surface. Note that the quantity $1/K$ is a dimensionless squared frequency, which we can denote as $\hat{\omega}^2$, and the above equation is a quadratic which can be solved for

$$\hat{\omega}^2 \;=\; D_{n\ell} \pm \sqrt{D_{n\ell}^2 + \ell(\ell+1)} \;. \tag{7.47}$$

The positive roots are denoted p-modes, or pressure modes, and the negative roots (which in general models do not always result in $\hat{\omega}^2 < 0$, as they do above) are denoted g-modes, or gravity modes. The physical meaning behind these labels will be discussed below. Note that these solutions are degenerate for the $2\ell+1$ possible values of the azimuthal parameter m, but this degeneracy will be lifted when rotation is introduced.

One additional set of discrete modes is possible, and these correspond to the purely solenoidal (zero divergence) displacements of a homogeneous *incompressible* fluid sphere. In these modes, $\delta P = \delta\rho = 0$, and thus $\alpha = 0$. Chandrasekhar (1964) derived the associated eigenfrequencies for the homogeneous compressible model, and

$$\hat{\omega}^2 \;=\; \frac{2\ell(\ell-1)}{2\ell+1} \;. \tag{7.48}$$

These frequencies, originally identified in the incompressible case by Kelvin, are called f-modes, or fundamental modes. There is no oscillation for $\ell = 0$ or $\ell = 1$, and there is only one radial order, $n = 0$, with frequencies usually between those for the p-mode and g-mode $n = 0$ orders. Kelvin's incompressible f-modes have the unique property of possessing analytic solutions for the radial eigenfunctions:

$$\boldsymbol{\xi} \propto \nabla \left[r^\ell Y_{\ell m}(\theta,\phi) \right] \;, \qquad \text{hence,} \qquad u_\ell(r) \propto \ell r^{\ell+1} \;\;, \quad v_\ell(r) \propto r^\ell \;. \tag{7.49}$$

For general, centrally-condensed stellar models, the f-modes need not have zero divergence (i.e., constant volume), and in numerical computations the p, g, and f-modes all appear together in the solutions.

In general, the different classes of discrete modes can be described for the complete set of spheroidal (p, g, f) and toroidal solutions, as follows:

1. **f-modes:** The unique "pseudo-Kelvin" fundamental mode is characterized by the simplest radial eigenfunctions $\xi_r(r)$ and $\xi_h(r)$, usually with the fewest possible number of extrema. The f-mode frequency occurs at the unique value when the Brunt-Väisälä frequency ω_{BV} equals the horizontal acoustic, or "Lamb" frequency S_ℓ,

$$S_\ell^2 \equiv \frac{\ell(\ell+1)}{r^2} \frac{\Gamma_1 P_o}{\rho_o} = k_H^2 a^2 \ , \tag{7.50}$$

at the same point in the stellar interior. Because ω_{BV}^{-1} and S_ℓ^{-1} are the characteristic times a wave takes to move radially and horizontally (respectively) one oscillation wavelength, f-modes occur when these motions are the most comparable to each other.

2. **p-modes:** Because their motion is primarily radial, and only weakly horizontal ($K \lesssim 1$), the strong Eulerian variations in pressure and density create longitudinal acoustic waves, with pressure as the dominant restoring force. The p-mode frequencies are generally higher than those for f-modes, and $\hat{\omega}^2$ grows without bound for large n or ℓ. For $\ell = 0$, the p_n modes correspond to the purely *radial* oscillations exhibited by such stars as Cepheid and Mira type variables.

3. **g-modes:** These oscillations are primarily horizontal ($K \gtrsim 1$), and show only small Eulerian variations in pressure and density. The dominant restoring force in this case is gravity, and these oscillations can be likened to the "bobbing" of an object floating in water exhibiting transverse wave motion. The g-mode frequencies are generally lower than those for f-modes, and $\hat{\omega}^2 \to 0$ for a given ℓ as $n \to \infty$. In many stellar models, there exist both positive (g^+) and negative (g^-) modes, the former corresponding to oscillations and the latter corresponding to convective instability, with growth rate $-\hat{\omega}^2$.

4. **r-modes:** Also called "t" (toroidal) modes, these oscillations were described in detail by Papaloizou & Pringle (1978), and appear only for rotating stars. The motions are almost exclusively horizontal, and the inertial centrifugal and Coriolis forces are the dominant restoring forces. In this manner they are comparable to the Rossby waves which provide zonal motions in planetary atmospheres and oceans.

Figure 7.1 illustrates the discrete spheroidal (p, g, f) oscillation modes for an idealized nonrotating star. Solid lines link modes of the same radial order n, and the gray hatched regions represent waves which can *propagate* in the photosphere (see Section 7.2.5 below). The model star is a B supergiant, with $M_* = 20 M_\odot$ and $R_* = 22 R_\odot$, but the stellar interior has been scaled in radius from a zero-age main sequence

(ZAMS) configuration computed by the code of Hansen & Kawaler (1994). This, together with the simple Cowling-approximation pulsation code used to compute the eigenperiods (also from Hansen & Kawaler 1994), implies that these periods are only useful in an illustrative sense and should not be assumed quantitatively accurate.

Smeyers (1984) reviews several *asymptotic* treatments of adiabatic spheroidal modes, and presents the following useful approximations for high-order p- and g-modes. For the p-mode displaying $n \gg 1$ nodes in the radial displacement,

$$\omega_p \approx \frac{\pi}{2}\left(2n + \ell + n_p + \frac{1}{2}\right)\left[\int_0^{R_*} \frac{dr}{\sqrt{,\ {}_1 P_o/\rho_o}}\right]^{-1} , \qquad (7.51)$$

where n_p is a characteristic polytropic index of the photosphere. For high-order g-modes,

$$\omega_g \approx \frac{2}{\pi}\left(2n + \ell + n_p + \frac{5}{2}\right)^{-1} \sqrt{\ell(\ell+1)} \int_0^{R_*} |\omega_{BV}| \frac{dr}{r} . \qquad (7.52)$$

These approximate formulae have been derived by assuming a *wavelike* behavior of the radially dependent functions $u_\ell(r)$ and $v_\ell(r)$, and the above expressions are extensions of local wave dispersion relations to the global modes of the star.

Finally, we can note that the observationally-derived "period-mean density" relation,

$$Q = \Pi \sqrt{\bar{\rho}/\bar{\rho}_\odot} \qquad (7.53)$$

can be understood in terms of the dimensionless frequency $\hat{\omega}$. The period of the oscillations is $\Pi = 2\pi/\omega$, the mean density of a spherical star is given by

$$\bar{\rho} = \frac{M_*}{V_*} = \frac{3M_*}{4\pi R_*^3} , \qquad (7.54)$$

and $\bar{\rho}_\odot$ is the mean density of the sun. Thus, the "pulsation constant" Q can be written

$$Q = \frac{2\pi}{\hat{\omega}} \sqrt{\frac{R_\odot^3}{G M_\odot}} \approx \frac{0.1159 \text{ days}}{\hat{\omega}} . \qquad (7.55)$$

If a class of stars, over a wide range of spectral types (and radii and masses), all exhibit nearly the same oscillatory modes, then $\hat{\omega}$, and thus Q, will remain nearly constant.

7.1.5 The Effects of Rotation

Let us now consider a uniformly rotating star. By writing the equations of hydrodynamics in the corotating frame, we can preserve the condition of zero

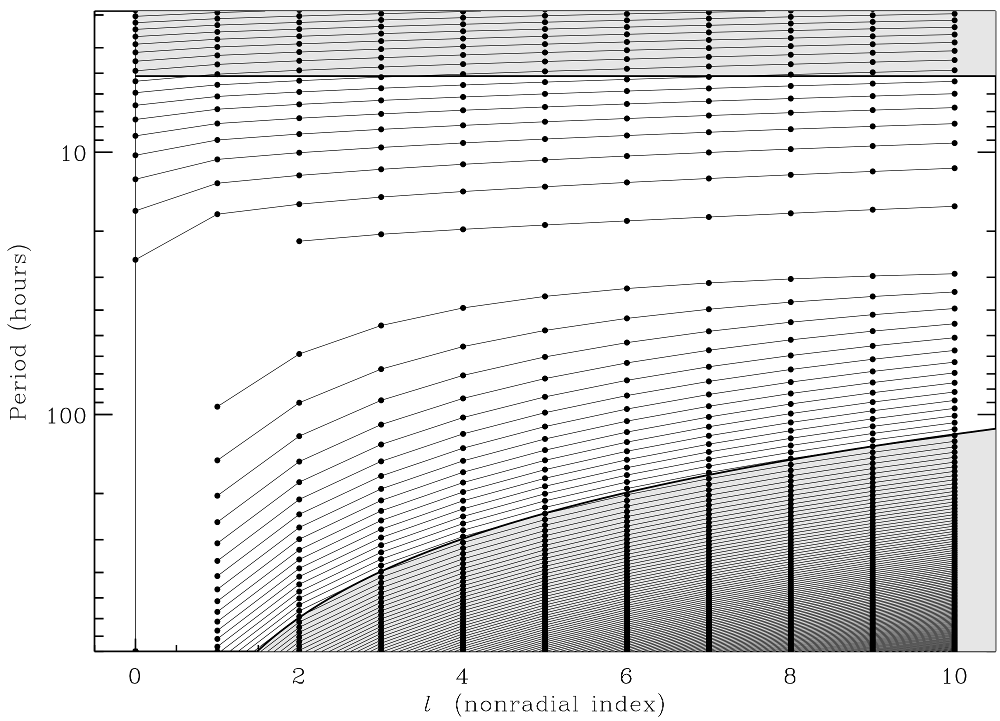

Figure 7.1: Discrete NRP eigenperiods (filled circles) for an idealized B supergiant model. For $\ell = 0$, only the short-period p-modes exist, and for $\ell = 1$, only the p and g-modes are present. The f-modes, with periods between the two, begin at $\ell = 2$. The gray hatched regions denote periods and horizontal wavenumbers ($k \approx \ell/R$) which can propagate radially in an isothermal photosphere.

equilibrium velocity ($\mathbf{v}_o = 0$). The equation of mass continuity is unchanged, while the momentum conservation equation contains two new terms (e.g., on the left side of eq. [7.6]), representing the non-inertial centrifugal force,

$$-\tfrac{1}{2}\nabla|\boldsymbol{\Omega} \times \boldsymbol{\xi}|^2 \tag{7.56}$$

and the Coriolis force,

$$2\boldsymbol{\Omega} \times \mathbf{v} \ , \tag{7.57}$$

where $\boldsymbol{\Omega} = \Omega\hat{\mathbf{e}}_z$ is the angular velocity of rotation. Most analyses of the effects of rotation on NRPs assume sufficiently *slow* rotation so that the $\mathcal{O}(\Omega^2)$ centrifugal term can be neglected, and that all rotational effects come from the $\mathcal{O}(\Omega)$ Coriolis term. Although we will often apply this approximation below, it is by no means universally applicable, even for modes with $(\Omega/\omega) < 1$. Fortunately, however, the primary effect of the centrifugal force is to affect the *equilibrium* state by means of oblateness and gravity darkening, and it has only negligible impact on small perturbations (Unno et al. 1989). The complete (centrifugal + Coriolis) problem is quite complicated, and the use of spherical harmonic eigenfunctions and distinct p, g, f, and r modes becomes less exact for rapidly rotating stars (Lee & Saio 1990; Clement 1994; Townsend 1996).

Using the above slow-rotation approximation, then, the oscillatory equation of motion can be written as

$$\omega^2\boldsymbol{\xi} - 2i\omega(\boldsymbol{\Omega} \times \boldsymbol{\xi}) \;=\; \nabla\chi + \mathbf{A}\frac{\Gamma_1 P_o}{\rho_o}\left(\frac{\delta\rho}{\rho_o}\right) \ , \tag{7.58}$$

and, to first order (and for a star still assumed to be *spherical*), we can make use of the same separation of variables used in Section 7.1.2 to find solutions. Note, however, that the vorticity equation (eq. [7.22]) is now

$$\omega^2\left(\nabla \times \boldsymbol{\xi}\right)_r \;=\; \frac{2i\omega\Omega}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\xi_r\sin^2\theta + \xi_\theta\sin\theta\cos\theta\right) + \cos\theta\frac{\partial\xi_\phi}{\partial\phi}\right] \ , \tag{7.59}$$

and it is no longer a simple matter to separate the spherical and toroidal type solutions from one another.

Aerts (1993) finds that the standard spheroidal modes (derived in Sections 7.1.2 and 7.1.4) are rotationally perturbed by the addition of Ω-dependent spheroidal terms with the same ℓ and m as the non-rotating mode, and two toroidal modes with the same m, but $\ell \pm 1$. These first order perturbations are proportional to (Ω/ω). The toroidal terms can be derived from the rotational vorticity equation (7.59), and written in terms of the zero order (non-rotating) eigenfunctions $u_\ell^{(0)}$ and $v_\ell^{(0)}$. Lee & Saio (1990) and Townsend (1996) discuss the procedure of constructing a new set of basis eigenfunctions for an arbitrarily rotating system, and in general, an infinite sum of toroidal and spheroidal modes proportional to $Y_{\ell m}$ (for $|m| \leq \ell < \infty$) is required to express *each* rotating mode. These higher order terms have a stronger impact on the mainly horizontal g and r modes, and little impact on the p-modes.

Along with the rotational perturbation in the eigenfunctions, the eigenvalues ω are similarly affected. Ledoux (1951) first derived the general rotational "splitting" of the $2\ell + 1$ frequency degeneracy for different values of m. Expanding equation (7.58) in terms of a non-rotating solution $\boldsymbol{\xi}^{(0)}$ and a small rotational perturbation $\boldsymbol{\xi}^{(1)}$, Ledoux noted that the right-hand side of equation (7.58) vanishes when the equation is dotted into the complex conjugate $\boldsymbol{\xi}^{*(0)}$ and integrated over the entire mass of the star, and the two terms that remain are

$$2\omega^{(0)}\omega^{(1)} \int_0^{R_*} \left(\boldsymbol{\xi}^{(0)} \cdot \boldsymbol{\xi}^{*(0)}\right) 4\pi r^2 dr - 2i\omega^{(0)} \int_0^{R_*} \left[\left(\boldsymbol{\Omega} \times \boldsymbol{\xi}^{(0)}\right) \cdot \boldsymbol{\xi}^{*(0)}\right] 4\pi r^2 dr \;=\; 0 \quad . \tag{7.60}$$

Using the terminology introduced in Section 7.1.2, we can solve for the rotationally perturbed part of the frequency,

$$\omega^{(1)} \;=\; m\Omega \frac{\int_0^{R_*} \left[2u_\ell(r)v_\ell(r)/r + v_\ell^2(r)\right] \rho(r) dr}{\int_0^{R_*} \left[2u_\ell^2(r)/r^2 + \ell(\ell+1)v_\ell^2(r)\right] \rho(r) dr} \;\equiv\; m\Omega\, C_{n\ell} \quad . \tag{7.61}$$

For the homogeneous compressible model of Section 7.1.4,

$$C_{n\ell} \;\approx\; \frac{K(2+K)}{1 + \ell(\ell+1)K^2} \quad , \tag{7.62}$$

but Ledoux finds that for more realistic polytropic indices, and specifically for low-order f and g-modes, $C_{n\ell}$ is typically of the order 0.1–0.2. Papaloizou & Pringle (1978) derive, for r-modes,

$$C_{n\ell} \;\approx\; \frac{2}{\ell(\ell+1)} \quad , \tag{7.63}$$

and in this case we have the unusual situation where $\omega^{(1)} > 0$, but $\omega^{(0)} = 0$.

Finally, we can now determine the overall effect that (sufficiently slow) rotation will have on a given intrinsic NRP frequency $\omega^{(0)}$. Because the spherical harmonics are defined in the rotating frame of the star, the *observed* inertial-frame frequency is found by making the transformation

$$\phi \;=\; \phi_{\text{inertial}} - \Omega t \quad , \tag{7.64}$$

so the overall dependence of any perturbed quantity will be

$$\begin{aligned} f \;&\propto\; \exp(im\phi + i\omega t) \\ &\propto\; \exp\left[im(\phi_{\text{inertial}} - \Omega t) + i(\omega^{(0)} + \omega^{(1)})t\right] \quad , \end{aligned} \tag{7.65}$$

and, effectively,

$$\omega_{\mathrm{observed}} = \omega^{(0)} - m\Omega(1 - C_{n\ell}) \quad . \tag{7.66}$$

Thus, the convention is to call $m < 0$ modes "prograde" and $m > 0$ modes "retrograde" because of their effect on the apparent frequency of features propagating across the stellar disk. In general, it is not a trivial matter to observationally identify rotationally-split pulsation modes of this kind. Although some well-sampled objects, such as the Sun and some white dwarf stars, exhibit the full $2\ell + 1$ splitting for many neighboring (n, ℓ) modes, early-type main sequence and supergiant stars usually only show one or two NRP periods at one epoch. Thus, the variables $\omega_{\mathrm{observed}}$ and Ω are known reasonably well (the latter from $V_{\mathrm{eq}} \sin i$), but $\omega^{(0)}$, m, and $C_{n\ell}$ are all virtually unconstrained. Even if an accurate theoretical NRP mode spectrum exists, eq. (7.66) allows many combinations of, e.g., ω and m to satisfy given observational values of $\omega_{\mathrm{observed}}$ and Ω.

7.2 Wave Propagation in Winds

The discrete spectrum of NRP "standing waves" derived above depends on the stellar interior being a *bounded* system. However, we are interested in the transition between the stellar interior and the "exterior" – i.e., the photosphere and accelerating wind. In order to ascertain whether this surrounding medium is significantly affected by underlying global oscillations we need to examine the physics of *wave dispersion* in a stratified and radiatively-accelerated gas. In this Section the linearized (first order perturbation) equations of mass, momentum, and energy conservation are described, and solved for wavelike variations of the density, pressure, and velocity in various circumstances. Although most low-order NRP modes are evanescent, or exponentially damped in the photosphere, we find that the presence of an accelerating stellar wind allows these modes to propagate radially in the transsonic and supersonic regions of the wind.

7.2.1 Basic Hydrodynamic Equations

Our goal is to derive linear analytic relations governing the propagation of modified acoustic waves in an accelerating stellar wind. For simplicity we will treat only the *two-dimensional* problem of the equatorial plane ($\theta = \pi/2$) around a star, with radius r and azimuthal angle ϕ being the two spatial independent variables. Thus, we assume no flow into or out of this plane, and that all latitudinal gradients vanish there.

Let us define the velocity components $v_r \equiv w$, $v_\theta \equiv v$, and $v_\phi \equiv u$. This choice is motivated by the correspondence we will draw below between spherical polar variables r, θ, ϕ and Cartesian variables z, y, x, respectively. Thus, we can write the equation of mass continuity,

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho}{\partial r} + \rho\frac{\partial w}{\partial r} + \frac{2\rho w}{r} + \frac{u}{r}\frac{\partial \rho}{\partial \phi} + \frac{\rho}{r}\frac{\partial u}{\partial \phi} = 0 \ . \tag{7.67}$$

The r and ϕ components of the equation of momentum conservation are

$$\rho\frac{\partial w}{\partial t} + \rho w\frac{\partial w}{\partial r} + \frac{\rho u}{r}\frac{\partial w}{\partial \phi} - \frac{\rho u^2}{r} = -\frac{\partial P}{\partial r} - \rho\,(g - g_{\rm rad}) \tag{7.68}$$

$$\rho\frac{\partial u}{\partial t} + \rho w\frac{\partial u}{\partial r} + \frac{\rho u}{r}\frac{\partial u}{\partial \phi} + \frac{\rho u w}{r} = -\frac{1}{r}\frac{\partial P}{\partial \phi} \ , \tag{7.69}$$

where g and $g_{\rm rad}$ represent the gravitational and radiative acceleration terms. Let us assume an ideal gas equation of state,

$$P = \frac{\rho k_B T}{\bar{m}} = (\gamma - 1)\rho e \ , \tag{7.70}$$

where k_B is Boltzmann's constant, $\bar{m}$ is the mean molecular mass of gas particles, γ is the ratio of specific heats c_P/c_V (usually 5/3 for a monatomic gas), and e is the specific internal energy of the gas.

This system of equations is closed by an equation of internal energy conservation, given by

$$\frac{DP}{Dt} - \left(\frac{\gamma P}{\rho}\right)\frac{D\rho}{Dt} = H(T) \ , \tag{7.71}$$

where the Lagrangian total derivative is defined here as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \ . \tag{7.72}$$

The function H is a temperature-dependent rate of net heating or cooling, and is most generally a function of ρ, P, and T. For simplicity, however, we restrict ourselves to only examine its behavior with temperature (see Mihalas & Mihalas 1984). Thus, the energy conservation equation is

$$\rho\frac{\partial P}{\partial t} + \rho w\frac{\partial P}{\partial r} + \frac{\rho u}{r}\frac{\partial P}{\partial \phi} - \gamma P\frac{\partial \rho}{\partial t} - \gamma P w\frac{\partial \rho}{\partial r} - \frac{\gamma P u}{r}\frac{\partial \rho}{\partial \phi} = \rho H \ . \tag{7.73}$$

7.2.2 Linearization

Let us assume the dynamical variables in the fluid equations can be separated into an equilibrium state (zero order) and a small-amplitude perturbation (first order):

$$u \equiv u_0(r) + u_1(r,\phi,t) \tag{7.74}$$

$$
\begin{aligned}
w &\equiv w_0(r) + w_1(r, \phi, t) && (7.75)\\
\rho &\equiv \rho_0(r) + \rho_1(r, \phi, t) && (7.76)\\
P &\equiv P_0(r) + P_1(r, \phi, t) && (7.77)\\
T &\equiv T_0(r) + T_1(r, \phi, t) \ . && (7.78)
\end{aligned}
$$

The gravitational acceleration g is assumed to be a zero-order function of r only, but the radiative acceleration $g_{\rm rad}$ is more complicated. In the most general radiation hydrodynamical formulation (Mihalas & Mihalas 1984), $g_{\rm rad}$ and H depend on angle-moments of the radiation field specific intensity I_ν, which is specified by the equation of radiative transfer. However, here we assume the circumstellar radiation field is dominated by the (known) "core" stellar intensity, that the radiative acceleration is given by the standard analytic Sobolev form, and that the wavelengths of perturbations are larger than the Sobolev length $L_{\rm Sob}$ (Owocki & Rybicki 1984, hereafter OR-I). Thus, $g_{\rm rad}$ depends on r, the radial velocity w, and the density ρ (as well as the velocity gradient $\partial w/\partial r$, but this is formally a function of w and r). Expanding to first order in a Taylor series,

$$
\begin{aligned}
g_{\rm rad} &\equiv g_{\rm rad,0} + g_{\rm rad,1}\\
&= g_{\rm rad}(w_0, \rho_0) + \left\{ (w - w_0) \left. \frac{\partial g_{\rm rad}}{\partial w} \right|_{w_0} + (\rho - \rho_0) \left. \frac{\partial g_{\rm rad}}{\partial \rho} \right|_{\rho_0} \right\} \ . \qquad (7.79)
\end{aligned}
$$

Thus we can define

$$
g_{\rm rad,1} = \left\{ w_1 \left. \frac{\partial g_{\rm rad}}{\partial w} \right|_{w_0} + \rho_1 \left. \frac{\partial g_{\rm rad}}{\partial \rho} \right|_{\rho_0} \right\} \equiv w_1 F_1 + \rho_1 F_3 \ . \qquad (7.80)
$$

Similarly, the heating/cooling rate H is expressed as the sum

$$
H \equiv H_0 + H_1 = H(T_0) + T_1 \left. \frac{\partial H}{\partial T} \right|_{T_0} \equiv H_0 - \frac{C \rho_0 k_B T_1}{\bar{m}} \ , \qquad (7.81)
$$

where C is defined as a cooling relaxation time-constant (see OR-I, eq. [58]). The variables C, F_1, and F_3 are considered to be zero-order functions of radius only, and are *not* the same as the variables of the same name in Chapter 2. Note that we leave room for a variable "F_2," but defer to Section 7.2.7 to define it formally.

Substituting in these expansions yields, in general, terms of zero through third order. Let us assume that the zero-order solutions for u_0, w_0, ρ_0, P_0, and T_0 are *known* functions of r, so that the zero-order terms in the fluid equations can be canceled exactly when writing any higher-order equations. The zero-order equation of mass continuity,

$$
w_0 \frac{\partial \rho_0}{\partial r} + \rho_0 \frac{\partial w_0}{\partial r} + \frac{2 \rho_0 w_0}{r} = 0 \ , \qquad (7.82)
$$

provides a constraint linking the zero-order density and radial velocity gradients. The zero-order equations of momentum conservation reduce to

$$w_0 \frac{\partial w_0}{\partial r} - \frac{u_0^2}{r} + \frac{1}{\rho_0} \frac{\partial P_0}{\partial r} + (g - g_{\rm rad,0}) \ = \ 0 \tag{7.83}$$

$$\frac{\partial u_0}{\partial r} + \frac{u_0}{r} \ = \ 0 \ , \tag{7.84}$$

which allows us to solve for the radial variation of the azimuthal velocity, $u_0(r) = V_{\rm rot} R_*/r$, as well as eliminate the zero-order net radial force $(g - g_{\rm rad,0})$. Similarly, the zero-order heating rate can be eliminated via the zero-order energy equation,

$$H_0 \ = \ w_0 \frac{\partial P_0}{\partial r} - w_0 \left(\frac{\gamma P_0}{\rho_0} \right) \frac{\partial \rho_0}{\partial r} \ . \tag{7.85}$$

Finally, we can write the *first order* fluid equations, neglecting second and third order terms because they are small when compared to those of first order. Thus, taking care to use the above zero-order constraints to simplify, the continuity equation becomes

$$\frac{\partial \rho_1}{\partial t} + w_1 \frac{\partial \rho_0}{\partial r} + w_0 \frac{\partial \rho_1}{\partial r} + \rho_1 \frac{\partial w_0}{\partial r} + \rho_0 \frac{\partial w_1}{\partial r} + \frac{2\rho_0 w_1}{r} + \frac{2\rho_1 w_0}{r} + \frac{u_0}{r} \frac{\partial \rho_1}{\partial \phi} + \frac{\rho_0}{r} \frac{\partial u_1}{\partial \phi} \ = \ 0 \ , \tag{7.86}$$

and the momentum equation components become

$$\rho_0 \frac{\partial w_1}{\partial t} + \rho_0 w_0 \frac{\partial w_1}{\partial r} + \rho_0 w_1 \frac{\partial w_0}{\partial r} - \frac{\rho_1}{\rho_0} \frac{\partial P_0}{\partial r} + \frac{\rho_0 u_0}{r} \frac{\partial w_1}{\partial \phi} - \frac{2\rho_0 u_0 u_1}{r}$$

$$+ \frac{\partial P_1}{\partial r} - \rho_0 w_1 F_1 - \rho_0 \rho_1 F_3 \ = \ 0 \tag{7.87}$$

$$\rho_0 \frac{\partial u_1}{\partial t} + \rho_0 w_0 \frac{\partial u_1}{\partial r} + \frac{\rho_0 u_0}{r} \frac{\partial u_1}{\partial \phi} + \frac{\rho_0 w_0 u_1}{r} + \frac{1}{r} \frac{\partial P_1}{\partial \phi} \ = \ 0 \ . \tag{7.88}$$

The ideal gas equation of state can be expressed in first order as

$$\frac{P_1}{P_0} \ = \ \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \tag{7.89}$$

and the energy equation can be expanded to obtain

$$\rho_0 \frac{\partial P_1}{\partial t} + \rho_0 w_0 \frac{\partial P_1}{\partial r} + \rho_0 w_1 \frac{\partial P_0}{\partial r} + \frac{\rho_0 u_0}{r} \frac{\partial P_1}{\partial \phi} - \gamma P_0 \frac{\partial \rho_1}{\partial t} - \gamma P_0 w_0 \frac{\partial \rho_1}{\partial r} - \gamma P_0 w_1 \frac{\partial \rho_0}{\partial r}$$

$$- \frac{\gamma P_0 u_0}{r} \frac{\partial \rho_1}{\partial \phi} - \gamma P_1 w_0 \frac{\partial \rho_0}{\partial r} + \left(\frac{\gamma P_0}{\rho_0} \right) \rho_1 w_0 \frac{\partial \rho_0}{\partial r} + \frac{C \rho_0^2 k_B T_1}{\bar{m}} \ = \ 0 \ . \tag{7.90}$$

To investigate the intrinsic behavior of first-order perturbations in a moving medium, a transformation is often made to a coordinate system *comoving* with the zero-order flow. In a constant-velocity medium, this simplifies the equations of motion greatly. In our case, however, the zero-order reference frame is accelerating, thus noninertial, and there is no immediate benefit gained in transforming into the wind's frame. Bogdan et al. (1996) discuss the proper nonflat metric needed to perform this transformation for a general analysis of waves in a radiating and accelerating medium, but this is definitely beyond the scope of this work.

Thus, our independent variables remain r, ϕ, and t. The latter two variables are "homogeneous," since the zero-order solutions vary only in the radial direction. Let us then assume oscillatory solutions of the normal-mode form

$$\frac{w_1}{\tilde{W}(r)} = \frac{u_1}{\tilde{U}(r)} = \frac{\rho_1/\rho_0}{\tilde{D}(r)} = \frac{P_1/\rho_0}{\tilde{\Pi}(r)} = \frac{T_1/T_0}{\tilde{\Theta}(r)} = \exp\left[i(\omega t + m\phi)\right] \quad , \tag{7.91}$$

(see, e.g., Mihalas & Mihalas 1984), where the denominators are complex, radially-dependent amplitudes of the respective perturbed variables, and the frequency ω and azimuthal mode number m are assumed real. Of course, the actual first-order variables u_1, w_1, ρ_1, P_1, and T_1 are obtained in the end by taking the *real part* of any complex quantity. Although global stellar NRPs usually demand m to be an integer, this restriction is not yet imposed. Note that, when differentiating first-order quantities with respect to r, there will be *two* terms for the $\tilde{\Pi}$, $\tilde{D}$, and $\tilde{\Theta}$ amplitudes, because of the assumed dependence on the zero-order density and temperature.

Upon substitution, and dividing all terms by $\rho_0 \exp[i(\omega t + m\phi)]$, the fluid equations become a system of first-order ordinary differential equations. For compactness of notation, let us define the acoustic sound speed,

$$a(r) \equiv \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} \quad , \tag{7.92}$$

and the density scale height in the wind,

$$H(r) \equiv -\frac{\rho_0}{\partial \rho_0/\partial r} \quad . \tag{7.93}$$

Let us further assume the sound speed a does not vary in the wind, implying an *isothermal* zero-order wind. Because radiative heating and cooling dominates the energy balance in hot-star winds, the time scale for the gas to gain or lose energy is short when compared to the time scale of the flow. This results in the near-constancy of T_0 in the wind (see Section 7.2.4, below, and, e.g., Klein & Castor 1978; Drew 1989). However, this does not prevent first-order temperature *perturbations* from existing.

Under these approximations, the continuity equation thus becomes expressible as

$$i\omega\tilde{D} - \frac{1}{H}\tilde{W} + w_0\frac{\partial\tilde{D}}{\partial r} + \frac{\partial\tilde{W}}{\partial r} + \frac{2}{r}\tilde{W} + \frac{imu_0}{r}\tilde{D} + \frac{im}{r}\tilde{U} \;=\; 0 \;\;, \tag{7.94}$$

and the momentum equations become

$$i\omega\tilde{W} + w_0\frac{\partial\tilde{W}}{\partial r} + \frac{\partial w_0}{\partial r}\tilde{W} + \frac{a^2}{\gamma H}\tilde{D} + \frac{imu_0}{r}\tilde{W} - \frac{2u_0}{r}\tilde{U} - \frac{1}{H}\tilde{\Pi}$$

$$+ \frac{\partial\tilde{\Pi}}{\partial r} - F_1\tilde{W} - \rho_0 F_3\tilde{D} \;=\; 0 \tag{7.95}$$

$$i\omega\tilde{U} + w_0\frac{\partial\tilde{U}}{\partial r} + \frac{imu_0}{r}\tilde{U} + \frac{w_0}{r}\tilde{U} + \frac{im}{r}\tilde{\Pi} \;=\; 0 \;\;. \tag{7.96}$$

The energy equation becomes (after eliminating $\tilde{\Theta}$ in favor of $\tilde{\Pi}$ and $\tilde{D}$, and dividing out another factor of the zero-order density)

$$i\omega\tilde{\Pi} + w_0\frac{\partial\tilde{\Pi}}{\partial r} + \frac{(\gamma-1)\,w_0}{H}\tilde{\Pi} + \left(\frac{\gamma-1}{\gamma}\right)\frac{a^2}{H}\tilde{W} + \frac{imu_0}{r}\tilde{\Pi} - i\omega a^2\tilde{D}$$

$$-\, a^2 w_0\frac{\partial\tilde{D}}{\partial r} - \frac{ima^2u_0}{r}\tilde{D} + C\tilde{\Pi} - \frac{Ca^2}{\gamma}\tilde{D} \;=\; 0 \;\;. \tag{7.97}$$

The above system of coupled first-order ordinary differential equations is solved most generally as an initial value problem, with values of the complex amplitudes $\tilde{U}$, $\tilde{W}$, $\tilde{D}$, and $\tilde{\Pi}$ specified at a known initial radius. A numerical scheme, such as Runge-Kutta, then integrates this system in radius to obtain the full solution for the first-order wave variations in the wind. Further, any of the radially-dependent complex amplitudes can be written at any radius as

$$\tilde{Z} \;=\; A_Z \exp(i\psi_Z) \;\;, \tag{7.98}$$

where here $\tilde{Z}$ represents either $\tilde{U}$, $\tilde{W}$, $\tilde{D}$, or $\tilde{\Pi}$, and its magnitude A_Z and phase ψ_Z are real. Below, it will be convenient to express various quantities, such as the wave phase speed, as functions of these real components of the complex amplitudes. As in geometrical optics, the function ψ_Z may be regarded, over a small region of space, as an *eikonal,* with the linear wave vector and frequency defined as

$$\mathbf{k} \;\equiv\; -\nabla\psi_Z \;\;, \qquad\qquad \omega \;\equiv\; \frac{\partial\psi_Z}{\partial t} \;\;. \tag{7.99}$$

Here, however, ψ_Z is a function of position only, and not of time. In general, when the properties of the medium at each point in space do not vary in time, the frequency remains *constant* along the wave path. Thus we assume that a stellar atmosphere "driven" at a certain frequency ω should induce coherent modulations throughout the wind at that same frequency.

7.2.3 Local Dispersion Analysis

Before exploring full global solutions to the linearized first-order ordinary differential equations (in Section 7.2.8), it is instructive to examine the limiting cases of various local environments that can be treated in a simpler way:

1. A **homogeneous medium** (constant u_0, w_0, ρ_0, P_0).
2. A **stratified atmosphere** (varying ρ_0 and P_0, $w_0 \ll a$, constant u_0).
3. A **subsonic wind** ($w_0 \ll a$), superimposed on a stratified plane-parallel atmosphere.
4. A **supersonic wind** ($w_0 \gg a$), with sufficiently high frequency or short wavelength modes.

In these cases, let us assume that the *local* radial variation of the complex amplitudes is given by

$$\tilde{Z}(r) \equiv Z \exp(-ik_r r) \ , \tag{7.100}$$

where the radial wavenumber k_r, in the most general formulation, is allowed to be complex. Note that there is a *single* and *constant* value of k_r defined for all four complex amplitudes; this is consistent only when the coefficients of U, W, D, and Π in the linearized equations are *constants* in radius as well. For more complicated situations, the equations demand that k_r itself vary in radius, and one must also question (and usually reject) the entire notion of a single k_r for all four complex amplitudes.

Given the above radial dependence for the first-order amplitudes, the equations of motion can be written in matrix form as

$$\begin{pmatrix} M_{1U} & M_{1W} & M_{1D} & M_{1\Pi} \\ M_{2U} & M_{2W} & M_{2D} & M_{2\Pi} \\ M_{3U} & M_{3W} & M_{3D} & M_{3\Pi} \\ M_{4U} & M_{4W} & M_{4D} & M_{4\Pi} \end{pmatrix} \begin{pmatrix} U \\ W \\ D \\ \Pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ , \tag{7.101}$$

where the numbers 1, 2, 3, and 4 refer to the continuity, r-momentum, ϕ-momentum, and energy equations, respectively. These coefficients are listed here for completeness, but note that only certain limiting cases (discussed below) are completely consistent with this localized analysis:

$$M_{1U} \ = \ \frac{im}{r} \tag{7.102}$$

$$\begin{aligned}
M_{1W} &= -ik_r - \frac{1}{H} + \frac{2}{r} && (7.103)\\
M_{1D} &= i\omega - ik_r w_0 + \frac{imu_0}{r} && (7.104)\\
M_{1\Pi} &= 0 && (7.105)\\
M_{2U} &= -\frac{2u_0}{r} && (7.106)\\
M_{2W} &= i\omega - ik_r w_0 + \frac{imu_0}{r} + \frac{\partial w_0}{\partial r} - F_1 && (7.107)\\
M_{2D} &= \frac{a^2}{\gamma H} - \rho_0 F_3 && (7.108)\\
M_{2\Pi} &= -ik_r - \frac{1}{H} && (7.109)\\
M_{3U} &= i\omega - ik_r w_0 + \frac{imu_0}{r} + \frac{w_0}{r} && (7.110)\\
M_{3W} &= 0 && (7.111)\\
M_{3D} &= 0 && (7.112)\\
M_{3\Pi} &= \frac{im}{r} && (7.113)\\
M_{4U} &= 0 && (7.114)\\
M_{4W} &= \frac{(\gamma-1)a^2}{\gamma H} && (7.115)\\
M_{4D} &= -i\omega a^2 + ik_r w_0 a^2 - \frac{imu_0 a^2}{r} - \frac{Ca^2}{\gamma} && (7.116)\\
M_{4\Pi} &= i\omega - ik_r w_0 + \frac{imu_0}{r} + \frac{(\gamma-1)w_0}{H} + C && (7.117)
\end{aligned}$$

Of course, all coefficients in this *homogeneous* system have been divided by $\exp(-ik_r r)$. In addition, the homogeneity of the system allows one of the four amplitudes to be chosen as a free parameter. We will assume hereafter that we can arbitrarily assign

$$D = \delta + 0i \ , \qquad (7.118)$$

where $\delta \ll 1$, and D is chosen to be *real* to allow convenient comparison with the other variables (which in general all lead or lag each other in complex phase). Also, the complex temperature amplitude is given immediately by

$$\Theta = \frac{\rho_0}{P_0}\Pi - D \ . \qquad (7.119)$$

From the equation of ϕ-momentum conservation, the pressure and horizontal velocity amplitudes are directly related by

$$\Pi = \left(\frac{iw_0 - \omega' r}{m} \right) U \equiv nU \ , \tag{7.120}$$

where $\omega' \equiv \omega - k_r w_0 + m u_0 / r$, and is an effective comoving or Doppler-shifted frequency. Note that ω' is the frequency in the locally-inertial frame defined by the translations $r \rightarrow (r + w_0 t)$ and $r\phi \rightarrow (r\phi - u_0 t)$. From the mass and r-momentum conservation equations, the remaining two amplitude relations are given schematically by

$$W = \left[\frac{(M_{2U} + nM_{2\Pi})M_{1D} - M_{1U}M_{2D}}{M_{1U}M_{2W} - (M_{2U} + nM_{2\Pi})M_{1W}} \right] D \tag{7.121}$$

$$U = -\frac{1}{M_{1U}} (M_{1W} W + M_{1D} D) \ . \tag{7.122}$$

In the "one-dimensional" limit of $m = 0$, the above relations contain canceling infinities, and it becomes easier to express them as follows:

$$U = 0 \ , \quad W = -\left(\frac{M_{1D}}{M_{1W}} \right) D \ , \quad \Pi = -\frac{(M_{2W} W + M_{2D} D)}{M_{2\Pi}} \ . \tag{7.123}$$

Finally, if the frequency ω and the azimuthal mode number m are specified arbitrarily, the allowed values for the radial wavenumber k_r can be determined uniquely by considering them eigenvalues of the coefficient matrix M. In other words, there can exist a nontrivial solution of eq. (7.101) only if the determinant of M vanishes. This determinant results in a fourth-order polynomial equation, and in general there are four allowed solutions for k_r. Let us examine this so-called *dispersion relation* for the four localized cases listed above.

7.2.4 A Homogeneous Medium

Let us assume all zero-order quantities, u_0, w_0, ρ_0, P_0 are constants with position, and that the local medium can be considered nearly Cartesian, or plane-parallel. Thus, the radial and azimuthal variations can be recast into the form

$$\exp(-ik_r r) \Longrightarrow \exp(-ik_z z) \tag{7.124}$$

$$\exp(im\phi) \Longrightarrow \exp(-ik_x x) \ , \tag{7.125}$$

where x (equivalent to $r[\phi - \phi_0]$) and z (equivalent to $[r - r_0]$) are horizontal and vertical displacement coordinates, respectively. In these conditions, the dispersion matrix becomes

$$\begin{vmatrix} -ik_x & -ik_z & i\omega' & 0 \\ 0 & i\omega' & 0 & -ik_z \\ i\omega' & 0 & 0 & -ik_x \\ 0 & 0 & -i\omega' a^2 - Ca^2/\gamma & i\omega' + C \end{vmatrix} = 0 \ , \qquad (7.126)$$

where, of course, all sphericity, stratification, and radiative acceleration terms vanish, and $\omega' \equiv \omega - k_z w_0 - k_x u_0 = \omega - \mathbf{k} \cdot \mathbf{v}_0$. The algebraic dispersion relation is

$$\omega'^3(\omega' - iC) - \omega' a^2(k_z^2 + k_x^2)(\omega' - iC/\gamma) = 0 \ . \qquad (7.127)$$

In a purely static ($u_0 = w_0 = 0$) and adiabatic ($C = 0$) medium, eq. (7.127) reduces to the pure acoustic dispersion relation,

$$\omega^2 = a^2(k_z^2 + k_x^2) \ , \qquad (7.128)$$

where two factors of the trivial solution $\omega = 0$ have been eliminated. Waves propagate parallel to the wave vector $\mathbf{k} = k_z \hat{\mathbf{e}}_z + k_x \hat{\mathbf{e}}_x \equiv (k_z^2 + k_x^2)^{1/2} \hat{\mathbf{e}}_k \equiv k \hat{\mathbf{e}}_k$. The phase speed and group velocity are defined as

$$v_p(\hat{\mathbf{n}}) \equiv \frac{\omega}{\mathbf{k} \cdot \hat{\mathbf{n}}} \ , \qquad \mathbf{v}_g \equiv \nabla_k \omega = \frac{\partial \omega}{\partial k_z} \hat{\mathbf{e_z}} + \frac{\partial \omega}{\partial k_x} \hat{\mathbf{e_x}} \ . \qquad (7.129)$$

Note that the phase speed (in the arbitrary $\hat{\mathbf{n}}$ direction) cannot be written as a vector velocity, since, e.g., the x and z phase speeds do not vectorially add to the phase speed in the direction of propagation $\hat{\mathbf{e}}_k$. This same property is what prevents the *wavelength* ($\lambda = 2\pi/k$) from being written as a vector quantity. Note that the phase speed in the direction of propagation and the magnitude of the group velocity are the same in this present case, and both equal to $\pm a$. The relative perturbation amplitudes can be found easily, from the individual equations in (7.101), and written in terms of the density amplitude as

$$\Pi = a^2 D \ , \qquad U = \frac{a^2 k_x}{\omega} D \ , \qquad W = \frac{a^2 k_z}{\omega} D \ , \qquad (7.130)$$

or as a general velocity amplitude

$$V \equiv \sqrt{U^2 + W^2} = \pm a\, \delta \ . \qquad (7.131)$$

Thus, for acoustic waves, the velocity, density, pressure, and temperature all vary *in phase* with one another.

If the adiabatic ($C = 0$) fluid is now examined in a *moving* inertial frame (with u_0, w_0 constant and nonzero), then ω is simply replaced by ω', and the nontrivial solution to the dispersion equation is

$$\omega = u_0 k_x + w_0 k_z \pm a\sqrt{k_x^2 + k_z^2} \ , \tag{7.132}$$

Note, however, that the trivial solution, now $\omega' = 0$, represents a modulation which is passively carried along with the mean fluid motion. This "entropy-mode" solution corresponds to a constant-pressure ($\Pi = 0$) wave which would occur in an incompressible fluid ($\nabla \cdot \mathbf{v} = 0$). This wave solution is incompatible with the arbitrary specification of the frequency ω at a given location in the wind, and we will not pursue these solutions at present.

Although the phase speeds in the x and z directions are not equal to the x and z components of the group velocity, the phase speed in the direction of propagation remains equal to the magnitude of the group velocity:

$$v_p(\hat{\mathbf{e}}_k) = v_g = \frac{u_0 k_x + w_0 k_z \pm ak}{k} \ . \tag{7.133}$$

In the limit of both $|u_0|, |w_0| \gg a$, as happens in some regions of a supersonic wind, the two oppositely-propagating wave solutions merge into one solution moving at the velocity of the medium. The velocity amplitudes can be written similar to eq. (7.130), with ω' replacing ω, or in vector form, as

$$\mathbf{V} = \frac{a^2 \mathbf{k}}{\omega'} D \ . \tag{7.134}$$

Heuristically, the uniformly moving medium effectively "stretches" an acoustic wave via the Doppler effect. To quantify this, assume a simple *one-dimensional* z-oscillation ($k_x = 0$), in a medium moving only in the z-direction ($u_0 = 0$), which has a frequency

$$\omega \equiv aK = w_0 k_z \pm a k_z \ , \tag{7.135}$$

with K defined as the vertical wavenumber (or inverse wavelength, $2\pi/\Lambda$) in the case of a stationary medium. Thus, the ratio of wavenumbers, or wavelengths, is

$$\frac{K}{k_z} = \frac{\lambda_z}{\Lambda} = \frac{w_0}{a} \pm 1 \tag{7.136}$$

which is precisely the wavelength shift due to Doppler motions.

Finally, let us examine the effect of net heating or cooling ($C \neq 0$) on waves in a homogeneous medium. For simplicity, consider a static medium. The nontrivial dispersion relation can then be written as

$$\omega^2 \,=\, a^2 k^2 \left[\frac{\omega - iC/\gamma}{\omega - iC} \right] \;\; . \tag{7.137}$$

If the cooling rate is negligibly slow or absent ($C \ll \omega$), waves propagate acoustically, but if the cooling rate is rapid ($C \gg \omega$), the phase speed approaches

$$v_p^2 \,=\, \frac{\omega^2}{k^2} \,\rightarrow\, \frac{a^2}{\gamma} \,=\, \frac{P_0}{\rho_0} \,\equiv\, a_T^2 \;\; , \tag{7.138}$$

where a_T is an effective *isothermal* sound speed (equivalent to setting $\gamma = 1$ in the definition of a). Thus, when the energy balance is dominated by rapid cooling, all perturbations are quickly brought into thermal equilibrium. For intermediate values of the cooling rate ($C \approx \omega$) waves will be damped by the imaginary component of the resulting wavenumber.

As an aside, let us examine several physical mechanisms that can be responsible for a net rate of heating or cooling. If the fluid allows *thermal conduction,* then

$$H(T) \,=\, \nabla \cdot (K_T \nabla T) \,\approx\, K_T \nabla^2 T \;\; , \tag{7.139}$$

where K_T is the coefficient of thermal conductivity. Thus, we can write

$$C \,=\, \chi \left(\frac{\gamma}{\gamma - 1} \right) \left(k_x^2 + k_z^2 \right) \,\equiv\, D k^2 \;\; , \tag{7.140}$$

where $\chi \equiv K_T / \rho_0 c_P$ is the thermal diffusivity, and the specific heat at constant pressure $c_P = \gamma k_B / (\gamma - 1) \bar{m}$. Mihalas & Mihalas (1984) investigate the solutions to the modified dispersion relation in this case, and find two classes of wave modes. One pair of solutions are standard acoustic modes, only slightly modified by conductive damping terms,

$$k \,\approx\, \begin{cases} \pm(\omega/a)[1 - i(\gamma - 1)\varepsilon/2] \; , & \varepsilon \ll 1 \\ \pm(\omega/a_T)[1 - i(\gamma - 1)/2\varepsilon\gamma^2] \; , & \varepsilon \gg 1 \; , \end{cases} \tag{7.141}$$

with $\varepsilon \equiv \omega\chi/a^2$, the relative strength of conduction. Thus, the characteristic damping (or e-folding) length L of oscillations scales as $1/\mathrm{Im}(k)$, and is *large* in the extreme limits $\varepsilon \ll 1$ and $\varepsilon \gg 1$. However, when $\varepsilon \approx 1$, the real and imaginary parts of k are comparable, and waves damp out over a small L. The second pair of solutions are the so-called *thermal waves,* with

$$k \,\approx\, \begin{cases} \pm(\omega/a)(2\varepsilon)^{-1/2}(1 - i) \; , & \varepsilon \ll 1 \\ \pm(\omega/a_T)(2\varepsilon)^{-1/2}(1 - i) \; , & \varepsilon \gg 1 \; , \end{cases} \tag{7.142}$$

which are always heavily damped (over a few wavelengths), and propagate with an extremely *slow* or *fast* phase speed, compared to a, in the two limits $\varepsilon \ll 1$ and $\varepsilon \gg 1$, respectively.

In hot-star winds, radiation can strongly damp out temperature fluctuations by providing an added "sink" for energy exchange, analogous to the transfer of wave energy into *entropy* for the above heat-conducting fluid. For optically thin disturbances ($\kappa \ll k$, see below), the net heat input to the gas is adequately described by Newtonian cooling, with the constant

$$C \equiv \frac{1}{t_{RR}} = \frac{16\sigma_B \kappa T_0^3}{\rho_0 c_V} , \tag{7.143}$$

where t_{RR} is a characteristic radiative relaxation time, σ_B is the Stefan-Boltzmann constant, κ is the opacity (in cm^2/g) of the gas, and c_V is the specific heat at constant volume. For an optically thick disturbance ($\kappa \gg k$), radiation and matter are in thermal equilibrium, and energy is exchanged via a conduction-like diffusion. The resulting wave propagation is then formally identical to the pure conduction case above, but with

$$K_T = K_T^{\mathrm{thermal}} + K_T^{\mathrm{radiation}} . \tag{7.144}$$

Note, however, that K_T in an ionized or radiating fluid is now a function of temperature, and the treatment of such *nonlinear conduction* is an ongoing subject of study in radiation hydrodynamics.

For simplicity, however, we will keep $C = 0$ in all subsequent cases, and either assume that first-order perturbations are adiabatic ($\gamma = 5/3$) or isothermal ($\gamma = 1$). In fact, in preliminary hydrodynamical simulations using VH-1, the strong radiative cooling which keeps the zero-order wind nearly isothermal *also* drives most perturbations to behave as if $\gamma = 1$ in most of the wind. See, e.g., OR-I for further discussion of this heating/cooling parameter in radiatively-driven winds.

7.2.5 A Stratified Atmosphere

The standard theoretical description of waves in a gravitationally-stratified, plane-parallel medium is that of *gravo-acoustic* waves (Lamb 1932; Mihalas & Mihalas 1984). However, there is a simpler formalism that should be understood prior to the general full gravo-acoustic analysis: the extreme limit of gravity forces being much stronger than pressure-gradient forces.

In a medium where *gravity* dominates as the restoring force on perturbations, we can neglect pressure-gradient forces, and write the dispersion matrix under the ansatz that $\Pi = 0$, i.e., all variations remain in pressure equilibrium, and thus all motions are *adiabatic*. If we also assume the mean state is at rest ($u_0 = w_0 = 0$), and radiative forces and cooling rates are negligible ($C = F_1 = F_3 = 0$), we obtain

$$\begin{vmatrix} -ik_x & -ik_z - 1/H & i\omega & 0 \\ 0 & i\omega & a^2/\gamma H & 0 \\ i\omega & 0 & 0 & 0 \\ 0 & (\gamma-1)a^2/\gamma H & -i\omega a^2 & 0 \end{vmatrix} = 0 \ , \qquad (7.145)$$

which is now effectively *uncoupled,* since the determinant is identically zero. Note also that the assumed isothermality of the medium implies that the zero-order equation of hydrostatic equilibrium,

$$\frac{\partial P_0}{\partial z} = \frac{a^2}{\gamma}\frac{\partial \rho_0}{\partial z} = -\rho_0 g \ , \qquad (7.146)$$

(which is the limit of $u_0 = w_0 = 0$ in eq. [7.83]) can be integrated to find

$$\rho_0(z) = \rho_0(0)\exp(-z/H) \ , \qquad (7.147)$$

and in a plane-parallel atmosphere, g and $H = a^2/\gamma g$ are *constants.* Thus, the upper-left three-fourths of the above dispersion matrix (ignoring all Π terms and the energy equation) can be solved for

$$\omega^2 = igk_z + \frac{g}{H} \ . \qquad (7.148)$$

Also note that $U = 0$ is mandated by the x-momentum equation, mainly because there are no restoring forces in the horizontal direction. Finally, if one takes $-a^2$ times the continuity equation, and compares it term-by-term with the energy equation, one obtains a unique solution for the vertical wavenumber, $k_z = ig/a^2$. A purely imaginary wavenumber implies no wave *propagation,* but rather an exponentially growing (unstable) or decaying (stable) oscillation in z. Substituting this imaginary k_z into the dispersion relation provides the definition for the specified frequency of buoyancy oscillations, called the Brunt-Väisälä frequency:

$$\omega_{BV}^2 \equiv \frac{g}{\rho_0}\left(\frac{1}{a^2}\frac{\partial P_0}{\partial z} - \frac{\partial \rho_0}{\partial z}\right) = (\gamma - 1)\frac{g^2}{a^2} \ , \qquad (7.149)$$

where the latter equality is valid only for our assumed constant-T_0 equilibrium model. This relation is also known as the Schwarzschild criterion for stability against convection: when $\omega_{BV}^2 > 0$ the medium is stably stratified, and when $\omega_{BV}^2 < 0$ the medium is unstable to convective overturning.

It is clear that if *both* the zero-order medium and the first-order perturbations are isothermal (implying $\gamma = 1$ everywhere), buoyancy oscillations cannot occur, and the medium is considered "neutrally stable." This condition can be also understood by examining the oscillation amplitudes, which are given by

$$U = \Pi = 0 \ , \qquad W = \frac{ig}{\omega} D \ , \qquad \Theta = -D \ , \tag{7.150}$$

and it is clear that buoyancy oscillations depend on a nonzero *density* perturbation. If the oscillations are adiabatic (in pressure equilibrium) *and* isothermal ($\gamma = 1$, and $\Theta = 0$), clearly the density oscillation amplitude must also be zero. Note also that for finite amplitude oscillations, the velocity and density are 90° out of phase, and the temperature and density are 180° out of phase.

A significantly more realistic treatment of oscillations in a gravitationally stratified medium takes both pressure-gradient forces and gravity into account. Let us write the dispersion matrix for a stratified, but unmoving mean state ($u_0 = w_0 = 0$), still ignoring the effects of radiative acceleration and net cooling/heating, and still assuming a plane-parallel geometry. In addition, assume the zero-order medium is isothermally stratified, with density given by eq. (7.147). Thus,

$$\begin{vmatrix} -ik_x & -ik_z - \gamma g/a^2 & i\omega & 0 \\ 0 & i\omega & g & -ik_z - \gamma g/a^2 \\ i\omega & 0 & 0 & -ik_x \\ 0 & (\gamma - 1)g & -i\omega a^2 & i\omega \end{vmatrix} = 0 \ , \tag{7.151}$$

which reduces to the (complex) dispersion relation

$$\omega^4 - \omega^2 \left[a^2(k_x^2 + k_z^2) - i\gamma g k_z \right] + g^2 k_x^2 (\gamma - 1) = 0 \ . \tag{7.152}$$

Because we are presently concerned with steady-state oscillations, we will assume a *real* frequency ω. Note, however, that the use of a *complex* frequency would allow some transient wave-energy transport to occur in ways not previously expected (see Wang, Ulrich, & Coroniti 1995). We will also assume that the horizontal wavenumber k_x is real, since the ambient medium is horizontally homogeneous, and no nonlocal variations in x are expected.

However, the vertical wavenumber k_z can be arbitrarily complex, because a wave can grow or decay in amplitude as it propagates vertically in the stratified medium. Defining $k_z \equiv k_r + ik_i$, the *imaginary* part of the dispersion relation can be written

$$\omega^2 \left(\gamma g k_r - 2a^2 k_i k_r \right) = 0 \ , \tag{7.153}$$

which solves simply for k_i (when $k_r \neq 0$)

$$k_i = \frac{\gamma g}{2a^2} = \frac{1}{2H} \ . \tag{7.154}$$

Thus, all propagating oscillatory solutions contain a factor $\exp(z/2H)$, implying *rapid growth* of wave amplitudes with height, or with decreasing ambient density. Using this value for k_i, the real part of the dispersion relation reduces to

$$\omega^4 - \omega^2 \left[a^2(k_x^2 + k_r^2) + \omega_a^2 \right] + \omega_g^2 a^2 k_x^2 = 0 \ , \tag{7.155}$$

where ω_a is the so-called "acoutstic cutoff" frequency,

$$\omega_a \equiv \frac{\gamma g}{2a} = \frac{a}{2H} \ , \tag{7.156}$$

and ω_g is the Brunt-Väisälä frequency for an isothermal medium (eq. [7.149]). Note that $\omega_a \gtrsim \omega_g$ for physically relevant values of γ (between 1 and 5/3). Solving the real dispersion relation for the vertical wavenumber,

$$k_r^2 = \frac{(\omega^2 - \omega_a^2)}{a^2} - \frac{(\omega^2 - \omega_g^2)\, k_x^2}{\omega^2} \ , \tag{7.157}$$

we find that, for purely vertical waves ($k_x = 0$), k_r approaches the pure acoustic limit of $(\pm\,\omega/a)$ for $\omega \gg \omega_a$, but grows smaller for finite values of ω_a. When $\omega = \omega_a$, $k_r = 0$, and there can be no vertical propagation. An imposed oscillation at this frequency coherently lifts and drops the entire medium, an effect known as "atmospheric resonance," first noted by Lamb. In numerical wind models, these Lamb oscillations represent a persistent "ringing" of the near-static base boundary conditions, and often hinder or prevent the formation of a steady-state solution (see Feldmeier 1995).

Frequencies below ω_a make k_r imaginary, implying amplitudes which vary exponentially with height. These *evanescent* waves have a formally infinite vertical phase velocity (since we define v_p as a function of only the real, i.e. propagating, part of $\mathbf{k}$), implying *zero* group velocity (since for gravo-acoustic waves, $v_p v_g = a^2$), and thus they transport no energy vertically.

The general two-dimensional case ($k_x \neq 0$) similarly allows three types of solutions for k_r (propagating, resonant, and evanescent), but with these regions separated into distinct areas of (ω, k_x) space (see Figure 7.1). Two "propagation boundary curves" (on which $k_r = 0$) split up the space into three regions, bounded by

$$\omega_{pbc}^2(k_x) = \frac{\omega_a^2 + a^2 k_x^2 \pm \sqrt{(\omega_a^2 + a^2 k_x^2)^2 - 4a^2 k_x^2 \omega_g^2}}{2} \ . \tag{7.158}$$

These solutions for ω_{pbc} reduce to the two values $\{0, \omega_a\}$ in the limit $k_x = 0$, and approach the asymptotic limits $\{\omega_g, ak_x\}$ in the limit $k_x \to \infty$. Above the upper branch, $k_r^2 > 0$, and solutions propagate as gravity-modified acoustic waves. Below the lower branch, $k_r^2 > 0$ as well, and waves appear more and more like the gravitationally-dominated buoyancy waves defined above, and here are called *internal gravity waves.* Between the two propagation boundary curves, $k_r^2 < 0$, and this is an evanescent region. Note that in this region, $k_i \neq 1/2H$, and in general, there are two solutions for k_i, one larger and one smaller than $1/2H$ (Wang et al. 1995).

The phase and group speeds of the two varieties of propagating gravo-acoustic waves are given in full by Mihalas & Mihalas (1984), as are the amplitudes and phases of the first-order fluid variables U, W, D, Π, and Θ. In the limits of high and low frequency these quantities approach the purely acoustic or buoyancy wave solutions found previously, and become more complicated for intermediate values. The phase speed in the z-direction is easily computed from eq. (7.157), however, and is

$$\frac{\omega}{k_r} = \pm \frac{a\omega^2}{\sqrt{\omega^2(\omega^2 - \omega_a^2) - a^2 k_x^2(\omega^2 - \omega_g^2)}} \quad . \tag{7.159}$$

Because most observed pulsation modes in hot stars have relatively low frequencies and long horizontal wavelengths, it may be instructive to analyze gravo-acoustic waves in this limit, as well as the limit of a purely isothermal medium. When $\gamma = 1$, the complex wavenumber k_z is given by

$$\begin{aligned} 2k_z H &= i \pm \sqrt{\left(\frac{\omega}{\omega_a}\right)^2 - 1 - 4H^2 k_x^2} && (7.160) \\ &\approx 2i \left\{ \left[1 + H^2 k_x^2 - \frac{1}{4}\left(\frac{\omega}{\omega_a}\right)^2\right] , \left[-H^2 k_x^2 + \frac{1}{4}\left(\frac{\omega}{\omega_a}\right)^2\right] \right\} \quad , && (7.161) \end{aligned}$$

where the latter approximation is in the limit of $\omega \ll \omega_a$ and $k_x H \ll 1$. This region is clearly evanescent, as no internal gravity waves are possible when $\gamma = 1$. Note the existence of a rapidly-varying solution ($k_i H \approx 1$) and a slowly-varying solution ($|k_i H| \ll 1$), which take the place of the upward/downward propagating solutions in the evanescent limit. The velocity amplitudes in the $\gamma = 1$ limit are

$$U = \left(\frac{k_x a^2}{\omega}\right) D \quad , \qquad W = \frac{iH}{\omega}\left(\frac{\omega^2 - a^2 k_x^2}{ik_z H + 1}\right) D \quad , \tag{7.162}$$

and the rapidly-varying (+) solution for k_z results in a wave mode with $|W| \gg |aD|$, while the slowly-varying (−) solution results in $|W| \ll |aD|$.

7.2.6 A Subsonic Wind

Finally, consider a plane-parallel stratified medium with a slow (subsonic) wind. In the limit of $w_0 \ll a$, the isothermal density stratification is still exponential, and the zero-order continuity equation demands

$$w_0(z) \;=\; w_0(0)\exp(+z/H) \;\;, \qquad \frac{\partial w_0}{\partial z} \;=\; \frac{\partial w_0}{\partial r} \;=\; \frac{w_0}{H} \;\; . \tag{7.163}$$

Thus, we can generalize the gravo-acoustic wave model above to include this nascent flow. For simplicity, let us neglect any horizontal flow ($u_0 = 0$) and restrict ourselves to a purely isothermal medium ($\gamma = 1$). The nontrivial dispersion relation is then

$$\omega'^2 - a^2(k_x^2 + k_z^2) + igk_z + \frac{iw_0}{H}\left(\frac{a^2 k_x^2 - \omega'^2}{\omega'}\right) \;=\; 0 \;\; , \tag{7.164}$$

where $\omega' \;=\; \omega \,-\, w_0 k_z$. After this substitution, this equation becomes a cubic in complex k_z, and can be solved analytically. Much simpler, however, is the vertical propagation limit of $k_x = 0$, which only results in a quadratic in k_z. This is solved by

$$(w_0^2 - a^2)k_z \;=\; \omega w_0 - \frac{i(w_0^2 + a^2)}{2H} \pm \sqrt{\omega^2 a^2 - \frac{2i\omega w_0 a^2}{H} - \frac{(w_0^2 + a^2)^2}{4H^2}} \;\; , \tag{7.165}$$

and reduces, in the limit $\epsilon \equiv w_0/a \ll 1$, to

$$2k_z H \;\approx\; i - \epsilon\left(\frac{\omega}{\omega_a}\right) \pm \sqrt{\left(\frac{\omega}{\omega_a}\right)^2 - 1 - 4i\epsilon\left(\frac{\omega}{\omega_a}\right)} \;\; . \tag{7.166}$$

Note that, even in the evanescent limit, there will always be a *real* part of k_z, and thus the presence of the wind helps all wave modes to propagate. This result is significant because it provides low-order NRP modes, many of which are evanescent in the photosphere, a mechanism to begin to "tunnel" their way out into the wind. The requirement, of course, is for these modes to have sufficient *amplitude* to survive the static evanescent photosphere.

Acoustic modes are only slightly modified by the wind, while evanescent modes are strongly affected. In the high-frequency limit ($\omega \gg \omega_a$),

$$2k_z H \;\approx\; \frac{\omega}{\omega_a}(\mp 1 - \epsilon) + i(1 \pm 2\epsilon) \;\; , \tag{7.167}$$

and in the low-frequency limit ($\omega \ll \omega_a$),

$$2k_z H \;\approx\; \left\{\left[2i - 3\epsilon\left(\frac{\omega}{\omega_a}\right)\right] \; , \; \left[\epsilon\left(\frac{\omega}{\omega_a}\right)\right]\right\} \;\; . \tag{7.168}$$

For these quasi-evanescent waves, then, the vertical phase speeds are large, but not infinite:

$$v_p \equiv \frac{\omega}{\mathrm{Re}(k_z)} \approx \left\{ -\frac{a^2}{3w_0} , \frac{a^2}{w_0} \right\} . \qquad (7.169)$$

Note that it is the slowly-varying evanescent solution (here with $k_i H \approx 0$) which corresponds to the outwardly-propagating wave, and the rapidly-varying evanescent solution (here with $k_i H \approx 1$) which corresponds to the inwardly-propagating wave.

7.2.7 A Supersonic Wind

Abbott (1980) examined the physical significance of the critical point in CAK line-driven winds by a linearized wave analysis in the *comoving* frame of the wind. In order for the localized "wavenumber" picture to be valid, however, one must assume wavelengths shorter than the zero-order density and velocity scale lengths, but longer than the Sobolev length L_{Sob} (see Section 2.4). Abbott (1980) assumed that perturbations to the radiative acceleration are dependent only on perturbations in the radial *velocity gradient.* We thus re-cast the first-order expansion variable F_1, which represents the velocity dependence of the force (see eq. [7.80]) as

$$F_1 = -ik_z F_2 , \qquad \text{where} \quad F_2 \equiv (\partial g_{\mathrm{rad}}/\partial[\partial w/\partial r])_0 , \qquad (7.170)$$

which is a completely equivalent means of expressing the perturbed radiative acceleration (in the approximation that there is no *other* dependence on the velocity w). Thus, ignoring nonlocal sphericity and stratification effects ($1/H \rightarrow 0, \partial w_0/\partial r \rightarrow 0$) and transforming into a comoving frame ($u_0 = w_0 = 0$), the dispersion matrix becomes

$$\begin{vmatrix} -ik_x & -ik_z & i\omega & 0 \\ 0 & i\omega + ik_z F_2 & -\rho_0 F_3 & -ik_z \\ i\omega & 0 & 0 & -ik_x \\ 0 & 0 & -i\omega a^2 & i\omega \end{vmatrix} = 0 . \qquad (7.171)$$

Abbott (1980) also assumed that $F_3 = 0$, which is valid for a point-star CAK force when $\omega \gg \partial w_0/\partial r$ (which is consistent with the assumption of wavelengths shorter than the local scale heights). Thus, the dispersion relation reduces to

$$\omega^3 + \omega^2(k_z F_2) - \omega(a^2 k_x^2 + a^2 k_z^2) - a^2 k_x^2 k_z F_2 = 0 , \qquad (7.172)$$

and is equivalent to Abbott's eq. (45).

Because the radiation is directed in the vertical direction, horizontal ($k_z = 0$) perturbations propagate purely acoustically, but vertical ($k_x = 0$) perturbations propagate at modified real *radiative-acoustic* phase speeds

$$C_{\pm} \equiv \frac{\omega}{k_z} = -\frac{F_2}{2} \pm \sqrt{\left(\frac{F_2}{2}\right)^2 + a^2} \ . \tag{7.173}$$

In most regions of a line-driven wind, $F_2 \gg a$, so that $C_+ \approx a^2/F_2$, and $C_- \approx -F_2$, signifying slower (subsonic) outward propagation and faster (supersonic) inward propagation in the frame of the wind. At the mCAK critical point, $C_- = -w_0$, implying zero net wave propagation in the inertial frame at this point. Note that Abbott's $C_{\pm}$ modes are dispersionless (i.e., $C_{\pm}$ is not a function of k_z) and non-damped (i.e., k_z is real, even for frequencies that would be evanescent in the photosphere), and that wave amplitudes do not vary appreciably on local length scales. This is to be expected: if the local density and velocity stratification is *ignored,* the medium is effectively homogeneous, and no gravo-acoustic evanescence arises. However, we find below (Section 7.2.8) that on the global scale of the entire wind, such modes do vary systematically in amplitude.

Rybicki, Owocki, & Castor (1990) generalized Abbott's linear analysis to the more general case of a perturbed radiative acceleration of the form

$$g_{\text{rad},1,i} = T_{ij} \, v_{1,j} \ , \tag{7.174}$$

where the subscripts i and j refer to radiative acceleration and velocity perturbations in the x, y, and z directions ($i, j = 1, 2, 3$), and T_{ij} is in general a second-rank tensor. An important new result for intermediate-wavelength (i.e., stable) perturbations is that, for the radiative acceleration from a *finite stellar disk,* the horizontal and radial perturbations interact with one another. Purely horizontal ($k_z = 0$) waves propagate at a modified phase speed

$$\frac{\omega}{k_x} = \pm\sqrt{a^2 + \left(\frac{R_*^2}{4r^2} F_2\right)^2} \ , \tag{7.175}$$

and purely radial ($k_x = 0$) waves now have three possible modes: two with phase speeds very close to Abbott's $C_{\pm}$, and one slow inward "interaction" mode,

$$C_{int} \equiv \frac{\omega}{k_z} = -\frac{R_*^2}{4r^2} F_2 \ . \tag{7.176}$$

These modes are highly dependent on the sphericity of the stellar *source* of radiation, but not on the overall spherical geometry of the medium. Thus, even in a near-star Cartesian analysis, the limit $r \to R_*$ retains these important interaction terms.

Because high-frequency radiative-acoustic waves occur in an assumed "homogeneous" background, the effect of a moving medium is included by the simple substitution of ω for ω'. For the case of purely vertical Abbott (1980) waves (with $k_x = 0$), the three solutions for the vertical wavenumber are simply

$$k_z = \left(\frac{\omega}{w_0 + C_\pm} \right) \quad , \quad \left(\frac{\omega}{w_0} \right) \quad , \tag{7.177}$$

the latter unphysical solution given by the formerly degenerate case $\omega' = \omega - w_0 k_z = 0$. The C_- solution above creates an asymptote at the CAK critical point, below which $k_z < 0$ (inward propagation), and above which $k_z > 0$ (outward propagation). Owocki, Castor, & Rybicki (1988) found that the most robust wave modes (in the extreme short-wavelength, unstable regime) start out on the stellar surface on the C_+ branch, then switch to the C_- branch further out in the wind, thus always remaining outwardly-propagating.

However, Owocki & Rybicki (1986) also determined that the supersonic inward C_- characteristic carries no true *information,* but is merely a local reconstruction of a smooth (long-wavelength) wave pattern. This effect is a manifestation of the small-scale line-driven instability, and is an example of the generalized case of an unstable medium where information need not necessarily travel at the local group velocity (Bers 1983). In fact, in Owocki & Rybicki's (1986) pure-absorption approximation, the sound speed a remains the true speed of inward information propagation. Thus, Abbott's (1980) interpretation of the CAK critical point as a "dam" below which information cannot propagate is called into question. Macroscopic perturbations in a hot-star wind, however, may still result in some structures advecting along the C_+ and C_- characteristics, but it is still unclear how this subtle technical issue is to be resolved in a more realistic wind model.

Although it is difficult to apply the localized "wavenumber" picture to an accelerating *and* stratified medium, it is possible to examine the simple case of 1D wave propagation ($u_0 = 0$, $k_x = 0$) in an isothermal ($\gamma = 1$) wind, in the absence of radiative forces ($F_1 = F_2 = F_3 = 0$). Let us define the velocity scale height $-J$,

$$\frac{1}{J} = -\frac{1}{w_0}\frac{\partial w_0}{\partial r} = \left(\frac{2}{r} - \frac{1}{H} \right) \quad , \tag{7.178}$$

and write the (3×3) dispersion matrix, neglecting U-terms:

$$\begin{vmatrix} -ik_z + 1/J & i\omega' & 0 \\ i\omega' - w_0/J & a^2/H & -ik_z - 1/H \\ 0 & -i\omega' a^2 & i\omega' \end{vmatrix} = 0 \quad , \tag{7.179}$$

which reduces to

$$(w_0^2 - a^2)\, k_z^2 - 2\omega w_0 k_z - \frac{ik_z}{J}(w_0^2 + a^2) + \omega^2 + \frac{i\omega w_0}{J} \;=\; 0 \;. \tag{7.180}$$

This equation can be solved for k_z, but it is a complicated expression. In the *supersonic* limit of $w_0 \gg a$, however, the solution simplifies to

$$k_z \;\approx\; \frac{\omega}{w_0} + \frac{i}{2J} \pm \frac{i}{2J} \;\approx\; \frac{\omega}{w_0} + \left\{ -\frac{i}{w_0}\frac{\partial w_0}{\partial r} \, , \; 0 \right\} \;. \tag{7.181}$$

This solution obviously neglects the acoustic propagation terms which depend on a, and the real part of k_z gives a trivial phase speed of w_0 (instead of $w_0 \pm a$). However, the imaginary part of k_z contains a solution with a decreasing ($k_i < 0$) amplitude in radius. Let us derive this amplitude, and ambitiously apply it to the *global* variation of wave amplitude with radius.

Because wave amplitudes (D, W, etc.) all vary locally as $\exp(k_i r)$ in addition to their sinusoidal oscillations, we can define the nonlocal amplitude as proportional to

$$A(r) \;\propto\; \exp\left[\int_{r_0}^{r} k_i(r')\, dr'\right] \;. \tag{7.182}$$

Note, however, that there is no guarantee that *all* wave amplitudes will have this same radial dependence. We will find, however, that short-wavelength or high-frequency waves come close to this idealized limit. For the above variation of k_i, then,

$$A(r) \;\propto\; \exp\left[-\int_{r_0}^{r} \frac{\partial}{\partial r'}(\ln w_0)\, dr'\right] \;\propto\; \frac{1}{w_0(r)} \;. \tag{7.183}$$

This inverse velocity-amplitude dropoff occurs only in the supersonic wind, and acts as a natural limiting factor to the exponential growth of wave amplitude in the near-static photosphere (Section 7.2.5). Preliminary numerical models of supersonic hot-star winds (see Figure 7.2) indeed show this behavior, at least for the highest acoustic frequencies. However, global variations in the zero-order medium result in the macroscopic wave variables ($\tilde{D}$, $\tilde{W}$, etc.) all behaving slightly *differently* in radius, thus invalidating the assumption of a single universal value of k.

7.2.8 Large-Scale Wave Propagation Analysis

Our eventual goal is to incorporate all the above effects – gravity, radiative forces, and a moving medium – into a model of wave propagation in an accelerating stellar wind. Grzędzielski (1971) initially investigated some of these effects in the solar wind, but the neglect of the energy equation (in favor of a universally-applied polytropic index) in that work makes even the results for a stationary medium differ from ours above. In the limit of a static, plane-parallel atmosphere, Grzędzielski's (1971) dispersion relation predicts purely *acoustic* waves, as in eq. (7.128) above, for *all* frequencies in the isothermal propagation limit of $\gamma = 1$ (i.e., no evanescence at all). This is clearly unphysical, but the remainder of Grzędzielski's predictions (such as the existence of zero-density-amplitude "gravity-shear" waves) still remain to be examined.

A useful formalism for analyzing the global variation of wave amplitudes is the *conservation of wave action.* This is a generalization of the conservation of wave energy density in the case of a moving medium. Also, it is the fluid-continuum analogue of Hamilton's classical principle of least action for, e.g., the motion of a discrete harmonic oscillator.

Landau & Lifshitz (1987) give the wave energy density for acoustic waves in a moving medium,

$$\mathcal{E} \;=\; \frac{1}{2}\rho_0 v_1^2 + \rho_1 \mathbf{v}_0 \cdot \mathbf{v}_1 + \frac{a^2 \rho_1^2}{2\rho_0} \quad , \tag{7.184}$$

where the vector velocity $\mathbf{v} = u\hat{\mathbf{e}}_x + w\hat{\mathbf{e}}_z$, and the above is a second-order perturbation quantity. Using eq. (7.134) above for $\mathbf{v}_1$, this energy density can be rewritten as

$$\mathcal{E} \;=\; \mathcal{E}_0 \left(\frac{\omega}{\omega - \mathbf{v}_0 \cdot \mathbf{k}} \right) \quad , \tag{7.185}$$

where $\mathcal{E}_0$ is the wave energy density in the comoving frame (i.e., when $\mathbf{v}_0 = 0$). We can conveniently define the wave action $\mathcal{S} \equiv \mathcal{E}_0/(\omega - \mathbf{v}_0 \cdot \mathbf{k})$, and note that it is this quantity that is conserved throughout the wind. Landau & Lifshitz (1987) interpret this from a quantum standpoint, as the number of wave "phonons" being independent of the choice of inertial reference frame.

Bretherton & Garrett (1968) and Jacques (1977) derive the equation of conservation of wave action,

$$\frac{\partial \mathcal{S}}{\partial t} + \nabla \cdot (\mathbf{v}_g \mathcal{S}) \;=\; 0 \quad , \tag{7.186}$$

which reduces in our case of a steady-state and spherical mean wind to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(v_{g,r} \mathcal{S} r^2 \right) \;=\; 0 \quad . \tag{7.187}$$

The quantity in parentheses is constant in radius. For pure acoustic waves propagating in the radial direction, this quantity is

$$v_{g,r} \mathcal{S} r^2 \;=\; (w_0 \pm a) \left[\frac{\rho_0 w_1^2}{\omega} \left(1 \pm \frac{w_0}{a} \right) \right] r^2 \tag{7.188}$$

$$\propto \;\pm \frac{(w_0 \pm a)^2 \, w_1^2}{w_0} \quad , \tag{7.189}$$

and this implies

$$|w_1| \propto \sqrt{\frac{w_0}{(w_0 \pm a)^2}} \ , \tag{7.190}$$

or that, in the supersonic limit, $|w_1| \propto w_0^{-1/2}$. Parker (1966) derived this result in the WKB limit for outwardly-propagating waves in an isothermal wind, but the wave action picture is potentially more general. Note that this result is qualitatively similar, but not identically the same as eq. (7.183), which predicted wave amplitudes to drop off as w_0^{-1}.

Figure 7.2 shows the radial variation of the density amplitude D for preliminary numerical VH-1 models of winds perturbed by radial ($k_x = 0$) gravo-acoustic base oscillations. The density and pressure at the base are varied in time according to the static isothermal gravo-acoustic modes derived in Section 7.2.5, with $\gamma = 1$, and the radial velocity is allowed to "float" so the wind can determine a unique mass flux. The hydrodynamical simulation was allowed to evolve in time until all transient variations propagated away, leaving only the periodic oscillations. The lower boundary of the model was driven with a small linear amplitude ($D = 0.01$), and the straight lines at negative values of z in Figure 7.2 show the ideal (static limit) growth of the imaginary part of the wavenumber k_i. Note that for evanescent modes, the more *slowly* growing solution matches the numerical variation of amplitude in the wind, and in Section 7.2.5 we determined that this solution was *outwardly-propagating* in the subsonic wind. Note also that, past the sonic point of the flow (at $z/R_* \approx 0.008$), the amplitude variations for all modes behave similarly, and begin to drop off as w_0^{-1}, as approximated above in Section 7.2.7.

Another important consideration in global wind wave propagation is the effect of first-order waves on the *mean* zero-order flow. In general, the wave action equation and the zero-order equations of motion are a coupled system of differential equations, but in many cases the effect of waves can be modeled by a "wave pressure tensor" which provides a source of momentum to the mean flow. This formalism has been applied to solar wind models by, e.g., Jacques (1978) and Leer, Holzer, & Flå (1982); to general Alfvén-wave driven winds by Hartmann & MacGregor (1980), Velli (1993), and MacGregor & Charbonneau (1994); and to early-type stellar winds by Koninx & Hearn (1992). However, for acoustic waves of reasonable base amplitude, there is only a minimal wave-pressure impact on the mean wind, and we will not explicitly include these terms in our linear analysis. Of course, the full hydrodynamic solution of the equations of motion may contain a slightly-altered "mean" state (w_0, ρ_0), and it will be interesting to see how important radiative-acoustic waves are in altering the unperturbed zero-order wind.

Another type of wave-impact on the mean flow has been suggested for Be stars

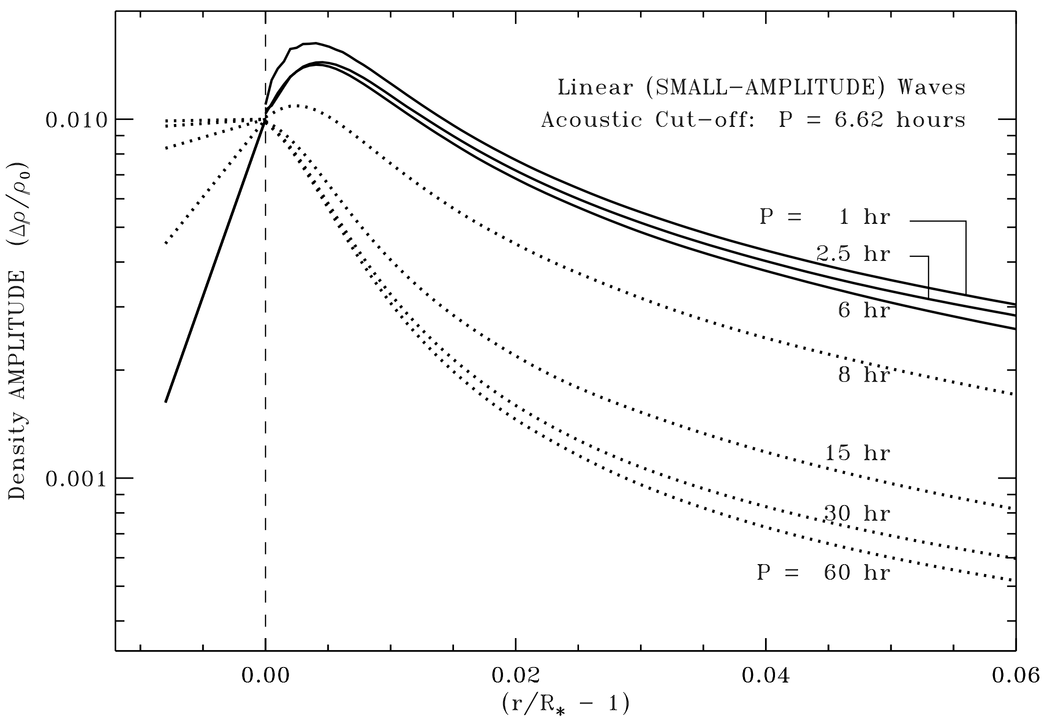

Figure 7.2: Fractional density amplitudes for near-star wind oscillations of a B supergiant (see, e.g., $\ell = 0$ modes in Figure 7.1). Solid lines represent modes that are able to propagate in the static isothermal photosphere, and dotted lines represent evanescent modes. Note that as the wind accelerates, the solid and dotted lines behave similarly and only differ by a constant factor in the supersonic wind.

by Osaki (1986): angular momentum transport from a rotating star. In the corotating equatorial plane, the radial flux of angular momentum is given by $r\rho_0\langle u_1 w_1\rangle$, where the angle-brackets denote an average over one oscillation period. Stars exhibiting adiabatic nonradial pulsations, however, have u_1 and w_1 varying 90° out of phase – implying zero mean angular momentum flux. Saio (1994) suggests that prograde nonaxisymmetric ($m < 0$) and *nonadiabatic* modes can transport significant amounts of angular momentum outward to form a dense "decretion" disk around the star. In our present wind models, nonadiabaticity is *not* necessary to produce a phase difference other than 90° between U and W. The acceleration, stratification, and radiative terms can produce this effect. It will be interesting to see if self-consistent wave models can be made to exhibit azimuthal velocities different enough from $u_0 \sim 1/r$ to show this transport.

7.3 Nonlinear Wave Effects

The theory of wave motion developed above applies well to small-amplitude (linear, first-order) perturbations in a fluid. However, *larger* amplitude disturbances are affected by the *nonlinear* terms in the hydrodynamic equations. Finite-amplitude acoustic waves "steepen" into shock discontinuities, while finite-amplitude internal gravity waves steepen and "break," like ocean waves, into dissipative turbulent motions. Also, since we know that linear wave trains must eventually steepen into nonlinear structures, we need to analyze what types of discontinuities can occur in the fluid variables and their gradients in an accelerating stellar wind. For example, the "kinks" in the radial velocity and density in the CIR models of Chapter 6 are a qualitatively new phenomenon which we hope to better understand by analyzing these nonlinear effects.

7.3.1 Weak-Shock Wave Steepening

Let us examine the principal steepening effects of nonlinear waves in a heuristic way. It is possible, via the theory of Riemann invariants (see, e.g., Landau & Lifshitz 1987), to derive a nonlinear correction term for the phase speed of a wave in the direction of propagation. For dispersionless acoustic waves, the nonlinear generalization of the simple amplitude relation

$$V = \pm aD \tag{7.191}$$

is the integral expression

$$\int dv = \pm \int a(\rho)\frac{d\rho}{\rho} \ , \tag{7.192}$$

where for large amplitudes, the phase speed $\pm a$ of such "simple waves" is now a function of the instantaneous velocity or density. This relation is coupled with the kinematic fact that the true definition of the "phase speed" in a disturbance is now given, for different points in the wave profile, by slightly different values:

$$v_p(\hat{\mathbf{n}}) \equiv \frac{\omega}{\mathbf{k}\cdot\hat{\mathbf{n}}} + [\mathbf{v}_1(\mathbf{r},t)\cdot\hat{\mathbf{n}}] \quad . \tag{7.193}$$

For simplicity, assume purely radial propagation. We can then write, schematically,

$$v_{p,r}(\text{nonlinear}) = v_{p,r}(\text{linear}) + \zeta w_1 \quad , \tag{7.194}$$

where ζ is an order-unity correction factor that takes into account the higher-order dependence of the phase speed on the density or velocity. For pure-acoustic and adiabatic waves, $\zeta = (\gamma+1)/2$ (Landau & Lifshitz 1987). For Abbott (1980) waves, in the supersonic limit of $F_2 \gg a$, the Riemann invariant theory gives $\zeta \approx 1 \pm \alpha$, for the $C_\pm$ modes (see Chapter 2 for the CAK definition of α).

In the linear limit of $|w_1/v_{p,r}| \ll 1$, this correction is negligible. When w_1 is larger, however, the nonlinear phase speed can be strongly affected. For $w_1(r) \propto \text{Re}[\exp(-ik_r r)]$, the "crests" of this sinusoid propagate *faster* than the linear phase speed, and the "troughs" propagate *slower*. In a finite time, or over a finite distance, the crests eventually overtake the troughs and form a sawtooth-pattern train of weak shocks. This nonlinear effect competes, in some cases, with the exponential growth of wave amplitude in a stratified photosphere. When, say, $D \sim \Theta \sim 1$, or when $|U|$ or $|W|$ surpass the local sound speed, then the linearized fluid equations no longer are valid. We find, however, that the presence of the wind naturally can limit this wave growth, and in fact causes amplitudes to begin to decrease around the sonic radius.

Let us estimate the distance Z over which a wave of a specified amplitude will steepen into a shock. Consider an initial waveform propagating in the radial, or $+z$ direction, with

$$w_1(z) = \frac{\tilde{w}_1}{2}\cos\left(\frac{2\pi z}{\lambda}\right) \quad , \tag{7.195}$$

where $k_z = 2\pi/\lambda$. A wave crest will meet and overtake its next highest "zero-point" (where $w_1 = 0$) when the crest travels a distance $(Z + \lambda/4)$ in the same time the zero-point travels a distance Z. This occurs when

$$\frac{\lambda}{4} = \int_0^T \Delta v_p(t)\,dt \approx \int_0^Z \Delta v_p(z)\,\frac{dz}{v_p^{(0)}} \quad , \tag{7.196}$$

where $v_p^{(0)}$ is the linear (unperturbed) value of the phase speed, and

$$\Delta v_p = \zeta\;[w_1(0) - w_1(\lambda/4)] = \frac{\zeta\tilde{w}_1}{2} \quad . \tag{7.197}$$

In a *homogeneous* medium, $\tilde{w}_1$ is constant in z, and it is trivial to solve for

$$Z = \frac{v_p^{(0)}\lambda}{2\zeta\tilde{w}_1} \ . \tag{7.198}$$

Similarly, for an exponentially-stratified static atmosphere, we find that

$$\tilde{w}_1 = \tilde{w}_1(0)\exp(z/2H) \tag{7.199}$$

and

$$Z = 2H \ln\left[\frac{v_p^{(0)}\lambda}{4H\zeta\tilde{w}_1(0)} + 1\right] \ . \tag{7.200}$$

Castor (1987) developed a different scaling estimate for the height Z of shock formation, which roughly agrees with the above analysis. By assuming the medium is disturbed by the motion of an upward-moving "piston," the creation of a shock is incumbent upon there being a net *change* in the piston's acceleration. If the piston were constantly accelerating, it would only produce an atmospheric "standing wave," with the atmosphere's acceleration matching that of the piston. For the piston position given by $z(t) = z_0 + z_1\cos(\omega t)$, its velocity amplitude is $W \propto \omega z$, and its rate of change of acceleration is

$$\frac{\partial^3 z}{\partial t^3} = \omega^2 W \ . \tag{7.201}$$

At each instant during the piston's motion, it emits sound waves, where later-departing waves have a larger absolute velocity (due to the exponential growth of gravo-acoustic modes). Thus, waves necessarily begin to cross each other when a "cusp" forms in the r-t diagram, and a shock forms at the height

$$Z \approx \frac{1}{3}\sqrt{\frac{a^3}{\partial^3 z/\partial t^3}} \approx \frac{\gamma g}{3\omega\sqrt{aW}} H \ , \tag{7.202}$$

for acoustic waves, with $v_p^{(0)} = a$. For either this or the above analysis, Z is of the order $(1-10)H$ for plausible early-type star values.

Gravo-acoustic waves, which inevitably grow with height, have an additional *local* means of reaching nonlinear amplitudes. This provides us with further ways to estimate where either shock formation or wave breaking occurs. First of all, the first-order linearized fluid equations break down when the amplitudes grow too large:

$$|D| \approx 1 \ , \qquad |\Pi| \approx \frac{a^2}{\gamma} \ , \qquad |\Theta| \approx 1 \ , \tag{7.203}$$

and, for acoustic-like waves, when the velocity amplitudes $|U|$ and $|W|$ surpass the sound speed a. For internal gravity waves, Mihalas & Toomre (1981) discuss various criteria for the onset of nonlinearity. The most clear condition is when the second-order *buoyancy* terms in the energy equation grow to the magnitude of the first-order terms, or when

$$\frac{|\partial\rho_1/\partial z - (1/a^2)\partial P_1/\partial z|}{|\partial\rho_0/\partial z - (1/a^2)\partial P_0/\partial z|} \approx \left| \frac{g}{\omega_{BV}^2} \left(\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} - ik_z \right) \left(D - \frac{\Pi}{a^2} \right) \right| \approx 1 \ . \qquad (7.204)$$

When this condition is satisfied, the convective stability is significantly increased in some parts of the atmosphere, and decreased in others, leading to local instabilities and eventual dissipative mixing.

7.3.2 Discontinuities in Fluid Variables

In order to determine whether a given fluid quantity can be discontinuous across a small spatial interval at radius r, we must integrate the conservation equations of these quantities across this interval. Because this interval is assumed to be vanishingly small, any smoothly-varying function f_0 (such as a function of radius only) is considered *constant* across the interval. Thus, between r_1 and r_2 (where $\Delta r \equiv (r_2 - r_1) \rightarrow 0$),

$$\int_{r_1}^{r_2} f_0(r)dr \approx f_0\Delta r \rightarrow 0 \ . \qquad (7.205)$$

Variables allowed to be discontinuous across Δr can be approximated by the piecewise continuous form

$$f \approx \begin{cases} f_1 & , \quad r \leq r_1 \\ f_1 + (r - r_1)(\Delta f/\Delta r) & , \quad r_1 < r < r_2 \\ f_2 & , \quad r \geq r_2 \end{cases} \ , \qquad (7.206)$$

where $\Delta f \equiv f_2 - f_1$. The integral of such a variable across Δr is similarly negligible,

$$\int_{r_1}^{r_2} \left[f_1 + \frac{\Delta f}{\Delta r}(r - r_1) \right] dr = \left(\frac{f_1 + f_2}{2} \right) \Delta r \rightarrow 0 \ . \qquad (7.207)$$

However, the integral of the *radial gradient* of this discontinuous variable does not vanish, and

$$\int_{r_1}^{r_2} \frac{\partial f}{\partial r} dr = \left(\frac{\Delta f}{\Delta r} \right) \Delta r = \Delta f \ . \qquad (7.208)$$

Also, if we assume that the discontinuity is moving with a velocity $s \equiv \delta r/\delta t$ (with $\delta r \ll \Delta r$ and $\delta t \equiv t_f - t_i$), we find that the integral of the time derivative of f is finite. We can approximate

$$\int_{r_1}^{r_2} \frac{\partial f}{\partial t}\, dr \;=\; \frac{\partial}{\partial t} \int_{r_1}^{r_2} f\, dr \;\approx\; \frac{\int f(t_f)dr - \int f(t_i)dr}{t_f - t_i} \quad , \tag{7.209}$$

and for t_i and t_f occurring when the assumed pattern (eq. [7.206]) is a distance $\delta r/2$ below and above its given location, one obtains

$$\int_{r_1}^{r_2} f(t_i)dr \;=\; \frac{\Delta r}{2}(f_1 + f_2) + \frac{\delta r}{2}\Delta f - \frac{(\delta r)^2}{8\Delta r}\Delta f \tag{7.210}$$

$$\int_{r_1}^{r_2} f(t_f)dr \;=\; \frac{\Delta r}{2}(f_1 + f_2) - \frac{\delta r}{2}\Delta f + \frac{(\delta r)^2}{8\Delta r}\Delta f \;. \tag{7.211}$$

Thus,

$$\int_{r_1}^{r_2} \frac{\partial f}{\partial t}\, dr \;\approx\; -s\Delta f + \frac{1}{4} s\Delta f \left(\frac{\delta r}{\Delta r}\right) \;\approx\; -s\Delta f \;. \tag{7.212}$$

The 1D (spherically-symmetric) mass continuity equation (eq. [7.67], neglecting the $\partial/\partial\phi$ terms), when integrated over Δr, becomes

$$-\, s\Delta\rho + \Delta(\rho v) \;=\; 0 \;\;, \tag{7.213}$$

or in a more familiar form,

$$(v_2 - s)\,\rho_2 \;=\; (v_1 - s)\,\rho_1 \;\;. \tag{7.214}$$

The momentum equation can be integrated similarly, but special care must be taken when integrating the radiative acceleration term. For a simple Sobolev force from a point star, for example,

$$g_{rad} \;=\; \frac{GM,\, k}{r^2}\left(\frac{1}{\sigma_e v_{th}\rho}\frac{\partial v}{\partial r}\right)^{\alpha} \;\equiv\; C(r)\left(\frac{1}{\rho}\frac{\partial v}{\partial r}\right)^{\alpha} \quad , \tag{7.215}$$

and

$$\int_{r_1}^{r_2} \rho\, g_{rad}\, dr \;=\; C(r)\int_{r_1}^{r_2} \rho^{1-\alpha}\left(\frac{\partial v}{\partial r}\right)^{\alpha} dr \tag{7.216}$$

$$\approx\; C(r)\int_{r_1}^{r_2} \left[\rho_1 + \frac{\Delta\rho}{\Delta r}(r - r_1)\right]^{1-\alpha} \left(\frac{\Delta v}{\Delta r}\right)^{\alpha} dr \tag{7.217}$$

$$\approx\; \frac{C(r)}{2-\alpha}\,(\Delta v)^{\alpha}\,\frac{\Delta(\rho^{2-\alpha})}{\Delta\rho}\,(\Delta r)^{1-\alpha} \quad , \tag{7.218}$$

which, because $\Delta r \to 0$, vanishes for $\alpha < 1$. The slowly-varying finite disk correction factor $\eta = \eta(r, v, \partial v/\partial r)$ does not alter this result, since if the velocity v is represented by the piecewise continuous eq. (7.206), η becomes a simple function of r only:

$$\eta \equiv \frac{1-(1-\chi)^{1+\alpha}}{\chi(1+\alpha)} \ , \tag{7.219}$$

$$\chi = \frac{R_*^2}{r^2}\left(1 - \frac{v}{r\partial v/\partial r}\right) \approx \frac{R_*^2}{r^3}\left(\frac{r_1 \Delta v - v_1 \Delta r}{\Delta v}\right) \approx \frac{R_*^2 r_1}{r^3} \ , \tag{7.220}$$

the last approximation being that $\Delta r \ll r_1 \Delta v / v_1$. Thus, for our present purposes, $\eta(r)$ can be assimilated into $C(r)$, and the finite disk factor ignored.

Thus, the momentum jump condition is unaffected by the radiative acceleration, and is given by integrating eq. (7.68) over Δr,

$$-\, s\Delta(\rho v) + \Delta[\rho(v^2 + a^2)] = 0 \ , \tag{7.221}$$

or, using eq. (7.214) to rewrite,

$$\rho_2[(v_2 - s)^2 + a^2] = \rho_1[(v_1 - s)^2 + a^2] \ . \tag{7.222}$$

The assumed isothermality (i.e., constancy of a) admits the simplified solutions,

$$(v_1 - s)(v_2 - s) = a^2 \ , \qquad \frac{\rho_2}{\rho_1} = \left(\frac{v_1 - s}{a}\right)^2 \ , \tag{7.223}$$

or, if the inertial-frame velocities v_1 and v_2 are known, the shock velocity is given by

$$s = \left(\frac{v_1 + v_2}{2}\right) \pm \sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + a^2} \ . \tag{7.224}$$

The density contrast ρ_2/ρ_1 can reach arbitrarily high values, as shown by the fact that the limiting value of this ratio for adiabatic shocks, $(\gamma + 1)/(\gamma - 1)$, diverges for $\gamma = 1$. Note also that *contact discontinuities,* where $v_1 = v_2 = s$, $P_1 = P_2$, but $\rho_1 \neq \rho_2$, are not allowed in the isothermal case, because if $P_1 = P_2$, then $\rho_1 = \rho_2$ automatically.

7.3.3 Discontinuities in Gradients

Abbott (1980) expressed the equations of motion in "quasi-linear" form by taking the radial derivative $\partial/\partial r$ of every term. Defining the velocity and density gradients as two new variables, the differentiated equations of motion become

$$\frac{\partial v}{\partial r} = x \tag{7.225}$$

$$\frac{\partial \rho}{\partial r} = y \tag{7.226}$$

$$\frac{\partial y}{\partial t} + \rho\frac{\partial x}{\partial r} + v\frac{\partial y}{\partial r} = -2xy - \frac{2\rho x}{r} - \frac{2vy}{r} + \frac{2\rho v}{r^2} \tag{7.227}$$

$$\frac{\partial x}{\partial t} + v\frac{\partial x}{\partial r} - \frac{\partial g_{rad}}{\partial x}\frac{\partial x}{\partial r} + \frac{a^2}{\rho}\frac{\partial y}{\partial r} = -x^2 + \frac{a^2y^2}{\rho^2} - \frac{\partial g}{\partial r} + \frac{\partial g_{rad}}{\partial r} + x\frac{\partial g_{rad}}{\partial v} + y\frac{\partial g_{rad}}{\partial \rho} \tag{7.228}$$

Let us now consider discontinuities in the gradient variables x and y, but only those cases where v and ρ themselves remain *continuous*. This ansatz isolates weak discontinuities, or "kinks" in the flow density and velocity, and distinguishes their behavior from the shocks discussed above.

Integrating over the small interval Δr is done in a similar way as for the original equations of motion, and and the above eqns. (7.227) and (7.228) are written such that their left-hand sides contain the "jump" terms which remain finite upon integration. The quasi-linear mass continuity equation becomes

$$\rho\Delta x + (v - s)\Delta y = 0 \ , \tag{7.229}$$

and the quasi-linear momentum equation becomes

$$(v - s)\Delta x + \frac{a^2}{\rho}\Delta y - \Delta g_{rad} = 0 \ , \tag{7.230}$$

where

$$\Delta g_{rad} \equiv g_{rad}(r, v, \rho, x_2) - g_{rad}(r, v, \rho, x_1) \ . \tag{7.231}$$

Note that these jump-condition equations result from an integral of the differential of the original conservation equations, and thus have the same dimensionality as the original equations. It is, however, the choice of which variables to keep continuous (ρ and v) and which to let "jump" (x and y), that distinguishes these equations from the original conservative forms. Finally, if we define the "nonlinear Abbott speed" V_A as

$$V_A \equiv \frac{\Delta g_{rad}}{\Delta x} \ , \tag{7.232}$$

then the momentum equation can be written as

$$\left[(v - s)^2 - a^2 - V_A(v - s)\right] \Delta x = 0 \tag{7.233}$$

(where Δy has been eliminated via eq. [7.229]). This equation provides a constraint on the kink propagation velocity s, since in order for a nontrivial kink to exist at all ($\Delta x \neq 0$), we must have

$$(v - s)^2 - a^2 - V_A(v - s) = 0 \ , \tag{7.234}$$

or

$$s = v - \frac{V_A}{2} \pm \sqrt{\left(\frac{V_A}{2}\right)^2 + a^2} \tag{7.235}$$
$$= v + C_\pm \ . \tag{7.236}$$

This is an alternate means of proving that weak discontinuities must propagate along fluid characteristics. These characteristics are specified by V_A, which is the nonlinear generalization of the linear "Abbott speed" F_2 for small perturbations. It is simple to see that for a vanishingly small value of Δx, V_A would approach the partial derivative of the unperturbed g_{rad} with respect to x, which is precisely the linear F_2.

Note that the quasi-linear mass continuity equation can now be written simply as

$$\frac{\Delta y}{\Delta x} = \frac{\rho}{C_\pm} \ , \tag{7.237}$$

which helps to determine the sense of the gradient discontinuities. For kinks propagating along the positive ($C_+ > 0$) characteristics, Δx and Δy must have the same sign. For kinks propagating along the negative ($C_- < 0$) characteristics, Δx and Δy must be of opposite sign. This is a nonlinear generalization of the velocity-density correlation of simple linear acoustic (or modified radiative-acoustic) waves. Outwardly propagating (C_+) waves have their velocity and density oscillations *in phase*, and inwardly propagating (C_-) waves have their velocity and density oscillations 180° *out of phase.* This is in qualitative agreement with the sharp velocity and density kinks produced in the models of corotating stream structure in Chapter 6.

With the present development of velocity-gradient discontinuities as "shocks" in the quasi-linear dynamical variables x and y, we hope to bring to bear the extensive theory of shock waves developed over the last century to understand better these important and possibly *observable* features. Whether these idealized structures survive in the actual unstable and non-Sobolev winds is still to be determined. This is an example, however, of the value of the study of hot-star winds to the field of radiation hydrodynamics as a whole: providing a "laboratory" to explore regions of physics that are inaccessible under terrestrial conditions.