# A Theoretical Comparison Between Ion-Cyclotron Wave Damping and the Radiative Transfer in Line-Driven Stellar Winds

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This is a parallel development between two physical situations that seem to obey the same basic governing equations. Similarities are stated in each paragraph, in the same row. Significant differences are emphasized in boxes.

The steps in the development below are not extensively referenced. For the inquisitive reader, though, further references are given here for **hot-star line-driving theory** (Sobolev 1960; Lucy & Solomon 1970; Castor 1970, 1974; Castor, Abbott, & Klein 1975; Abbott 1980, 1982; Owocki & Rybicki 1984, 1985, 1986; Poe, Owocki, & Castor 1990; Owocki 1992; Gayley & Owocki 1994; Cranmer 1996) and **ion-cyclotron wave damping** (Rowlands, Shapiro, & Shevchenko 1966; Hollweg & Turner 1978; Dusenbery & Hollweg 1981; Marsch, Goertz, & Richter 1982; McKenzie & Marsch 1982; Isenberg & Hollweg 1983; Gomberoff & Elgueta 1991; McKenzie, Banaszkiewicz, & Axford 1995; Siregar & Goldstein 1996; Fletcher & Huber 1997; Tu & Marsch 1997; Cranmer, Field, & Kohl 1999).

| Hot-Star Wind Photons | Ion-Cyclotron Waves |
|---|---|
| Let us consider the case of a single photon from the star, which eventually comes into "resonance" with opacity from a single line transition in the wind. | Let us consider the case of a monochromatic (single frequency), outward-propagating Alfvén wave, which eventually comes into "resonance" with the ion-cyclotron frequency of a single ion in the wind. |
| [Box] The opacity is primarily of a **scattering** nature. (Let us only deal with the direct scattering of photons *out* of the radial direction.) | [Box] The ion-cyclotron interaction is primarily of a damping, or **absorption** nature. |
| The photon has frequency $\nu$, which remains constant in the inertial, or stellar rest frame. | The wave has (angular) frequency $\omega$, which remains constant in the inertial, or stellar rest frame. |
| The wind has a radially varying outflow speed $u(r)$, so in the frame comoving with the wind, the local photon frequency is *redshifted* to be $\nu(1-u/c)$. Because $u \ll c$, we often represent this as $\nu/(1+u/c)$. | The wind has a radially varying outflow speed $u(r)$, so in the frame comoving with the wind, the local wave frequency is "Doppler shifted" to be $\omega - uk_{\parallel}$, where $k_{\parallel}$ is the local wavenumber. |
| The **opacity** is centered on a frequency $\nu_0$. Resonance between the photon and opacity source occurs when: $$\nu = \nu_0 \left( 1 + \frac{u}{c} \right)$$ | The **ion-cyclotron motions** take place at a frequency $\Omega_i$. Resonance between the wave and particle oscillations occur when: $$\omega - uk_{\parallel} = \Omega_i$$ |
| [Box] The main radial variation comes from the photon redshift (in the $u/c$ term). The opacity frequency $\nu_0$ is constant with radius. | [Box] The main radial variation comes from the ion resonance frequency $\Omega_i$, which is proportional to the rapidly decreasing $B$-field strength. The Doppler shifted wave frequency ($\omega - uk_{\parallel}$) is approximately constant with radius. |

The resonance is broadened around $\nu_0$ by random thermal motions, with most-probable speed $w$, of the ions that give rise to the line.

The frequency-dependent opacity (in units of 1/cm) is given by

$$\chi_\nu = \frac{\rho_i \kappa_L}{\pi^{1/2}} \exp\left[ - \left( \frac{\nu - \nu_0(1+u/c)}{\nu_0 w/c} \right)^2 \right]$$

where $\kappa_L$ is the mass absorption coefficient (cm$^2$/g) at the center of the transition, and $\rho_i$ is the mass density of scattering ions.

The photon flux $F_\nu$, which normally varies in radius to conserve photon number (i.e., as the inverse square of distance), is also attenuated by an extinction term due to the opacity:

$$F_\nu(r) = \underbrace{F_\nu(R_*) \left( \frac{R_*}{r} \right)^2}_{\equiv F_0(r)} e^{-\tau_\nu}$$

where the optical depth is given by an integration of the opacity from the star to a given radius:

$$\tau_\nu(r) = \int_{R_*}^r dr' \, \chi_\nu$$

For a given photon propagating away from the star, $\tau_\nu$ is zero until the resonance is encountered. Then $\tau_\nu$ rapidly rises to an asymptotic value that we will determine below. This variation is an erf function in radius, which approaches a step function for an infinitely sharp resonance.

In the supersonic part of the wind, $u \gg w$, and thus the most rapidly varying part of the opacity is the Doppler shift in the resonant exponential term. Let us make the **Sobolev approximation** and assume the other quantities in $\chi_\nu$ vary slowly over the resonance zone, and thus can be pulled out of the optical depth integral:

$$\tau_\nu = \kappa_L \rho_i \int_{R_*}^r \frac{dr'}{\pi^{1/2}} \exp\left[ - \left( x - \frac{u}{w} \right)^2 \right]$$

where $x = (\nu - \nu_0)/(\nu_0 w/c)$.

The ion-cyclotron interaction is broadened around $\Omega_i$ by random thermal motions in the direction of wave propagation. The relevant *parallel* most-probable speed of the gyrating ions is denoted by $w_\parallel$.

The frequency-dependent "opacity," or momentum transfer rate per unit wave power (in 1/cm), is given by

$$\mathcal{R}_i = \rho_i \frac{\Omega_i^2 F_i}{B^2 k_\parallel} \exp\left[ - \left( \frac{\omega - u k_\parallel - \Omega_i}{w_\parallel k_\parallel} \right)^2 \right]$$

where $F_i$ is a dimensionless factor (proportional to $\Omega_i / w_\parallel k_\parallel$ and the ion temperature anisotropy ratio $T_{\perp i}/T_{\parallel i}$) which may be likened to an "oscillator strength."

The wave power $P(\omega)$, which normally varies in radius to conserve WKB wave action, is also attenuated by an extinction term due to the wave-particle interaction:

$$P(\omega, r) = \underbrace{P(\omega, R_*) \, f_{\rm WKB}(r)}_{\equiv P_0(r)} e^{-\tau_\nu}$$

where the "optical depth" is given by an integration of the opacity from the star to a given radius:

$$\tau_\nu(r) = \int_{R_*}^r dr' \, \mathcal{R}_i$$

For a given wave propagating away from the star, $\tau_\nu$ is zero until the resonance is encountered. Then $\tau_\nu$ rapidly rises to an asymptotic value that we will determine below. This variation is an erf function in radius, which approaches a step function for an infinitely sharp resonance.

For $\Omega_i \gg w_\parallel k_\parallel$, or an Alfven speed large compared to the thermal speed, the most rapidly varying part of the opacity is the variation of $\Omega_i$ in the resonant exponential term. Let us make a Sobolev-like approximation and assume the other quantities in $\mathcal{R}_i$ vary slowly over the resonance zone, and thus can be pulled out of the optical depth integral:

$$\tau_\nu = \rho_i \frac{\pi^{1/2} \Omega_i^2 F_i}{B^2 k_\parallel} \int_{R_*}^r \frac{dr'}{\pi^{1/2}} \exp\left[ - \left( x - \frac{\Omega_i}{w_\parallel k_\parallel} \right)^2 \right]$$

where $x = (\omega - u k_\parallel)/(w_\parallel k_\parallel)$.

Becuase the Doppler shifted frequency is really the primary variable in the resonance zone, let us change variables in the integral, defining:

$$x' = x - \frac{u}{w}$$

$$dx' = -\frac{1}{w}\left(\frac{\partial u}{\partial r}\right) dr'$$

The optical depth then becomes

$$\tau_\nu = \tau_{\rm sob}\Phi(x,r)$$

where

$$\tau_{\rm sob} = \rho_i \kappa_L L_{\rm sob} \ \ ,$$

the Sobolev length is given by

$$L_{\rm sob} = \frac{w}{\partial u/\partial r} \ \ ,$$

and

$$\Phi(x,r) = \int_{x-u/w}^{\infty} \frac{dx'}{\pi^{1/2}} \exp(-x'^2)$$

The Sobolev optical depth $\tau_{\rm sob}$ is the asymptotic value of $\tau_\nu$ above the resonance zone, and it is a purely *local* measure of how strongly the opacity wants to attenuate the photons.

The energy lost by the photons is transferred into particle momentum and energy. The acceleration exerted on each particle is given by:

$$g_{rad} = \int_0^\infty d\nu \, \frac{\chi_\nu}{\rho}\frac{F_\nu}{c}$$

Note that the quantity $F_\nu/c$ has the units of momentum flux per unit frequency, or in cgs, g cm$^{-1}$ s$^{-1}$.

Converting the integration variable from $\nu$ to $x$, and taking out all slowly varying quantities (in accordance with the Sobolev approximation), one obtains

$$g_{rad} = \frac{\nu_0 w \kappa_L F_0}{c^2} \int_{-\infty}^{+\infty} \frac{dx}{\pi^{1/2}} \, e^{-x'^2} e^{-\tau_{\rm sob}\Phi(x,r)}$$

Becuase the Doppler shifted frequency is really the primary variable in the resonance zone, let us change variables in the integral, defining:

$$x' = x - \frac{\Omega_i}{w_\| k_\|}$$

$$dx' = -\frac{1}{w_\| k_\|}\left(\frac{\partial \Omega_i}{\partial r}\right) dr'$$

The optical depth then becomes

$$\tau_\nu = \tau_{\rm sob}\Phi(x,r)$$

where

$$\tau_{\rm sob} = \left(\frac{\rho_i \Omega_i^2 F_i \pi^{1/2}}{B^2 k_\|}\right) L_{\rm sob} \ \ ,$$

the Sobolev length is given by

$$L_{\rm sob} = \frac{w_\| k_\|}{\partial \Omega_i/\partial r} \ \ ,$$

and

$$\Phi(x,r) = \int_{x-\Omega_i/w_\| k_\|}^{\infty} \frac{dx'}{\pi^{1/2}} \exp(-x'^2)$$

The Sobolev optical depth $\tau_{\rm sob}$ is the asymptotic value of $\tau_\nu$ above the resonance zone, and it is a purely *local* measure of how strongly the opacity wants to attenuate the photons.

The energy lost by the waves is transferred into particle momentum and energy. The acceleration exerted on each particle is given by

$$D_{res} = \int_0^\infty d\omega \, \frac{\mathcal{R}_i}{\rho_i} P(\omega)$$

Note that the wave power $P(\omega)$ has the units of momentum flux per unit frequency, or in cgs, g cm$^{-1}$ s$^{-1}$. (The *heating* rates have additional phase velocity terms in the integrand.)

Converting the integration variable from $\omega$ to $x$, and taking out all slowly varying quantities (in accordance with the Sobolev approximation), one obtains

$$D_{res} = \frac{w_\| \Omega_i^2 F_i \pi^{1/2} P_0}{B^2} \int_{-\infty}^{+\infty} \frac{dx}{\pi^{1/2}} \, e^{-x'^2} e^{-\tau_{\rm sob}\Phi(x,r)}$$

Noting that $e^{-x'^2}\pi^{-1/2}$ is equal precisely to $d\Phi(x,r)/dx$, the integral becomes analytic:

$$g_{rad} = \frac{\nu_0 w \kappa_L F_0}{c^2} \left( \frac{1 - e^{-\tau_{\rm sob}}}{\tau_{\rm sob}} \right)$$

For the case of an **optically thick** source of opacity ($\tau_{\rm sob} \gg 1$), the quantity in parentheses above is just ($1/\tau_{\rm sob}$), and the acceleration reduces to

$$g_{rad} = \frac{\nu_0 F_0}{\rho_i c^2} \frac{\partial u}{\partial r}$$

Note that in this limit, $g_{rad}$ does not depend on the thermal speed $w$ or the line absorption coefficient $\kappa_L$; The extraction of momentum is *saturated* by the velocity gradient. The fact that the acceleration is proportional to the velocity gradient (itself, really) introduces a complicated nonlinear feedback into the physics of line driving.

Winds from hot stars are driven by a large number of lines, so the total acceleration is the sum of hundreds of $g_{rad}$ terms as derived above. A key development in the theory was the insight of Castor, Abbot, & Klein (1975) that the opacity of a large ensemble of lines can be represented as a **power law distribution** in $\kappa_L$. The sum over many lines can be replaced with much a simpler acceleration term which is a function of the distribution parameters.

Noting that $e^{-x'^2}\pi^{-1/2}$ is equal precisely to $d\Phi(x,r)/dx$, the integral becomes analytic:

$$D_{res} = \frac{w_{\parallel} \Omega_i^2 F_i \pi^{1/2} P_0}{B^2} \left( \frac{1 - e^{-\tau_{\rm sob}}}{\tau_{\rm sob}} \right)$$

For the case of an **optically thick** source of opacity ($\tau_{\rm sob} \gg 1$), the quantity in parentheses above is just ($1/\tau_{\rm sob}$), and the acceleration reduces to

$$D_{res} = \frac{P_0}{\rho_i} \frac{\partial \Omega_i}{\partial r}$$

This extremely simple form was used by Tu & Marsch (1997) for their model of proton heating by continual "erosion" of the high frequency wave spectrum. We have found, though, that there should be an additional mechanism replenishing the spectrum on reasonably rapid time scales, so the usefulness of this approach is unclear.

Would it be useful to express the large number of minor-ion cyclotron resonances as a CAK-like power law distribution? One problem is that the "opacity" ($\mathcal{R}_i/\rho_i$ without the Gaussian term) does not span a large range of values like line opacity does. More work needs to be done.

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