

On the Synthesis of Coronal White-Light Polarization Diagnostics

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1. Introduction

This set of notes has been compiled as an introduction to some of the basic theory behind the formation of Thomson-scattered white-light emission in the solar K-corona. Although, for the most part, the derivations in these notes can be found elsewhere (see References), it is important to gather them together in a reasonably *ab initio* format at least once, if only for ease of later reference (and if only for the author!).

The remainder of these notes are organized as follows. In §2 we define the required terminology and derive the basic equations of polarized radiative transfer. This section is condensed mainly from Chapter 3 of Cranmer (1996). In §3 we apply the general equations to the specific case of the solar corona.

2. Definitions

Well before Maxwell's theoretical exposition of classical electromagnetism in 1864, the nature of light propagating via transverse waves was generally accepted from empirical evidence. Stokes (1852) showed that any collection of such waves can be characterized by four independent parameters which describe its intensity, geometry, and phase distribution; i.e., the *polarization* state of the wave. If the intensity I of a light beam is measured with respect to a given set of axes, its projected components in two perpendicular planes (their normals also perpendicular to the direction of propagation) can be denoted I_ℓ and I_r . The addition of two or more electric field vectors of the radiation components will, in the most general case, result in a vector whose tip traces out an elliptically helical spiral as it propagates through space. If this ellipse is oriented with an angle θ to the ℓ -reference axis and has an ellipticity (ratio of semiminor to semimajor axes) of $\tan \phi$, the four intensity-like Stokes parameters can be defined as

$$I = (I_\ell + I_r) \quad (1)$$

$$Q = (I_\ell - I_r) \quad (2)$$

$$U = (I_\ell - I_r) \tan 2\theta \quad (3)$$

$$V = (I_\ell - I_r) \sec 2\theta \tan 2\phi \quad (4)$$

(see, e.g., Collins 1989). This represents a single *elliptically polarized* and monochromatic beam, expressible most generally as a function of only three independent variables (I, θ, ϕ) since I_ℓ and I_r can be written in terms of each other via

$$\frac{I_\ell}{I_r} = \frac{1 + \cos 2\theta \cos 2\phi}{1 - \cos 2\theta \cos 2\phi} . \quad (5)$$

For $\phi = \pm\pi/4$ (or $Q = U = 0$) the beam traces out a circular path, and is considered *circularly polarized*. For $\phi = 0$ or $\pm\pi/2$ (or $V = 0$) the beam traces out a one-dimensional, but inclined path, and is considered *linearly polarized*. For all values of θ and ϕ , this "complete" polarization of a monochromatic beam results in $I^2 = Q^2 + U^2 + V^2$.

However, when observing an extended source such as a star, we observe an unresolved collection of many electromagnetic waves, each with its own polarization, convolved together into a single time-averaged intensity. By passing this radiation through, e.g., polarizing filters and quarter-wave plates, we can obtain some

information about its overall polarization state. Redefining the above Stokes parameters as the appropriate averages over the incident beams (see Rybicki & Lightman 1979), we obtain the inequality

$$I^2 \geq Q^2 + U^2 + V^2 \quad , \quad (6)$$

and in the extreme limit of a totally random distribution, $Q = U = V = 0$, and the beam is considered *unpolarized*. Anticipating observations of polarization from the sun and stars, let us define the degree of linear polarization P and the position angle Ψ ,

$$P \equiv \frac{\sqrt{Q^2 + U^2}}{I} \quad , \quad \tan 2\Psi \equiv \frac{U}{Q} \quad , \quad (7)$$

and for a completely polarized beam, the position angle Ψ and the ellipse angle θ are equivalent. The so-called *polarization brightness* pB is defined as the total linearly polarized intensity $(Q^2 + U^2)^{1/2}$, or the product PI .

Note that the Stokes parameters I and V are invariant under coordinate rotations about the axis of propagation, but Q and U are not. Thus, when computing the emergent Stokes intensities from a heterogeneous and extended collection of emitters, it becomes necessary to refer the four components due to each emitter to the same set of axes. This is achieved by rotating the vector-like intensity through an angle ψ ,

$$\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \quad , \quad (8)$$

where ψ is defined in practice as the angle between the chosen Q -axis for the entire region and the individual normal vectors to the scattering planes which produce the bulk of the polarized radiation.

The observation of polarization from stars is a difficult task. Although Chandrasekhar (1946) predicted that Thomson scattering in stellar atmospheres can give rise to as much as 12% local linear polarization at the limb, this effect averages to zero over a spherical unresolved star, for which there is no preferred plane on the sky. It was not until Kemp et al. (1983) observed a phase-dependent variation of polarization from the eclipsing binary Algol that this effect was actually confirmed from individual *portions* of an occulted star. A rapidly rotating star will also exhibit a small degree of atmospheric polarization ($\sim 0.1\%$) because oblateness and gravity darkening create a preferred plane on the sky (Rucinski 1970; Collins, Truax, & Cranmer 1991). In addition, Collins & Cranmer (1991) predicted that rapidly rotating stars, which produce Doppler-broadened absorption line profiles, should exhibit a slightly stronger variation in linear polarization across these “spatially filtered” lines.

In this work, however, we are mainly concerned with the polarization due to the circumstellar gas, and we will assume the incident light from the star is initially unpolarized. Many classes of hot stars which exhibit emission lines (Be, Of, Wolf-Rayet stars) are also observed to have significant ($\sim 1\text{--}2\%$) linear polarization, and this is assumed to come from an asymmetric outer envelope (see, e.g., Zellner & Serkowski 1972; Coyne & McLean 1982; Schmidt 1988). The polarization of Be stars, for example, is commonly interpreted as arising from Thomson scattering of photospheric radiation in a rotationally-flattened envelope or disk. Because the circumstellar environments of most stars are *optically thin* to continuum radiation, let us follow Brown et al. (1978) and Wood et al. (1993) in treating this region as a single-scattering medium, thus ignoring absorption, local emission, and multiple-scattering of photons (see, however, Wood et al. 1996). The formal solution to the equation of radiative transfer, in the limit of the scattered radiation dominating the direct stellar radiation, reduces in the optically thin limit to

$$I_\nu \approx \int_0^{\tau_\nu} S_\nu(t_\nu) dt_\nu \approx \int_{-\infty}^{+\infty} \sigma_T n_e S_\nu dx \quad , \quad (9)$$

where σ_T is the Thomson-scattering cross section, n_e is the electron number density, and all intensity-like quantities are here represented by four-component Stokes vectors.

The four-component Stokes source function for Thomson scattering must take into account the geometrical effects of scattering between the ℓ and r polarization planes. Assuming coherent scattering, the source function can be written as

$$\mathbf{S}_\nu(\Omega) = \frac{1}{4\pi} \oint \mathbf{R}(\Omega, \Omega') \mathbf{I}_\nu^*(\Omega') d\Omega' \quad , \quad (10)$$

where the incident intensity is assumed to be unpolarized,

$$\mathbf{I}_\nu^* = \begin{pmatrix} I_\nu^{\text{inc}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (11)$$

and the 4×4 redistribution matrix \mathbf{R} is the *product* between a Rayleigh phase matrix which takes into account local scattering between the four Stokes components, and the rotation matrix given above (eq. [8]) which affixes each scattering into a single coordinate frame. When this matrix is multiplied by the unpolarized incident intensity vector (see Wood et al. 1993 for details), the source function becomes

$$\mathbf{s}_\nu = \begin{pmatrix} S_\nu^I \\ S_\nu^Q \\ S_\nu^U \\ S_\nu^V \end{pmatrix} = \frac{3}{16\pi} \oint I_\nu^{\text{inc}} \begin{pmatrix} 1 + \cos^2 \chi \\ \sin^2 \chi \cos 2\psi \\ -\sin^2 \chi \sin 2\psi \\ 0 \end{pmatrix} d\Omega' \quad , \quad (12)$$

where $\chi = \cos^{-1}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')$ is the scattering angle between the direction of incident radiation $\hat{\mathbf{n}}'$ and the direction pointing to the observer $\hat{\mathbf{n}}$. Note that all V -components are thus zero, implying *linear* polarization of circumstellar envelopes. (Stars with magnetic fields, however, will have a non-zero V polarization due to the Zeeman effect; see Collins 1988.)

3. Coronal Geometry

For observations of the solar corona, we must evaluate the scattering angles χ and ψ for arbitrary radiation vectors $\hat{\mathbf{n}}'$ and $\hat{\mathbf{n}}$. Let us define the x coordinate direction as that pointing to the observer (at an assumed infinite distance from the source volume at the origin). Let us also assume the observing ray passes through the positive z axis at its minimum distance from the center of the sun, and that this minimum “impact parameter” is denoted ρ . The vector $\hat{\mathbf{n}}$ thus is equal to the $\hat{\mathbf{e}}_x$ unit vector, and is constant for all points along the observing ray. The vector $\hat{\mathbf{n}}'$ depends on three coordinates: one that specifies the *location* of a given scattering center along the ray, and two that specify what point on the *solar disk* gave rise to the incoming photon. The first coordinate can be denoted as the polar colatitude θ of the point from the z axis (and thus the corresponding azimuthal angle $\phi = 0$ because the ray is in the x - z plane).

Let us define the subsequent coordinates required for $\hat{\mathbf{n}}'$ as a spherical colatitude η and azimuth ξ , of the vector $\hat{\mathbf{n}}'$ in a *rotated coordinate system* with its new z axis pointing directly outward from the center of the sun. The solid-angle integration over Ω' will thus involve integrating η from 0 to a critical value of

$$\eta_* = \sin^{-1} \left(\frac{R_\odot}{r} \right) \quad (13)$$

at the limb, and ξ from 0 to 2π . We can relate the Cartesian components of $\hat{\mathbf{n}}'$, in the original coordinate system, to both the original scattering-plane coordinates χ and ψ and to the new rotated coordinates:

$$\begin{aligned}\hat{n}'_x &= \sin \chi \cos \psi = \cos \eta \sin \theta + \sin \eta \cos \xi \cos \theta \\ \hat{n}'_y &= \sin \chi \sin \psi = \sin \eta \sin \xi \\ \hat{n}'_z &= \cos \chi = \cos \eta \cos \theta - \sin \eta \cos \xi \sin \theta\end{aligned}$$

(see also Wood et al. 1993).

Thus, the Stokes vector intensity is given by

$$\mathbf{I}_\nu = \begin{pmatrix} I_\nu \\ Q_\nu \\ U_\nu \\ V_\nu \end{pmatrix} = \frac{3\sigma_T}{16\pi} \int_{-\infty}^{+\infty} dx \oint d\Omega' n_e I_\nu^{\text{inc}} \begin{bmatrix} 1 + (\hat{n}'_x)^2 \\ (\hat{n}'_z)^2 - (\hat{n}'_y)^2 \\ 2\hat{n}'_y \hat{n}'_z \\ 0 \end{bmatrix}. \quad (14)$$

The integration over the solid angle, $d\Omega' = \sin \eta d\eta d\xi$, is complicated, but it is analytically tractable for simple forms of the solar limb darkening. Let us assume the common linear form:

$$I_\nu^{\text{inc}}(\eta) = I_\nu^{\text{central}} \left[1 - u + u \sqrt{\frac{\mu^2 - \mu_*^2}{1 - \mu_*^2}} \right], \quad (15)$$

where $\mu = \cos \eta$, $\mu_* = \cos \eta_*$, and the limb-darkening constant u has been empirically determined to be ~ 0.63 in the white-light wavelengths of interest ($\lambda \approx 4500\text{--}6000 \text{ \AA}$). The disk-center ($\eta = 0$) intensity I_ν^{central} can be related to a *disk-averaged* white-light intensity,

$$I_\nu^{\text{avg}} \equiv I_\nu^{\text{central}} (1 - u/3) \approx 1.97 \times 10^{10} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ sr}^{-1}, \quad (16)$$

and both I_ν^{central} and I_ν^{avg} have been denoted as the solar “surface brightness” B_\odot in various references; care should be taken that one is not misinterpreted as the other.

Upon integration over the source solid angle, the intensity vector becomes

$$\begin{pmatrix} I_\nu \\ Q_\nu \\ U_\nu \\ V_\nu \end{pmatrix} = \frac{3\sigma_T I_\nu^{\text{central}}}{16\pi} \int_{-\infty}^{+\infty} dx n_e \left\{ \begin{array}{c} [(1-u)Y(r) + uZ(r)] - (\rho^2/r^2) [(1-u)A(r) + uB(r)] \\ (\rho^2/r^2) [(1-u)A(r) + uB(r)] \\ 0 \\ 0 \end{array} \right\}, \quad (17)$$

where

$$A(r) = \mu_* - \mu_*^3 \quad (18)$$

$$B(r) = \frac{1}{4} - \frac{3}{8}\mu_*^2 + \frac{f(r)}{2} \left[\frac{3}{4}\mu_*^2 - 1 \right] \quad (19)$$

$$Y(r) = \frac{8}{3} - 2\mu_* - \frac{2}{3}\mu_*^3 \quad (20)$$

$$Z(r) = \frac{3}{2} - \frac{1}{4}\mu_*^2 + f(r) \left[\frac{1}{4}\mu_*^2 + 1 \right] \quad (21)$$

and

$$f(r) = \frac{\mu_*^2}{\sqrt{1 - \mu_*^2}} \ln \left(\frac{\mu_*}{1 + \sqrt{1 - \mu_*^2}} \right). \quad (22)$$

Figure 1 displays the two composite functions in eq. (17) as a function of radius. The dependence on the colatitude θ along the observing ray is embedded in the local value of

$$\frac{\rho^2}{r^2} = 1 - \frac{x^2}{r^2} = \cos^2 \theta. \quad (23)$$

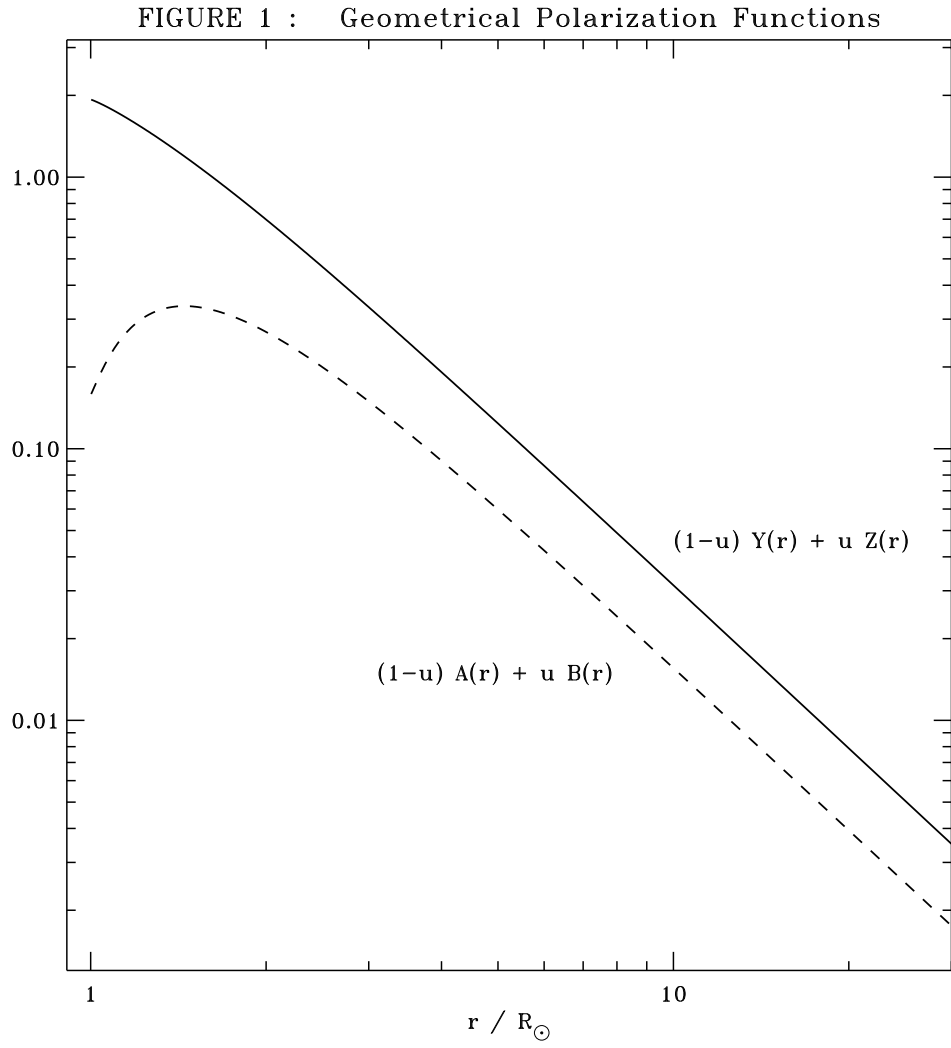


Fig. 1.— Geometrical polarization functions for linear limb darkening (see equations in text), and $u = 0.63$.

Also note that in the chosen coordinate system, the coronal polarization is “oriented” solely in the Q_ν direction, and $U_\nu = 0$. Alternate formulations of eq. (17) are given by Minnaert (1930), van de Hulst (1950), and Altschuler & Perry (1972).

4. Electron Density Inversion

Observations of pB along a given observing ray (with a constant value of ρ) sample the electron density n_e at many radii, and it is the main goal of these measurements to solve for the actual three-dimensional spatial distribution of density. A simple first approximation is to assume a *spherically-symmetric*, purely radial density dependence. Thus, we can make the transformation of variables:

$$pB(\rho) = K \int_{-\infty}^{+\infty} dx n_e(r) \frac{\rho^2}{r^2} [(1-u)A(r) + uB(r)] \quad (24)$$

$$= 2K \int_{\rho}^{+\infty} \frac{r dr}{\sqrt{r^2 - \rho^2}} n_e(r) \frac{\rho^2}{r^2} [(1-u)A(r) + uB(r)] , \quad (25)$$

where $K \equiv (3\sigma_T I_\nu^{\text{central}})/16\pi$. If we also assume that the function

$$F(r) \equiv n_e(r) [(1-u)A(r) + uB(r)] \quad (26)$$

behaves as a power series in radius, i.e., if

$$F(r) = a_1 \left(\frac{r}{R_\odot}\right)^{-b_1} + a_2 \left(\frac{r}{R_\odot}\right)^{-b_2} + \dots , \quad (27)$$

then eq. (25) can be integrated analytically. The dependence of pB on the impact parameter ρ thus follows a similar power series:

$$pB(\rho) = c_1 \left(\frac{\rho}{R_\odot}\right)^{-d_1} + c_2 \left(\frac{\rho}{R_\odot}\right)^{-d_2} + \dots , \quad (28)$$

where

$$c_i = a_i K R_\odot \sqrt{\pi} \frac{\Gamma([b_i + 1]/2)}{\Gamma([b_i + 2]/2)} ; \quad d_i = b_i - 1 . \quad (29)$$

Fits to observations can thus provide empirical values of the c_i and d_i coefficients, and eqs. (26)–(29) can invert these to determine $n_e(r)$. Often, because of the presence of the A and B functions in $F(r)$, $n_e(r)$ itself is fit with another power series for simplicity of presentation.

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