CHAPTER 6
LIMB DARKENING

6.1 *Introduction. The Empirical Limb-darkening*

The Sun is not equally bright all over, but it is darkened towards the limb. The effect is more pronounced at the blue end of the spectrum and less pronounced at the red. A reasonably good empirical representation of the form of the limb darkening is given by an equation for the specific intensity of the form

$$I(r) = I(0)\left[1 - u\left(1 - \sqrt{\frac{a^2 - r^2}{a^2}}\right)\right] \qquad 6.1.1$$

Here, a is the radius of the solar disc, r is radial distance from the centre of the disc and u is the *limb darkening coefficient.* This is often written in terms of θ (see figure VI.1) or of $\mu = \cos\theta$:

$$I(\theta) = I(0)\left[1 - u(1 - \cos\theta)\right] = I(0)[1 - u(1 - \mu)] \qquad 6.1.2$$

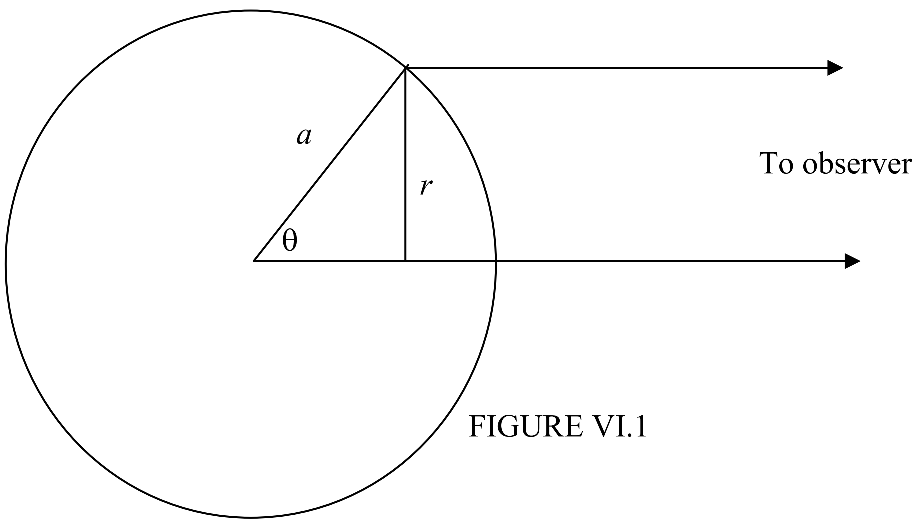

FIGURE VI.1

Whether written in the form of equation 6.1.1 or 6.1.2, $I(0)$ is the specific intensity at the centre of the disc. The specific intensity at the limb (where $r = a$ or $\theta = 90^{\circ}$) is $I(0)(1-u)$. The limb darkening coefficient can be written as $u = [I(\text{centre}) - I(\text{limb})]/I(\text{centre})$. Equation 6.1.1 is shown in graphical form in figure VI.2 for different values of the limb darkening coefficient.

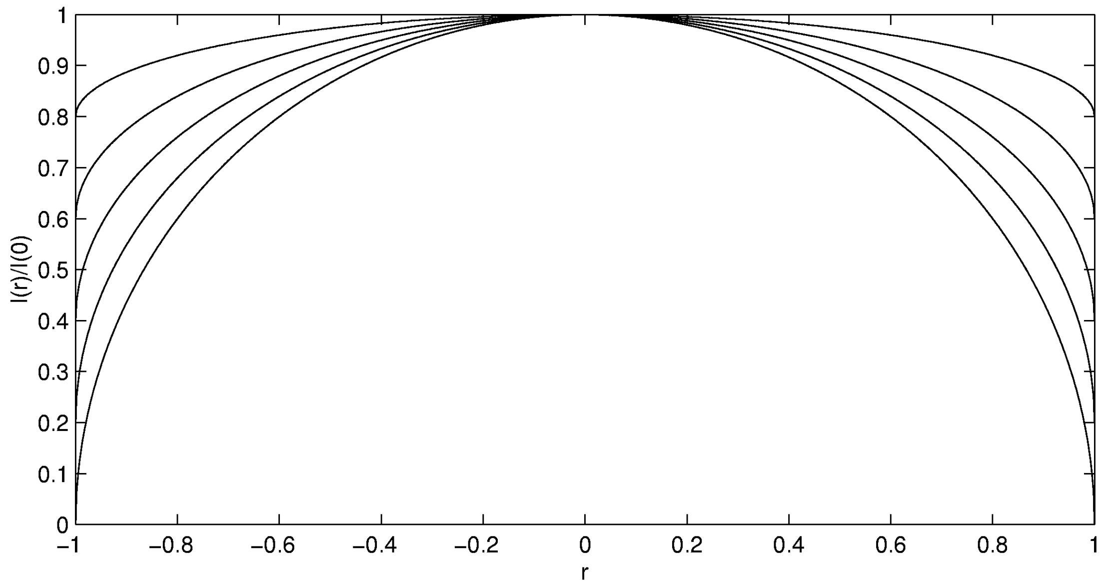

FIGURE VI.2

Equation 6.1.1 for six limb darkening coefficients, from the lowest curve upwards, u = 1.0, 0.8, 0.6, 0.4, 0.2 and 0.0. The "curve" for the last of these (no limb darkening) is formed from three of the boundary lines. The curve for $u = 1$ is a circle. The radius of the disc is taken to be 1, $r = 0$ is the centre of the disc and $r = \pm 1$ is the limb.

Limb-darkening is much greater in the violet and near ultraviolet than in the red. For example, at a wavelength of 600 nm, $u = 0.56$, whereas at 320 nm $u = 0.95$.

A slightly better empirical representation of the limb darkening can be obtained with two parameters, u' and υ':

$$I(\theta) = I(0)\left[1 - u'(1 - \cos\theta) - \upsilon'\sin^2\theta\right] \qquad 6.1.3$$

Why is the Sun darkened towards the limb?

We may perhaps imagine that the surface of the Sun radiates like a black body with uniform lambertian radiance, but it is surrounded by an absorbing atmosphere. Light from near the limb has to traverse a greater length of atmosphere than light from near the centre of the disk and this accounts for the limb darkening. If that is the explanation, we should be able to calculate what form of limb darkening to expect, and see how well it agrees with what is observed. If the agreement is only moderately good, perhaps we could assume that the atmosphere is not only an absorbing atmosphere, but it also emits radiation of its own, and we could see if we could adjust the ratio of emission to extinction (the source function) to obtain good agreement with the model and the observations. Or perhaps, rather than thinking of a uniform radiating surface sharply separated from a surrounding atmosphere, we may imagine that there is no such sharp boundary, but, rather, the density and temperature of the solar gases increase continuously with depth. In that case, suppose that we can see everywhere to a given optical depth, say to $\tau = 1$. Near the limb, an optical depth of unity does not take us very deep (in terms of kilometres) into the atmosphere, because we are looking almost tangentially at the surface of the Sun, so we reach only relatively high levels in the atmosphere where the temperature is relatively cool. Near the centre, on the other hand, where we are peering down perpendicularly into the Sun, an optical depth of one reaches deep down (in terms of kilometres) to places where the atmosphere is very hot. Thus the centre appears brighter than the limb.

At any rate, the point is that, by making precise measurements of the form of the limb darkening and comparing these measurements with the predictions of different models, we should in principle be able to deduce something about the run of density and temperature with optical depth in the atmosphere.

One practical difficulty of doing this is that it turns out that it is necessary to make quite precise measurements of the exact form of the limb darkening very close to the limb to be able to distinguish convincingly between different models.

Are there any prospects of being able to measure the limb darkening of stars other than the Sun? The future will tell whether advances in technology, such as adaptive optics, may enable us to observe the limb darkening of other stars directly. Other methods are possible. For example, the detailed light curve of an eclipsing binary star undoubtedly gives us information on the limb darkening of the star that is being eclipsed. There are many factors that affect the form of the light curve of an eclipsing binary star, and the detailed interpretation of light curves is not at all easy - but no one ever claimed that astronomy was easy. In principle lunar or asteroidal occultations of stars might enable us to determine the limb darkening of a star. Another possible method is from a careful examination of the line profiles in the spectrum of a rotating star. If a star is of uniform brightness and is rotating rapidly, the intensity profiles of its spectrum lines are broadened and have a semi-elliptical profile. However, if the star is darkened towards the limb, the line profile is affected. If the star's disk is completely limb darkened ($u = 1$, so that the specific intensity at the limb is zero), it is an interesting exercise to show that the line profile is parabolic. For intermediate limb darkening, the profile is neither elliptical nor parabolic; an exact analysis of its shape could in principle tell us the limb darkening coefficient.

6.2 *Simple Models of the Atmosphere to Explain Limb Darkening.*

1. The Sun consists of a spherical body emitting continuous blackbody radiation of radiance (specific intensity) B_ν surrounded by a shallow ("plane parallel") atmosphere which absorbs light and is of optical thickness $\tau(\nu)$ but does not emit. See figure VI.3.

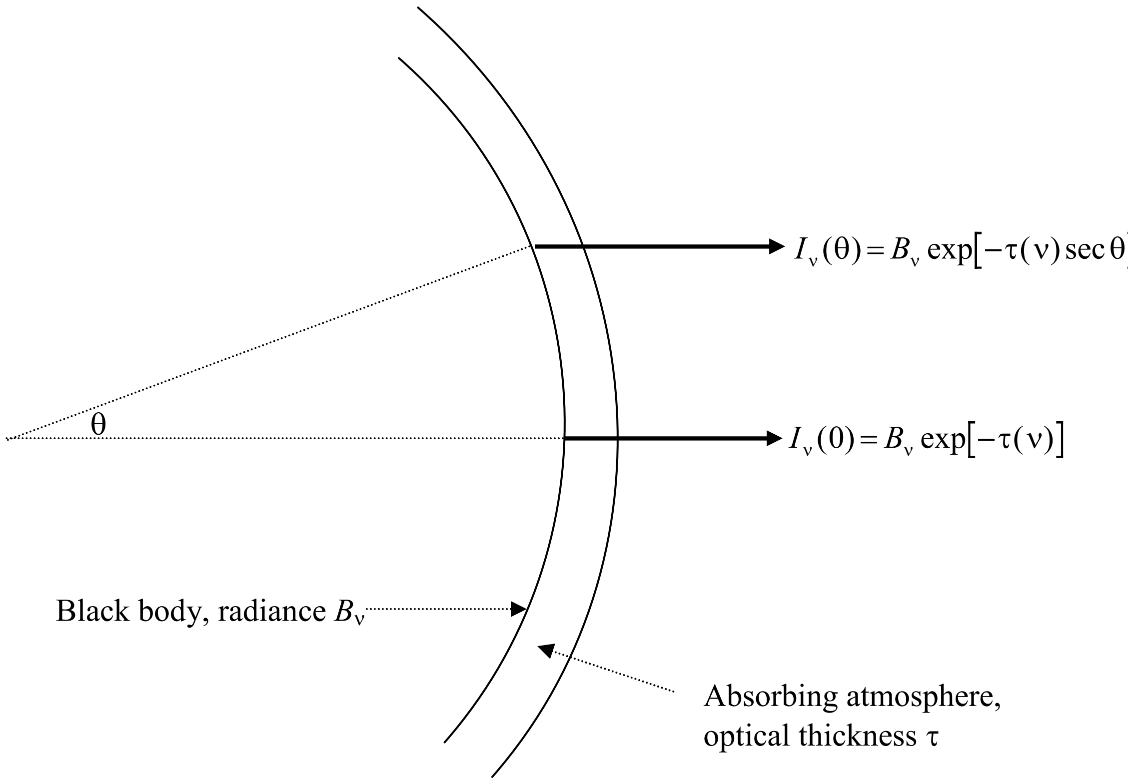

FIGURE VI.3

The emergent specific intensity at the centre of the disc is

$$I_\nu(0) = B_\nu e^{-\tau(\nu)} \qquad 6.2.1$$

and at a position on the disc given by θ is

$$I_\nu(\theta) = B_\nu e^{-\tau(\nu)\sec\theta} \qquad 6.2.2$$

so that the limb darkening is given by

$$I_\nu(\theta) = I_\nu(0) e^{-\tau(\nu)(\sec\theta - 1)} \qquad 6.2.3$$

If the limb darkening is indeed like this, then a graph of $\ln[I_\nu(\theta)/I_\nu(0)]$ versus $1 - \sec\theta$ will be a straight line whose slope will be the optical thickness of the atmosphere. However, in practice such a graph does not yield a straight line, and a comparison of equation 6.2.3, which is shown in figure VI.4, with the observed limb darkening shown in figure VI.2, suggests that this is not at all a promising model.

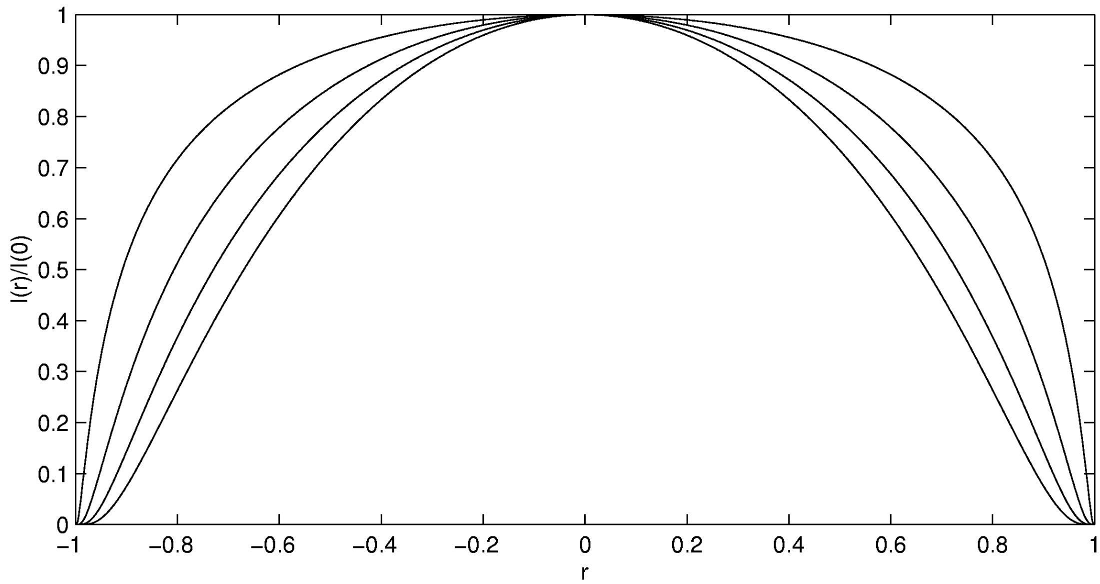

FIGURE VI.4

Equation 6.2.3 for four values of the optical thickness τ of the atmosphere. The curves are drawn for τ = 0.2, 0.4, 0.6 and 0.8. The curves do not greatly resemble the empirical, observed curves of figure VI.2, suggesting that this is not a very good atmospheric model.

2. This second model is similar to the first model, except that the atmosphere emits radiation as well as absorbing it. We suppose that the surface of the Sun is a black body of specific intensity B_1. The subscript 1 refers to the surface of the Sun. I have omitted a subscript ν. The argument is the same whether we are dealing with the specific intensity per unit frequency interval (Planck function) or the integrated specific intensity (Stefan's law). Suppose that the atmosphere, of optical thickness τ, is an emitting, absorbing atmosphere, of source function B_2, being a Planck function corresponding to a cooler temperature than the surface. The emergent specific intensity will be the sum of the emergent intensity of the atmosphere (see equation 5.7.2) and the specific intensity of the surface reduced by its passage through the atmosphere. At the centre of the disc, this will be

$$I(0) = B_1 e^{-\tau} + B_2\left(1 - e^{-\tau}\right) \qquad 6.2.4$$

and at a position θ on the disc it will be

$$I(\theta) = B_1 e^{-\tau \sec\theta} + B_2\left(1 - e^{-\tau\sec\theta}\right) \qquad 6.2.5$$

and so the limb darkening will be given by

$$\frac{I(\theta)}{I(0)} = \frac{(B_1 - B_2)e^{-\tau\sec\theta} + B_2}{(B_1 - B_2)e^{-\tau} + B_2} \qquad 6.2.6$$

In attempting to find a good fit between equation 6.2.6 and the observed limb darkening, we now have two adjustable parameters, τ and the ratio B_2/B_1. In figure VI.5 we show the limb darkening for τ = 0.5, 1.0, 1.5 and 2.0 for a representative ratio $B_2/B_1 = 0.5$. If we are dealing with radiation integrated over all wavelengths, this would imply an atmospheric temperature equal to $(0.5)^{1/4}$ = 0.84 times the surface temperature. There is no combination of the two parameters that gives a limb darkening very similar to the observed limb darkening, so this model is not a specially good one.

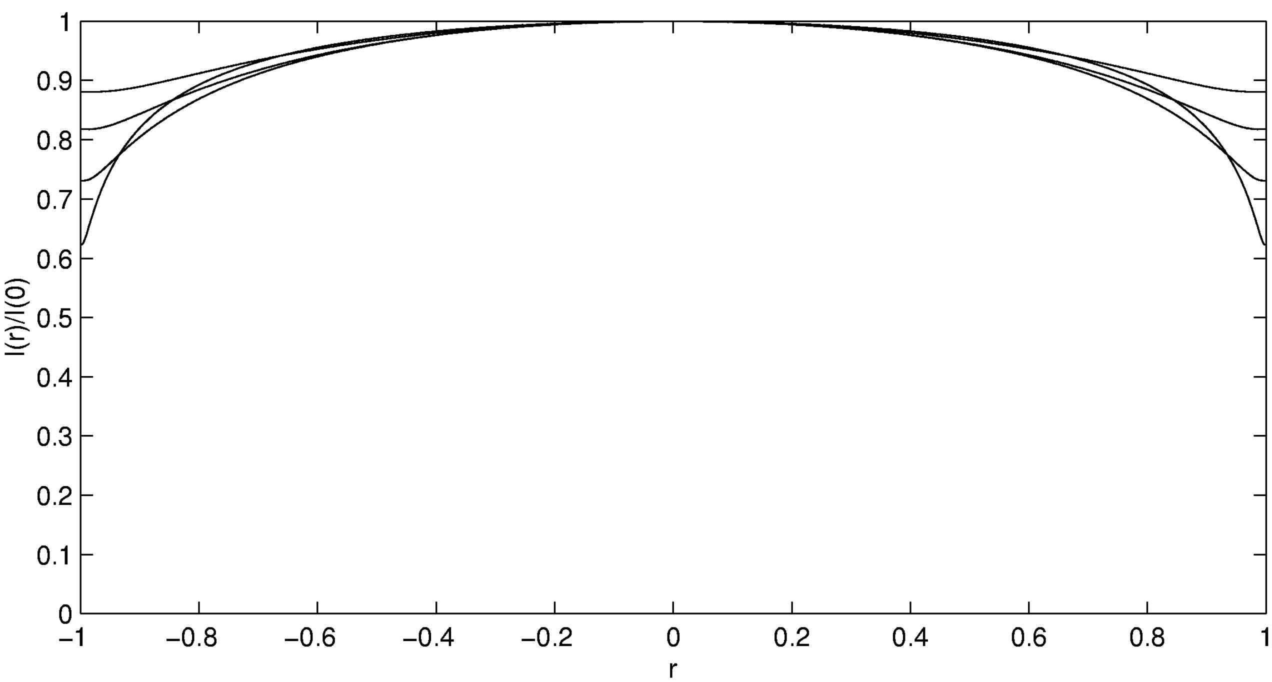

FIGURE VI.5
Equation 6.2.6 for $B_2 / B_1 = 0.5$, and τ = 0.5, 1.0, 1.5 and 2.0.

3. In this model we do not assume a hard and fast photosphere surrounded by an atmosphere of uniform source function; rather, we suppose that the source function varies continuously with depth. In figure VI.6 we draw two levels in the atmosphere, at optical depths τ and $\tau + d\tau$. The reader should recall that we are dealing only with a "plane parallel" atmosphere - i.e. one that is shallow compared with the radius of the star. The geometric distance between the two levels is therefore much exaggerated in the figure.

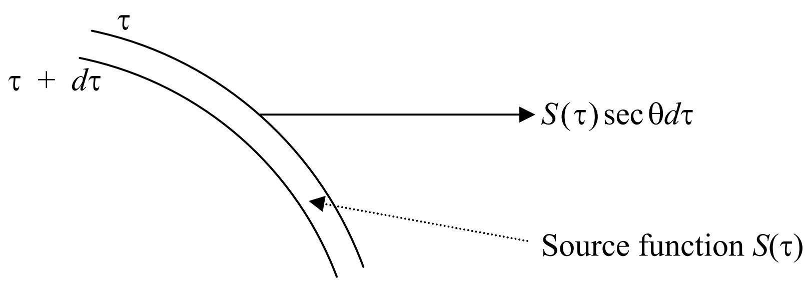

FIGURE VI.6

The source function of the shell between optical depths τ and $\tau + d\tau$ is $S(\tau)$. In the direction θ the radiance (specific intensity) of an elemental shell of optical thickness $d\tau$ is $S(\tau)\sec\theta d\tau$.. (I have not explicitly indicated in this expression the dependence on frequency or wavelength of the source function or optical depth.) By the time the radiation from this shell reaches the outermost part of the atmosphere (i.e. where $\tau = 0$), it has been reduced by a factor $e^{-\tau\sec\theta}$. The specific intensity resulting from the addition of all such elemental shells is

$$I(\theta) = \sec\theta \int_0^\infty S(\tau) e^{-\tau\sec\theta} d\tau \qquad 6.2.7$$

This important equation, attributed to Karl Schwarzschild, gives the limb darkening as a function of the way in which the source function varies with optical depth. The usual situation is that it is the limb darkening that is known and it is required to find $S(\tau)$, so that equation 6.2.7 has to be solved as an integral equation. This, however, is not as difficult as it may first appear because it will be noticed that if we write $s = \sec\theta$, the equation is merely a Laplace transform:

$$I(s) = s\mathcal{L}[S(\tau)] \qquad 6.2.8$$

so that the source function is the inverse Laplace transform of the limb darkening.

If we assume that the source function can be expressed as a polynomial in the optical depth:

$$S(\tau) = I(0)(a_0 + a_1\tau + a_2\tau^2 \ldots) \qquad 6.2.9$$

we find for the limb darkening (remembering that $1/s = \cos\theta$)

$$I(\theta) = I(0)\left(a_0 + a_1\cos\theta + 2a_2\cos^2\theta ...\right) \qquad 6.2.10$$

If we compare this with the empirical limb darkening equation 6.3 we find the *a* coefficients in terms of the limb darkening coefficients, as follows:

$$a_0 = 1 - u' - \upsilon' \qquad 6.2.11$$

$$a_1 = u' \qquad 6.2.12$$

$$a_2 = \tfrac{1}{2}\upsilon' \qquad 6.2.13$$

If we extend this analysis a little further, we find that if the source function is given by

$$S(\tau) = I(0)\sum_0^N a_n\tau^n \qquad 6.2.14$$

the limb darkening is

$$I(\theta) = I(0)\sum_0^N u_n\mu^n \qquad 6.2.15$$

where $\mu = \cos\theta$ and it is left to the reader to determine a general relation between the a_n and the u_n.

Problem. If the limb darkening is given by equation 6.1.2, calculate the mean specific intensity (radiance) $\bar{I}$ over the solar disc in terms of $I(0)$ and u. If the limb darkening is given by equation 6.1.3, what is the mean specific intensity in terms of $I(0)$, u' and υ'? This is an important calculation because if, for example, you need to calculate the irradiance of a planet or a comet by the Sun, the intensity of the Sun is the mean radiance times the projected area of the solar disc.