GRAVITATIONAL WAVES .... February 2016 edition

This is a heuristic derivation of the magnitude of *spacetime strain* (measurable with instruments like LIGO) expected from a distant source of gravitational radiation.

Hat-tip to Teviet Creighton (http://www.tapir.caltech.edu/~teviet/Waves/) for the broad outlines of this (mostly tensor-free!) derivation.

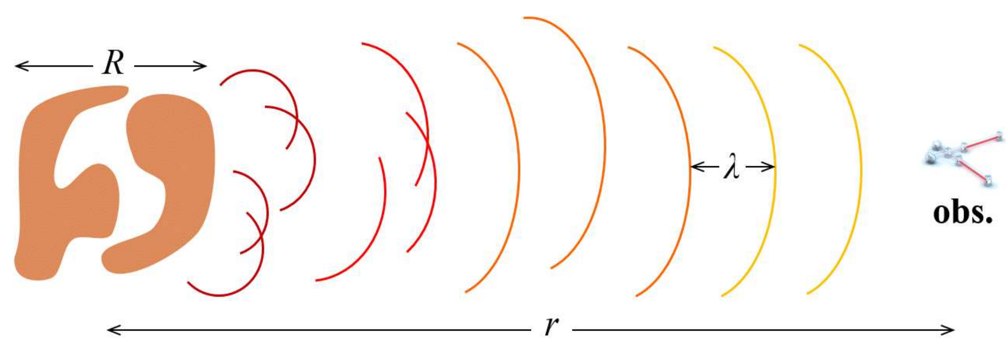
Classical field theories have 3 key ingredients:

- **Sources** (charges in E&M; masses in gravity)
- **Field equations** that show how the sources give rise to continuous fields that permeate all space (Maxwell's eqns in E&M; Einstein eqn in GR)
- **Laws of motion** that show how the fields affect the sources (Lorentz force in E&M; geodesic eqn in GR)

In a vacuum, one can often rewrite the field equations as **wave equations.** I won't derive them.

However, the upshot is that variability in a source (in space or time) gives rise to variability in the continuous fields that *propagate* out from the sources... at speed c for both E&M and GR.

Radiation zones:



Near the source, r may be $\lesssim \lambda$ or R, and life is complicated.

In the far-field zone, $r \gg \{\lambda, R\}$ and we can ignore field fluctuations that drop off *steeply* with increasing r. Only the "flattest" fluctuations survive.

. . . .

Let's first consider electric-field fluctuations:

Consider a distribution of charge density $\rho_c(\mathbf{r}, t)$ at the source. The electric field it produces is, more or less,

$$\mathbf{E}(\mathbf{r}, t) \;=\; -\nabla \left[\int \frac{d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \; \rho_c\!\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \right) \right]$$

and the following Legendre polynomial expansion is often used:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \;=\; \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \cdots \;=\; \frac{1}{r}\sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta) \qquad \text{where} \;\; \cos\theta = \mathbf{r}\cdot\mathbf{r}' \;\; .$$

Thus, the electric field is given as a sum over spatial *multipole moments* of the charge density distribution, evaluated "in the past" to account for the travel-time for fluctuations to reach the observer at $\mathbf{r}$.

Below, we'll just write down relevant terms for the magnitude E, in the far-field limit ($r \gg r'$), rather than following the full vector form.

Define the 0th spatial moment:

$$\text{Total Charge:} \qquad Q \;=\; \int d^3\mathbf{r}' \; \rho_c(\mathbf{r}')$$

In the far-field zone, Maxwell's equations say that the electric field we see has a magnitude

$$E \;\approx\; \frac{Q}{r^2}$$

(i.e., independent of internal structure).

How do we take account of the time evolution of the charge distribution? Once the fluctuations reach the far-field zone, they are propagating as waves with phase speed c.

At the source, let's say we know the charge distribution at a given time $t = 0$. If we know all time derivatives at $t = 0$, we can write the distribution at a subsequent time $t = \tau$ using the Taylor expansion,

$$Q(\tau) \;=\; Q(0) \;+\; \frac{\dot{Q}(0)}{1!}\tau \;+\; \frac{\ddot{Q}(0)}{2!}\tau^2 \;+\; \frac{\dddot{Q}(0)}{3!}\tau^3 \;+\; \cdots$$

It may sound a bit hand-wavy, but one can show that the fluctuations in the far-field limit contain terms like those above, but with the "delay time" τ replaced by the travel-time from the source to the observer r/c.

Thus, the time-variable electric field is expressible using terms that scale as

$$E \sim \frac{Q}{r^2} \; , \quad \frac{\dot{Q}}{cr} \; , \qquad \text{and so on.}$$

One can also show this by *dimensional analysis.* If E depends somehow on $\dot{Q}$, then the only combination of variables that produces the right units is $\dot{Q}/vr$. The natural value for the velocity v is c, since that is the speed of information propagation in this system.

Does it make sense to look at the higher derivatives of Q? **NO.** If we tried, the radial drop-off of the amplitude $E(r)$ would be less steep than $1/r$.

That means the dropoff in *energy* $\propto E^2$ would be less steep than $1/r^2$, and thus the total power enclosed in concentric spheres would be *increasing* with increasing r.

Thus, for $E(r) \propto 1/r^n$, we know that $n < 1$ is unphysical.

Also, we care *most* about fluctuations that are able to carry energy "to infinity." Thus, fields that drop off with $n > 1$ have a total power enclosed in concentric spheres that *decays to zero* as $r \rightarrow \infty$.

Thus, we will consider only amplitude fluctuations that drop off exactly as $1/r$ (i.e., $n = 1$, no more, no less).

By the way, in the real world:

- $Q = 0$ because most macroscopic gases & plasmas have overall *charge neutrality.*
- $\dot{Q} = 0$ because of charge conservation. If the "source region" is a closed system, the net charge inside it cannot be created or destroyed.

$\Longrightarrow$ {Monopole terms don't contribute to electromagnetic waves.}

. . . .

There are, however, higher moments of the charge distribution. 1st moment:

$$\text{Dipole Moment:} \quad \mathbf{D} \,=\, \int d^3\mathbf{r}' \; \rho_{\rm c}(\mathbf{r}')\,\mathbf{r}'$$

In E&M theory one can show that the far-field impact of a static dipole moment goes as

$$E \,\sim\, \frac{D}{r^3}$$

(worrying about its magnitude only, not the angular dependence).

Recalling the Taylor series argument from above, fluctuations in the dipole moment propagate away with terms proportional to

$$E \,\sim\, \frac{D}{r^3} \;\;,\;\; \frac{\dot{D}}{cr^2} \;\;,\;\; \frac{\ddot{D}}{c^2 r} \;.$$

As before, we care the most about the $1/r$ term that eventually dominates at large distances.

If the charges that make up the dipole moment are being "moved around" within the source (of size R), then one can estimate

$$D \sim QR \qquad\qquad \ddot{D} \sim Q\frac{d^2R}{dt^2} \;\sim\; Q \times \{\text{acceleration}\} \;.$$

This tells us that, on a micro-scale, the dominant source of electromagnetic radiation must come from *accelerated charges.*

(In reality, most accelerated charges are **oscillating,** with $\dot{D} = \omega D$, $\ddot{D} = \omega^2 D$, and so on.)

Let's now consider gravity. We know that we're going to be looking for tiny "ripples in the fabric of spacetime" caused by large, distant moving masses that generate gravitational fields.

The gravitational fields must be *time-varying* for us to get propagating fluctuations.

When we set up gravitational wave laboratories, are we trying to detect the actual gravitational acceleration $\mathbf{g}$ due to these distant masses?

No. Any "test masses" will accelerate along with the rest of the laboratory.

Instead, we want to measure **relative differences** in the gravitational fields felt by two (separated) test masses.

We've seen this before: The quantity we want is the **tidal acceleration** (i.e., spatial derivative of $\mathbf{g}$),

$$g' \approx \frac{GM}{r^3}$$

where the total mass of the source is another kind of 0th moment,

$$M = \int d^3\mathbf{r}' \; \rho(\mathbf{r}') \; .$$

We can repeat the analysis of small fluctuations, just like for a charge distribution in E&M. For a supposedly time-variable M we have terms like

$$g' \approx \frac{GM}{r^3} \;\; , \;\; \frac{G\dot{M}}{cr^2} \;\; , \;\; \frac{G\ddot{M}}{c^2 r} \; .$$

Since we're integrating over a closed system at the source, mass is conserved, and $\dot{M} = 0$. Thus, all higher time derivatives of M are also 0.

$$\Longrightarrow \; \{\text{Monopole terms don't contribute to gravitational waves.}\}$$

The 1st spatial moment of the mass distribution is similar to the dipole moment in E&M,

$$\text{Gravitational Dipole Moment:} \quad \mathbf{D}_{\rm g} = \int d^3\mathbf{r}' \; \rho(\mathbf{r}')\, \mathbf{r}'$$

and its fluctuating terms look like

$$g' \sim \frac{GD_{\rm g}}{r^4} \;\; , \;\; \frac{G\dot{D}_{\rm g}}{cr^3} \;\; , \;\; \frac{G\ddot{D}_{\rm g}}{c^2 r^2} \;\; , \;\; \frac{G\dddot{D}_{\rm g}}{c^3 r} \; .$$

However,

$$D_{\rm g} \sim MR \qquad \dot{D}_{\rm g} \sim M\frac{dR}{dt} \;\sim\; M \times \{\text{velocity}\} \;=\; \text{momentum} \quad .$$

Thus, due to the conservation of the *total* momentum in this closed system, $\dot{D}_{\rm g} = \text{constant}$. Thus, $\ddot{D}_{\rm g} = 0$ and so are all higher time derivatives.

$$\Longrightarrow \; \{\text{Dipole terms don't contribute to gravitational waves.}\}$$

To get something that won't eventually be zero, we need to go to the next higher (2nd) spatial moment,

$$\text{Moment of Inertia:} \quad \mathbb{I} = \int d^3\mathbf{r}' \; \rho(\mathbf{r}')\, \mathbf{r}'\mathbf{r}'$$

which is a 3×3 tensor that, in general, describes how the distribution of mass in an object *departs from spherical symmetry.*

There are more specific versions of $\mathbb{I}$ that are used in general relativity, usually by subtracting out the spherically symmetric contribution. For now, let's just consider the dimensional scaling

$$\mathbb{I} \sim MR^2$$

where we're limiting ourselves to including just the nonspherical "bits" of the source in the mass M and radial size R.

The fluctuating tidal acceleration due to time-variability of the moment of inertia contains terms like

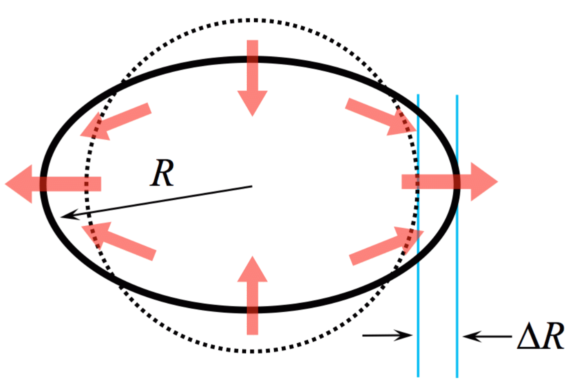
$$g' \sim \frac{G\,\mathbb{I}}{r^5} \;,\; \frac{G\,\dot{\mathbb{I}}}{cr^4} \;,\; \frac{G\,\ddot{\mathbb{I}}}{c^2r^3} \;,\; \frac{G\,\dddot{\mathbb{I}}}{c^3r^2} \;,\; \frac{G\,\ddddot{\mathbb{I}}}{c^4r} \;.$$

Thus, the term that survives to infinity fluctuates like

$$g' \sim \frac{G\,\ddddot{\mathbb{I}}}{c^4r}$$

What we actually measure is the spatial "strain" h between two test masses.

This is essentially the tidal acceleration integrated twice in time...



$$g' \sim \frac{\Delta g}{R} \sim \frac{\text{change in acceleration}}{\text{displacement}}$$

$$h \sim \frac{\Delta R}{R} \sim \frac{\text{change in displacement}}{\text{displacement}}$$

$$\sim \int dt \left(\int dt \; g' \right)$$

Thus, the basic scaling for the dimensionless strain due to gravitational radiation is

$$\boxed{h \sim \frac{G\ddot{\mathbb{I}}}{c^4\, r}\ .}$$

. . . .

Lastly, let's verify that the recently detected binary black hole inspiral event (GW150914) is what the LIGO team says it is.

We're told that LIGO detected a fractional strain of $h \sim 10^{-21}$. The black hole binary is also said to have had the following properties:

$$r \approx (230 \rightarrow 570)\ \text{Mpc}$$

$$M \approx 70\, M_{\odot} \qquad \text{(total mass)}\ .$$

Let us write

$$\ddot{\mathbb{I}} \sim \frac{MR^2}{t^2}\ .$$

Near the final moment of inspiral, the size of the "source" R can be considered roughly equal to the binary separation. In a Keplerian sense, the time t over which interesting changes occur is the orbital period, which for total mass M and separation R can be estimated as

$$t \sim \sqrt{\frac{R^3}{GM}} \qquad \text{and thus...} \qquad h \sim \frac{G^2M^2}{c^4\, r\, R}\ .$$

For black holes, we know that R should be of the same order of magnitude as the **Schwarzschild radius.** However, in the moments before the merger, the two original black holes are not quite in close contact. Let us introduce a fudge factor $f \gtrsim 1$, and

$$R = f\, R_{\text{Sch}} = \frac{2GMf}{c^2}\ .$$

Plugging in the numbers,

$$h \sim \frac{GM}{2fr\,c^2} \sim \frac{1}{4f}\frac{R_{\text{Sch}}}{r} \sim \frac{0.7 \rightarrow 2.4}{f} \times 10^{-21}$$

which for $f \approx 1$ agrees pretty nicely with the observation.