A selection of hopefully useful formulae

Steven R. Cranmer .... **Fall 2024**

The inspiration (and some of the LaTeX) for this document comes from the NRL Plasma Formulary, excellently curated by Joseph Huba. Generally, I add to this document when I find myself repeatedly looking up something that I probably should have memorized. C'est la vie!

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."

— Eugene Wigner (1960)

Other Sources of Useful Data

Wolfram Alpha .... https://www.wolframalpha.com/
Handbook of Space Astronomy & Astrophysics .... https://ads.harvard.edu/books/hsaa/
Fundamental Physical Constants .... https://www.nist.gov/pml/fundamental-physical-constants
Resources from Math Methods .... https://stevencranmer.bitbucket.io/ASTR_5540_2024/
Resources from RDP .... https://stevencranmer.bitbucket.io/ASTR_5120_2021/

1 Mathematics

1.1 On the Art of Approximation

You may be used to solving problems that have exact solutions. At some point, however, we run out of those, and we must rely increasingly on approximation/assumption...

$$\text{It's kind of an art to figure out} \begin{cases} \text{what to simplify} \\ \text{what to neglect} \\ \text{what to flat-out ignore} \end{cases}$$

Hopefully, by seeing it done in upper-level courses, you'll start to get a feel for doing it yourself. It takes a while...

$=$	the "exact equality" will often give way to
$\approx$	"is approximately equal to," or sometimes even
$\sim$	"very roughly equal to" (within an order of magnitude!?)
$\propto$	and sometimes we just care about which quantities are "proportional to" one another, ignoring normalizing constants.

Mahajan (2018) says a lot more about the mind-decluttering power of $\sim$ ("twiddle").

1.2 Algebra and Trigonometry

Quadratic Formulae

$$ax^2 + bx + c = 0 \qquad \Longrightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

where the latter is useful when $a \to 0$. For the two roots x_1 and x_2, Vieta's formulae are

$$x_1 + x_2 = -\frac{b}{a} \qquad \text{and} \qquad x_1 x_2 = \frac{c}{a} \ .$$

Logarithms

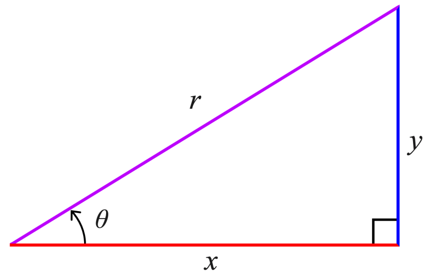
Definitions	$y = a^x \iff x = \log_a y$
Product	$\log_a(xy) = \log_a x + \log_a y$
Quotient	$\log_a(x/y) = \log_a x - \log_a y$
Power	$\log_a(y^n) = n \log_a y$
Change of base	$\log_a y = (\log_a b)(\log_b y)$

Trigonometric Identities

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



$$x^2 + y^2 = r^2$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \quad , \qquad \tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left(\frac{x \pm y}{1 \mp xy}\right)$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad , \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \quad , \qquad \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Undoing the inverse

$$\sin\left(\cos^{-1}x\right) = \cos\left(\sin^{-1}x\right) = \sqrt{1 - x^2}$$

$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1 + x^2}} \quad , \qquad \cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1 + x^2}}$$

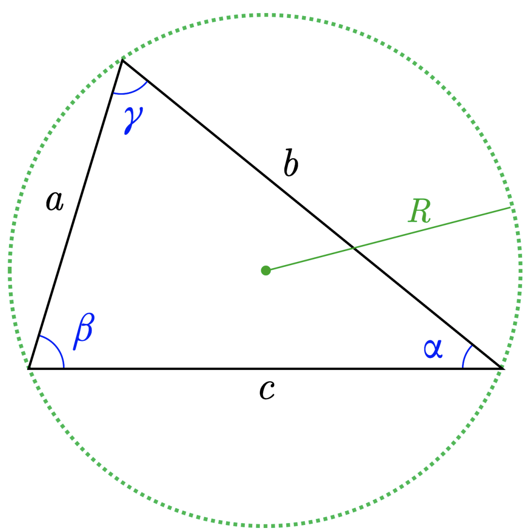
$$\tan\left(\sin^{-1}x\right) = \frac{x}{\sqrt{1 - x^2}} \quad , \qquad \tan\left(\cos^{-1}x\right) = \frac{\sqrt{1 - x^2}}{x}$$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



Law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

Sum of angles:

$$\alpha + \beta + \gamma = \pi$$

Angles 1 circle $= 360° = 2\pi$ radians (rad)
1 radian $= 360°/2\pi = 180°/\pi \approx 57.296°$
1 degree $= 60'$ (i.e., 60 arcminutes) $= 3600''$ (i.e., 3600 arcseconds)
1 arcminute $= 60''$ (i.e., 60 arcseconds)
Thus, 1 radian $\approx$ 206,265$''$ and 1 circle $=$ 1,296,000$''$
1 sphere $= 4\pi$ steradians (sr)

1.3 Vector Identities

Notation: f, g, are scalars; $\mathbf{A}$, $\mathbf{B}$, etc., are vectors.

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}=\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}=\mathbf{B}\times\mathbf{C}\cdot\mathbf{A}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}=\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\times\mathbf{B})\times\mathbf{A}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C}-(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg)=\nabla(gf)=f\nabla g+g\nabla f$

(7) $\nabla\cdot(f\mathbf{A})=f(\nabla\cdot\mathbf{A})+\mathbf{A}\cdot(\nabla f)$

(8) $\nabla\times(f\mathbf{A})=f(\nabla\times\mathbf{A})+(\nabla f)\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B})=(\nabla\mathbf{B})\cdot\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f=\nabla\cdot\nabla f$

(14) $\nabla^2\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f=0$ (i.e., if $\nabla\times\mathbf{F}=0$, it implies one can write $\mathbf{F}=\nabla\phi$)

(16) $\nabla\cdot\nabla\times\mathbf{A}=0$

(17) $\nabla^2(fg)=f\nabla^2 g+2(\nabla f)\cdot(\nabla g)+g\nabla^2 f$

(18) $\nabla(f/g)\,=\,(g\nabla f-f\nabla g)/g^2$

(19) $\nabla\times(\nabla\times\mathbf{A})\,=\,\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

Also, if vectors $\mathbf{A}$ & $\mathbf{B}$ depend on time t, then

$$\frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B})\,=\,\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial t}+\frac{\partial\mathbf{A}}{\partial t}\cdot\mathbf{B}\qquad\frac{\partial}{\partial t}(\mathbf{A}\times\mathbf{B})\,=\,\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial t}+\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{B}$$

1.4 Coordinate Systems

Cartesian $\longleftrightarrow$ Cylindrical (sometimes ρ or ϖ or R are used for r)

$$x = r\cos\phi \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\phi \qquad \phi = \tan^{-1}(y/x)$$
$$z = z \qquad z = z$$

Cartesian $\longleftrightarrow$ Spherical (assuming θ = colatitude, ϕ = longitude/azimuth)

$$x = r\cos\phi\sin\theta \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\phi\sin\theta \qquad \theta = \cos^{-1}(z/r)$$
$$z = r\cos\theta \qquad \phi = \tan^{-1}(y/x)$$

Vector & Differential Operators in CARTESIAN Coordinates

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$
$$\mathbf{A}\times\mathbf{B} = \mathbf{e}_x(A_yB_z - A_zB_y) + \mathbf{e}_y(A_zB_x - A_xB_z) + \mathbf{e}_z(A_xB_y - A_yB_x)$$

$$\nabla f = \mathbf{e}_x\frac{\partial f}{\partial x} + \mathbf{e}_y\frac{\partial f}{\partial y} + \mathbf{e}_z\frac{\partial f}{\partial z} \qquad \nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla\times\mathbf{F} = \mathbf{e}_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{e}_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{e}_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$[(\mathbf{A}\cdot\nabla)\mathbf{B}]_i \;=\; A_x\frac{\partial B_i}{\partial x} + A_y\frac{\partial B_i}{\partial y} + A_z\frac{\partial B_i}{\partial z} \qquad \text{for } i = x,y,z$$

Laplacian of a vector: $(\nabla^2\mathbf{A})_i \;=\; \nabla^2(A_i)$ for $i = x,y,z$

Divergence of a tensor: $(\nabla\cdot\mathbb{T})_i \;=\; \dfrac{\partial T_{ix}}{\partial x} + \dfrac{\partial T_{iy}}{\partial y} + \dfrac{\partial T_{iz}}{\partial z}$ for $i = x,y,z$

Volume element $dV \;=\; dx\,dy\,dz$

Line element displacement $d\mathbf{r} \;=\; (dx)\hat{\mathbf{e}}_x + (dy)\hat{\mathbf{e}}_y + (dz)\hat{\mathbf{e}}_z$

Vector & Differential Operators in CYLINDRICAL Coordinates (r, ϕ, z)

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} \;=\; (A_\phi B_z - A_z B_\phi)\hat{\mathbf{e}}_r \;+\; (A_z B_r - A_r B_z)\hat{\mathbf{e}}_\phi \;+\; (A_r B_\phi - A_\phi B_r)\hat{\mathbf{e}}_z$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\phi = \frac{1}{r}\frac{\partial f}{\partial \phi}; \quad (\nabla f)_z = \frac{\partial f}{\partial z}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}$$

Laplacian

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2}$$

$$(\nabla^2 \mathbf{A})_z = \nabla^2 A_z$$

Components of $(\mathbf{A} \cdot \nabla)\mathbf{B}$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_r = A_r \frac{\partial B_r}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_r}{\partial \phi} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_\phi = A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_\phi}{\partial \phi} + A_z \frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_z = A_r \frac{\partial B_z}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_z}{\partial \phi} + A_z \frac{\partial B_z}{\partial z}$$

Divergence of a tensor

$$(\nabla \cdot \mathbb{T})_r = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} - \frac{T_{\phi\phi}}{r}$$

$$(\nabla \cdot \mathbb{T})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rT_{r\phi}) + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{\phi r}}{r}$$

$$(\nabla \cdot \mathbb{T})_z = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z}$$

Volume element $\quad dV = r\,dr\,d\phi\,dz$

Line element displacement $\quad d\mathbf{r} = (dr)\hat{\mathbf{e}}_r + (r\,d\phi)\hat{\mathbf{e}}_\phi + (dz)\hat{\mathbf{e}}_z$

Transformation to/from Cartesian unit vectors

$$\begin{aligned} \hat{\mathbf{e}}_r &= \hat{\mathbf{e}}_x \cos\phi + \hat{\mathbf{e}}_y \sin\phi & \hat{\mathbf{e}}_x &= \hat{\mathbf{e}}_r \cos\phi - \hat{\mathbf{e}}_\phi \sin\phi \\ \hat{\mathbf{e}}_\phi &= -\hat{\mathbf{e}}_x \sin\phi + \hat{\mathbf{e}}_y \cos\phi & \hat{\mathbf{e}}_y &= \hat{\mathbf{e}}_r \sin\phi + \hat{\mathbf{e}}_\phi \cos\phi \\ \hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z & \hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z \end{aligned}$$

Vector & Differential Operators in SPHERICAL Coordinates (r, θ, ϕ)

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi$$

$$\mathbf{A} \times \mathbf{B} = (A_\theta B_\phi - A_\phi B_\theta)\hat{\mathbf{e}}_r + (A_\phi B_r - A_r B_\phi)\hat{\mathbf{e}}_\theta + (A_r B_\theta - A_\theta B_r)\hat{\mathbf{e}}_\phi$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}; \quad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

Laplacian

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Laplacian of a vector

$$(\nabla^2\mathbf{A})_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta A_\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2\mathbf{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2\mathbf{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}$$

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\theta = A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi} + \frac{A_\theta B_r}{r} - \frac{\cot\theta A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} + \frac{A_\phi B_r}{r} + \frac{\cot\theta A_\phi B_\theta}{r}$$

Divergence of a tensor

$$(\nabla\cdot\mathbb{T})_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi} - \frac{T_{\theta\theta}+T_{\phi\phi}}{r}$$

$$(\nabla\cdot\mathbb{T})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta T_{\phi\phi}}{r}$$

$$(\nabla\cdot\mathbb{T})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{\phi r}}{r} + \frac{\cot\theta T_{\phi\theta}}{r}$$

Volume element $\quad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

Line element displacement $\quad d\mathbf{r} = (dr)\hat{\mathbf{e}}_r + (r\,d\theta)\hat{\mathbf{e}}_\theta + (r\sin\theta\,d\phi)\hat{\mathbf{e}}_\phi$

Transformation to/from Cartesian unit vectors

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \hat{\mathbf{e}}_x\sin\theta\cos\phi + \hat{\mathbf{e}}_y\sin\theta\sin\phi + \hat{\mathbf{e}}_z\cos\theta \\
\hat{\mathbf{e}}_\theta &= \hat{\mathbf{e}}_x\cos\theta\cos\phi + \hat{\mathbf{e}}_y\cos\theta\sin\phi - \hat{\mathbf{e}}_z\sin\theta \\
\hat{\mathbf{e}}_\phi &= -\hat{\mathbf{e}}_x\sin\phi + \hat{\mathbf{e}}_y\cos\phi
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{e}}_x &= \hat{\mathbf{e}}_r\sin\theta\cos\phi + \hat{\mathbf{e}}_\theta\cos\theta\cos\phi - \hat{\mathbf{e}}_\phi\sin\phi \\
\hat{\mathbf{e}}_y &= \hat{\mathbf{e}}_r\sin\theta\sin\phi + \hat{\mathbf{e}}_\theta\cos\theta\sin\phi + \hat{\mathbf{e}}_\phi\cos\phi \\
\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta
\end{aligned}$$

1.5 Special Functions & Series Expansions

Binomial Series

$$(1+x)^n \;=\; 1 \,+\, nx \,+\, \frac{n(n-1)}{2!}x^2 \,+\, \frac{n(n-1)(n-2)}{3!}x^3 \,+\, \cdots \,+\, \frac{n(n-1)\cdots(n-(k-1))}{k!}x^k \,+\, \cdots$$

Trigonometric Functions

$$\sin x \;=\; x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \qquad \cos x \;=\; 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \qquad (x^2 < 1)$$

Exponential Functions

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad 10^x \;=\; \exp\left(2.30259x\right) \qquad (\ln 10 = 2.30259)$$

Full-width at half-maximum (FWHM):

$$\text{For} \quad y \;=\; \exp\left[-\frac{x^2}{2\sigma^2}\right] \;=\; \exp\left[-\left(\frac{x}{V_{1/e}}\right)^2\right] \;, \qquad \text{the FWHM} \;=\; V_{1/e}2\sqrt{\ln 2} \;\approx\; 1.66511V_{1/e} \;.$$

Euler's formulae:

$$e^{ix} = \cos x + i\sin x \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

Logarithmic Functions

$$\text{For } |x| \ll 1, \qquad \ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hyperbolic Trigonometric Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} = -i\sin(ix) \qquad \cosh x = \frac{e^x + e^{-x}}{2} = \cos(ix) \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\text{For } |x| \ll 1, \qquad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$$

$$\sinh^{-1} x = \text{arsinh } x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \text{arcosh } x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \text{artanh } x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\text{For } |x| \ll 1, \qquad \sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \cdots \qquad \tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\text{For } |x| \gg 1, \qquad \sinh^{-1} x = \ln(2x) + \frac{1}{4x^2} - \frac{3}{32x^4} + \cdots \qquad \cosh^{-1} x = \ln(2x) - \frac{1}{4x^2} - \frac{3}{32x^4} - \cdots$$

Dirac Delta Function

$$\int_{-\infty}^{+\infty} dx\; f(x)\,\delta(x-a) \;=\; f(a) \qquad\qquad \int_{-\infty}^{+\infty} dx\; f(x)\,\delta'(x-a) \;=\; -f'(a)$$

$$\delta[g(x)] \;=\; \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|} \qquad \text{where } x_i \text{ are the roots of } g(x).$$

$$\text{Limiting forms:} \qquad \delta(x) \;=\; \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\frac{\epsilon}{x^2+\epsilon^2}\right) \qquad\qquad \delta(x) \;=\; \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon\sqrt{\pi}}e^{-x^2/\epsilon^2}\right)$$

Identities:

$$\delta(-x) \;=\; \delta(x) \qquad \delta(x-a) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\; e^{i\omega(x-a)} \qquad x\,\delta(x) \;=\; 0 \qquad x\,\delta'(x) \;=\; -\delta(x)$$

3D vector delta function:

$$\int_V d^3\mathbf{r}\;\delta(\mathbf{r}-\mathbf{r}_0) \;=\; \begin{cases} 1, & \text{if } V \text{ contains } \mathbf{r}=\mathbf{r}_0, \\ 0, & \text{if } V \text{ does not contain } \mathbf{r}=\mathbf{r}_0. \end{cases} \qquad\qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \delta(\mathbf{r}_0-\mathbf{r})$$

$$\text{Cartesian:} \qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$\text{Cylindrical:} \qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \frac{1}{r}\,\delta(r-r_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0)$$

$$\text{Spherical:} \qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \frac{1}{r^2}\,\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)$$

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\right) \;=\; -\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \qquad\qquad \nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\right) \;=\; -4\pi\,\delta(\mathbf{r}-\mathbf{r}_0)$$

Error Function

$$\mathrm{erf}(x) \;=\; \frac{2}{\sqrt{\pi}}\int_0^x dt\, e^{-t^2} \qquad\qquad \mathrm{erfc}(x) \;=\; \frac{2}{\sqrt{\pi}}\int_x^\infty dt\, e^{-t^2} \;=\; 1-\mathrm{erf}(x)$$

For small arguments ($x \ll 1$),

$$\mathrm{erf}(x) \;\approx\; \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots\right)$$

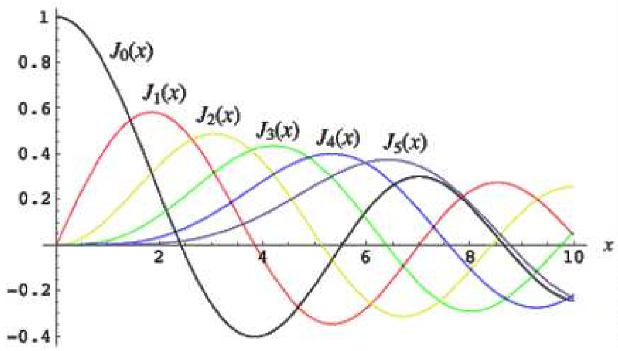
For large arguments ($x \gg 1$),

$$\mathrm{erf}(x) \;\approx\; 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{1}{x} - \frac{1}{2x^3} + \frac{3}{4x^5} - \frac{15}{8x^7} + \cdots\right)$$

The Chandrasekhar (1943) function:

$$G(x) = \frac{1}{2x^2}\left[\mathrm{erf}(x) \;-\; \frac{2x}{\sqrt{\pi}}\,e^{-x^2}\right] \qquad \text{small } x \;\left(G \approx \frac{2x}{3\sqrt{\pi}}\right) \;, \qquad \text{large } x \;\left(G \approx \frac{1}{2x^2}\right)$$

Bessel Functions of the First Kind



For small arguments ($x \ll \sqrt{n+1}$) and indices $n > 0$,

$$J_n(x) \;\approx\; \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n$$

For large arguments ($x \gg |n^2 - 1/4|$),

$$J_n(x) \;\approx\; \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$J_{-n}(x) \;=\; (-1)^n\, J_n(x) \qquad \int_0^x du\; u\; J_0(u) \;=\; x\, J_1(x) \qquad \frac{d}{dx}\left[x^n J_n(x)\right] \;=\; x^n J_{n-1}(x)$$

$$\frac{2n}{x}J_n(x) \;=\; J_{n-1}(x) + J_{n+1}(x) \qquad 2\frac{dJ_n}{dx} \;=\; J_{n-1}(x) - J_{n+1}(x)$$

Modified Bessel Functions

For small arguments ($z \to 0$) and fixed n indices,

$$I_n(z) \;\approx\; \frac{(z/2)^n}{\Gamma(n+1)} \qquad I_0(z) \;\approx\; 1 + \frac{z^2}{4} + \frac{z^4}{64} + \cdots$$

$$K_n(z) \;\approx\; \frac{\Gamma(n)}{2(z/2)^n} \qquad K_0(z) \;\approx\; -\ln z$$

For large arguments ($z \to \infty$) and fixed n indices,

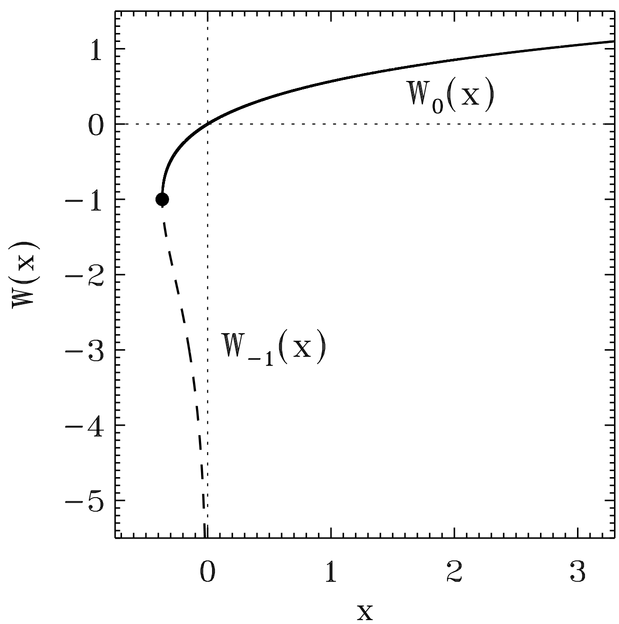
$$I_n(z) \;\approx\; \frac{e^z}{\sqrt{2\pi z}} \qquad K_n(z) \;\approx\; e^{-z}\sqrt{\frac{\pi}{2z}}$$

Lambert W Function

The Lambert W function is defined as the multivalued inverse of the function xe^x. Equivalently, the multiple branches of W are the multiple roots of the equation

$$W(z)e^{W(z)} = z \ ,$$

where z is in general complex. There are an infinite number of solution branches, labeled by convention by an integer subscript: $W_k(z)$, for $k = 0, \pm 1, \pm 2, \ldots$ If z is a real number x, the only two branches that take on real values are $W_0(x)$ and $W_{-1}(x)$. These branches are shown in the accompanying plot.



Numerous formulae for the differentiation, integration, and series expansion of W are given by Corless et al. (1996) and Valluri et al. (2000). A few useful results are given here. Near the branch cut point at $x = -1/e$, $W_0 = W_{-1} = -1$, and the two real branches can be approximated to lowest order by

$$\begin{aligned} W_0(x) &\approx -1 + \sqrt{2 + 2ex} \\ W_{-1}(x) &\approx -1 - \sqrt{2 + 2ex} \ . \end{aligned}$$

Some other useful ways of using W to write explicit solutions to standard families of of transcendental equations have been provided by Briggs (1999) and Chow (2023). For example,

$$\ln(A + Bx) + Cx = \ln D \qquad \text{is solved by} \qquad x = \frac{1}{C} W \left[\frac{CD}{B} \exp\left(\frac{AC}{B} \right) \right] - \frac{A}{B}$$

and

$$a^x + bx + c = 0 \qquad \text{is solved by} \qquad x = -\frac{c}{b} - \frac{1}{\ln a} W \left(\frac{a^{-c/b} \ln a}{b} \right) \ .$$

These solutions do not work for all values of the constants, and the choice of solution branch(es) often depends on physical arguments or boundary conditions.

1.6 Derivatives and Integrals

Definite Integrals of Gaussians

$$\int_0^\infty dx\, e^{-(x/\sigma)^2} = \frac{\sigma\sqrt{\pi}}{2} \qquad \int_0^\infty dx\, x\, e^{-(x/\sigma)^2} = \frac{\sigma^2}{2}$$

$$\int_0^\infty dx\, x^2\, e^{-(x/\sigma)^2} = \frac{\sigma^3\sqrt{\pi}}{4} \qquad \int_0^\infty dx\, x^3\, e^{-(x/\sigma)^2} = \frac{\sigma^4}{2}$$

$$\int_0^\infty dx\, x^4\, e^{-(x/\sigma)^2} = \frac{3\sigma^5\sqrt{\pi}}{8} \qquad \int_0^\infty dx\, x^5\, e^{-(x/\sigma)^2} = \sigma^6$$

$$\int_0^\infty dx\, x^6\, e^{-(x/\sigma)^2} = \frac{15\sigma^7\sqrt{\pi}}{16} \qquad \int_0^\infty dx\, x^7\, e^{-(x/\sigma)^2} = 3\sigma^8$$

$$\int_0^\infty dx\, x^8\, e^{-(x/\sigma)^2} = \frac{105\sigma^9\sqrt{\pi}}{32} \qquad \int_0^\infty dx\, x^n\, e^{-(x/\sigma)^2} = \frac{\sigma^{n+1}}{2}\Gamma\left(\frac{n+1}{2}\right)$$

$$\int_{-\infty}^\infty dx\, e^{-ax^2+bx} = e^{b^2/4a}\sqrt{\frac{\pi}{a}}$$

Definite Integrals relevant to the Planck function

$$\int_0^\infty dx\, \frac{x^n}{e^x-1} = \zeta(n+1)\,\Gamma(n+1) = \zeta(n+1)\,n! \qquad \text{(for integer } n\text{)}$$

$\zeta(n)$ is the Riemann zeta function, which ≈ 1 for $n \gtrsim 3$. In that case, $\displaystyle\int_0^\infty dx\, \frac{x^n}{e^x-1} \approx n!$

Integration by Parts

$$\int_a^b u\, dv = [u\, v]_a^b - \int_a^b v\, du \qquad \int u(x)\, \frac{dv}{dx}\, dx = u(x)\, v(x) - \int v(x)\, \frac{du}{dx}\, dx$$

and a vector version comes from Gauss' divergence theorem (Binney & Tremaine, eqn B.45):

$$\int d^3\mathbf{r}\; g\, \nabla\cdot\mathbf{F} = \oint g\mathbf{F}\cdot d^2\mathbf{S} - \int d^3\mathbf{r}\; (\mathbf{F}\cdot\nabla) g$$

1.7 Differential Equations

For a **first-order** linear ordinary differential equation (ODE) of the form

$$\frac{dy}{dx} + P(x)y(x) \;=\; Q(x)$$

we can use an integrating factor:

$$\mu(x) \;=\; \exp\left[\int^x P(x')\,dx'\right] \qquad \text{and the solution is} \qquad y(x) = \frac{1}{\mu(x)}\left[\int^x Q(x')\mu(x')\,dx' \;+\; C\right] \;.$$

For a **higher-order** ODE with constant coefficients,

$$a_n\frac{d^n y}{dx^n} \;+\; a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} \;+\; \cdots \;+\; a_2\frac{d^2y}{dx^2} \;+\; a_1\frac{dy}{dx} \;+\; a_0 y(x) \;=\; 0 \;\;,$$

the solutions are of the form $y(x) = e^{\lambda x}$, where the λ's are roots of the polynomial

$$a_n\lambda^n \;+\; a_{n-1}\lambda^{n-1} \;+\; \cdots \;+\; a_2\lambda^2 \;+\; a_1\lambda \;+\; a_0 \;=\; 0 \;\;.$$

If there exist complex-conjugate roots ($\lambda = \alpha \pm i\beta$), we can write the corresponding solutions as

$$y(x) \;=\; e^{\alpha x}\left[d_1\cos(\beta x) \;+\; d_2\sin(\beta x)\right] \;\;,$$

where the new constants d_1 and d_2 may be complex, too. If there are k copies of real roots with identical values λ, we must use

$$y(x) \;=\; \left(c_1 \;+\; c_2 x \;+\; c_3 x^3 \;+\; \cdots \;+\; c_k x^{k-1}\right)e^{\lambda x} \qquad \text{for that set of solutions.}$$

Sturm-Liouville ODEs have the general form

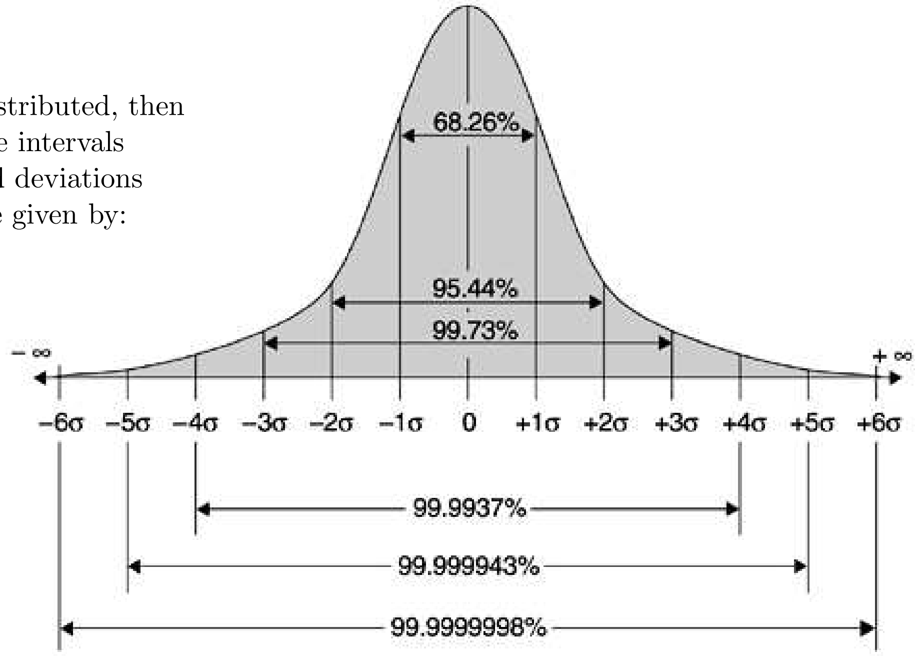
$$\frac{d}{dx}\left[p(x)\,\frac{dy}{dx}\right] \;+\; q(x)\,y \;=\; -\lambda\,w(x)\,y \;\;,$$

and some frequently-seen examples are:

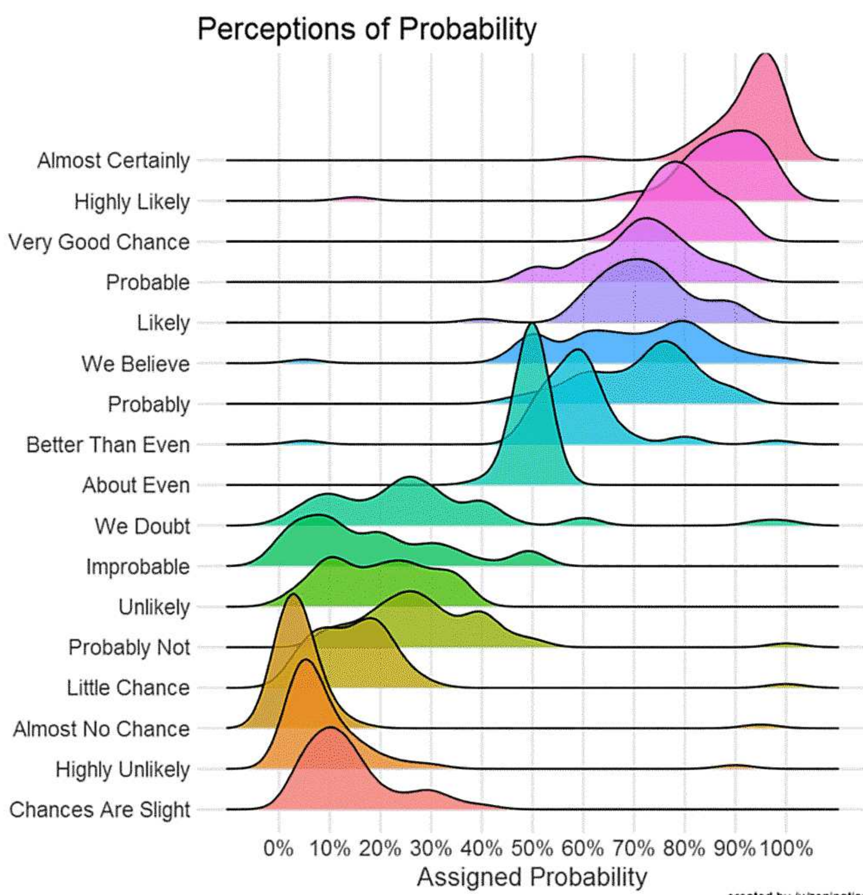
Name	$p(x)$	$q(x)$	$w(x)$	λ	Range
Simple harmonic (Fourier)	1	0	1	ω^2 or k^2	$-\infty < x < \infty$
Legendre, $P_\ell(x)$	$1 - x^2$	0	1	$\ell(\ell+1)$	$-1 \le x \le 1$
Associated Legendre, $P_\ell^m(x)$	$1 - x^2$	$-m^2/(1-x^2)$	1	$\ell(\ell+1)$	$-1 \le x \le 1$
Chebyshev, $T_n(x)$	$(1-x^2)^{1/2}$	0	$(1-x^2)^{-1/2}$	n^2	$-1 \le x \le 1$
Laguerre, $L_n(x)$	xe^{-x}	0	e^{-x}	n	$0 \le x < \infty$
Associated Laguerre, $L_n^\alpha(x)$	$x^{\alpha+1}e^{-x}$	0	$x^\alpha e^{-x}$	n	$0 \le x < \infty$
Hermite, $H_n(x)$	e^{-x^2}	0	e^{-x^2}	$2n$	$-\infty < x < \infty$
Bessel*, $J_\nu(\alpha x)$	x	$-\nu^2/x$	x	α^2	$0 \le x < \infty$

1.8 Probability and Statistics

If events are normally distributed, then the traditional confidence intervals (in units of $\pm N$ standard deviations away from the mean) are given by:



Perceptions of probability associated with common phrases (github link):



Permutations and Combinations

Consider a pile of n books that you want to read. How many different ways are there to order them? There are n ways of choosing the first one in your list. There are then $n-1$ ways of choosing the second one (because there's now one fewer book to choose from), $n-2$ ways of choosing the third one, and so on. The total number of uniquely ordered arrangements, or **permutations,** is thus given by

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \quad = \quad n! \qquad (\text{“}n \text{ factorial”}).$$

For $n \gg 1$, Stirling's approximation gives

$$n! \;\approx\; \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

or

$$\ln n! \;\approx\; \frac{1}{2}\ln(2\pi n) \,+\, n\ln n \,-\, n \;\sim\; n\ln n \,-\, n \;.$$

What if you don't have enough time to read all n books? If you have enough time to read only r books (where $r \leq n$), how many ways can you order them? Like before, you start with n ways of choosing the first one, $n-1$ ways of choosing the second one, and so on. But you stop when you reach the rth book, for which there are $n-r+1$ options. This number of permutations is denoted

$${}_nP_r \;=\; n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \;=\; \frac{n!}{(n-r)!} \;.$$

Another way of thinking about that last version of ${}_nP_r$ is the following: Because you're only reading r books, that means there are $(n-r)$ books that you *won't* be reading. Thus, you can divide out the $(n-r)!$ ways that those books could have been ordered from the overall total ($n!$).

Lastly, what if you wanted to compute the number of unique subsets of r books that can be extracted from the larger pool of n books, but *without regard to their ordering?* You first compute ${}_nP_r$, then divide it by the number of possible orderings of the r books that you will read (i.e., $r!$). This gives the number of **combinations,**

$${}_nC_r \;=\; \frac{n!}{(n-r)!\;\, r!} \;=\; \left(\begin{array}{c} n \\ r \end{array}\right)$$

These are also called binomial coefficients, since

$$(x+y)^n \;=\; \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) x^{n-k} y^k \;.$$

The above explanations were derived from similar ones by Spiegel (1975) and Arbuckle (2008).

2 Physics

Metric Prefixes

Multiple	Prefix	Symbol	Multiple	Prefix	Symbol
10^{-1}	deci	d	10	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E
10^{-21}	zepto	z	10^{21}	zetta	Z
10^{-24}	yocto	y	10^{24}	yotta	Y
10^{-27}	ronto	r	10^{27}	ronna	R
10^{-30}	quecto	q	10^{30}	quetta	Q

Physical Constants (SI)

Speed of light in vacuum .... $c = 2.99792458 \times 10^{8}$ m s^{-1}
Newton's gravitation constant .. $G = 6.67384 \times 10^{-11}$ N m^{2} kg^{-2}
Boltzmann's ideal gas constant . $k_{\mathrm{B}} = 1.3806488 \times 10^{-23}$ J K^{-1}
Stefan-Boltzmann constant ..... $\sigma = 5.670373 \times 10^{-8}$ W m^{-2} K^{-4}
Radiation pressure constant .... $a = 4\sigma/c = 7.5657314 \times 10^{-16}$ J m^{-3} K^{-4}
Planck's constant .... $h = 2\pi\hbar = 6.62606957 \times 10^{-34}$ J s
Permittivity of free space .... $\varepsilon_0 = 8.8542 \times 10^{-12}$ C^{2} N^{-1} m^{-2}
Permeability of free space ...... $\mu_0 = 4\pi \times 10^{-7}$ N s^{2} C^{-2}
Energy associated with 1 eV ... $E_{\mathrm{eV}} = 1.602176565 \times 10^{-19}$ J

Atomic Constants (SI)

Electron charge .... $e = 1.602176565 \times 10^{-19}$ C
Electron mass .... $m_e = 9.10938291 \times 10^{-31}$ kg
Proton mass .... $m_p = 1.672621777 \times 10^{-27}$ kg $\approx$ $1836\, m_e$
Neutron mass .... $m_n = 1.674927351 \times 10^{-27}$ kg $\approx$ $1839\, m_e$
Atomic mass unit .... 1 u = 1 Da = $m(^{12}\mathrm{C})/12 = 1.660538921 \times 10^{-27}$ kg
Bohr radius .... $a_0 = 4\pi\varepsilon_0\, \hbar^2/(m_e e^2) = 5.2917721092 \times 10^{-11}$ m
Classical electron radius .... $r_e = e^2/(4\pi\varepsilon_0\, m_e c^2) = 2.8179403267 \times 10^{-15}$ m
Thomson cross section .... $\sigma_{\mathrm{T}} = (8\pi/3) r_e^2 = 6.652458734 \times 10^{-29}$ m^{2}

Astronomical Constants (SI)

Solar mass .... $M_\odot = 1.989 \times 10^{30}$ kg
Solar radius .... $R_\odot = 6.963 \times 10^{8}$ m
Solar luminosity .... $L_\odot = 3.83 \times 10^{26}$ J s^{-1}
Solar effective temperature ..... $T_{\mathrm{eff}} = 5770$ K
Solar surface gravity .... $g_\odot = 273.79$ m s^{-2} $\log g_\odot$ (cgs) $= 4.4374$
Astronomical unit .... 1 AU $= 1.495978707 \times 10^{11}$ m $\approx$ $215\, R_\odot$
Parsec .... 1 $pc = 3.085677 \times 10^{16}$ m $=$ 3.2616 light years

Physical Constants (cgs)

Speed of light in vacuum	$c = 2.99792458 \times 10^{10}$ cm s^{-1}
Newton's gravitation constant ..	$G = 6.67384 \times 10^{-8}$ cm^3 g^{-1} s^{-2}
Boltzmann's ideal gas constant .	$k_{\rm B} = 1.3806488 \times 10^{-16}$ erg K^{-1}
Stefan-Boltzmann constant	$\sigma = 5.670373 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}
Radiation pressure constant	$a = 4\sigma/c = 7.5657314 \times 10^{-15}$ erg cm^{-3} K^{-4}
Planck's constant	$h = 2\pi\hbar = 6.62606957 \times 10^{-27}$ erg s
Energy associated with 1 eV ...	$E_{\rm eV} = 1.602176565 \times 10^{-12}$ erg

Atomic Constants (cgs)

Electron charge	$e = 4.803243 \times 10^{-10}$ esu
Electron mass	$m_e = 9.10938291 \times 10^{-28}$ g
Proton mass	$m_p = 1.672621777 \times 10^{-24}$ g $\approx$ $1836\, m_e$
Neutron mass	$m_n = 1.674927351 \times 10^{-24}$ g $\approx$ $1839\, m_e$
Atomic mass unit	1 u = 1 Da = $m(^{12}{\rm C})/12 = 1.660538921 \times 10^{-24}$ g
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Astronomical Constants (cgs)

Solar mass	$M_\odot = 1.989 \times 10^{33}$ g
Solar radius	$R_\odot = 6.963 \times 10^{10}$ cm
Solar luminosity	$L_\odot = 3.83 \times 10^{33}$ erg s^{-1}
Solar effective temperature	$T_{\rm eff} = 5770$ K
Solar surface gravity	$g_\odot = 2.7379 \times 10^4$ cm s^{-2} $\quad \log g_\odot = 4.4374$
Earth's mass	$M_\oplus = 5.9736 \times 10^{27}$ g $\approx$ $3 \times 10^{-3}\, M_{\rm J}$ $\approx$ $3 \times 10^{-6}\, M_\odot$
Earth's radius	$R_\oplus = 6.371 \times 10^8$ cm
Jupiter's mass	$M_{\rm J} = 1.8986 \times 10^{30}$ g $\approx$ $320\, M_\oplus$ $\approx$ $10^{-3}\, M_\odot$
Jupiter's radius	$R_{\rm J} = 6.9911 \times 10^9$ cm
Astronomical unit	1 AU = $1.495978707 \times 10^{13}$ cm $\approx$ $215\, R_\odot$
Parsec	1 $pc = 3.085677 \times 10^{18}$ cm = 3.2616 light years
Convenient mass-loss unit	1 $M_\odot$ yr^{-1} = 6.3029×10^{25} g s^{-1}

Some other units that you may encounter...

1 ångström	1 Å = 10^{-10} m = 0.1 nm
1 jansky	1 Jy = 10^{-26} W m^{-2} Hz^{-1}
1 rayleigh	1 R = $10^{10}/4\pi$ photons s^{-1} m^{-2} sr^{-1}
1 barn	1 b = 10^{-28} m^2 = 100 fm^2
1 bethe ("fifty one ergs")	1 foe = 10^{51} erg = 10^{44} J
1 rydberg	1 Ry = 13.605693123 eV
1 hartree	1 Ha = 2 Ry = 27.211386246 eV
1 ton of TNT	$\approx 4.184 \times 10^9$ J = 4.184×10^{16} erg
1 year	1 yr $\approx 365.25 \times 24 \times 60 \times 60$ = 3.15576×10^7 s

Converting between SI (mks) and Gaussian (cgs) Units

In the table below, $\{3\} = 2.99792458$ (exactly)

Physical quantity	Symbol	Amount in SI/mks	= Amount in Gaussian/cgs
Force	$\mathbf{F}$	1 newton (N)	$= 10^5$ dyne
Presure	P	1 pascal (Pa)	$= 10$ dyne cm^{-2} $= 10$ erg cm^{-3}
Energy	E	1 joule (J)	$= 10^7$ erg
Power, Luminosity	L	1 watt (W) $= 1$ J s^{-1}	$= 10^7$ erg s^{-1}
Energy flux	$\mathbf{F}$	1 W m^{-2}	$= 10^3$ erg s^{-1} cm^{-2}
Charge	q	1 coulomb (C)	$= \{3\} \times 10^9$ statcoul
Charge density	ρ_c	1 C m^{-3}	$= \{3\} \times 10^3$ statcoul cm^{-3}
Current	I	1 ampere (A)	$= \{3\} \times 10^9$ statamp
Current density	$\mathbf{J}$	1 A m^{-2}	$= \{3\} \times 10^5$ statamp cm^{-2}
Electric potential	V	1 volt (V)	$= 10^{-2}/\{3\}$ statvolt
Electric field	$\mathbf{E}$	1 V m^{-1}	$= 10^{-4}/\{3\}$ statvolt cm^{-1}
Magnetic induction	$\mathbf{B}$	1 tesla (T)	$= 10^4$ gauss (G)
Magnetic flux	Φ	1 weber (Wb) $= 1$ T m^2	$= 10^8$ maxwells (Mx) $= 10^8$ G cm^2

For a general way to convert equations from one system to another, see the NRL Plasma Formulary or Weibel (1968). If only magnetic quantities are present (i.e., no charges or electric fields), it's often sufficient to just replace μ_0 by 4π.

Specific Examples:

Expressions in SI

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0} \,, \qquad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left\{ \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right\}$$

$$U = \frac{\varepsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \,, \qquad \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

$$V_{\rm A} = \frac{B}{\sqrt{\mu_0 \rho}} \,, \qquad P_{\rm mag} = \frac{B^2}{2\mu_0}$$

Expressions in Gaussian

μ_0 and ε_0 not used

$$\mathbf{F} = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)$$

$$|\mathbf{E}| = \frac{q}{r^2}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho_c \,, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{4\pi \mathbf{J}}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$U = \frac{|\mathbf{E}|^2}{8\pi} + \frac{|\mathbf{B}|^2}{8\pi} \,, \qquad \mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$$

$$V_{\rm A} = \frac{B}{\sqrt{4\pi\rho}} \,, \qquad P_{\rm mag} = \frac{B^2}{8\pi}$$

Unit Conversions

Changing from one set of units to another is enabled by thinking of them as ratios that get multiplied together in a chain. Example:

$$\frac{\textbf{55 miles}}{\textbf{hour}} \times \frac{\text{5280 feet}}{\text{1 mile}} \times \frac{\text{12 inches}}{\text{1 foot}} \times \frac{\text{2.54 cm}}{\text{1 inch}} \times \frac{\text{1 m}}{\text{100 cm}} \times \frac{\text{1 hour}}{\text{3600 s}} = \textbf{24.59 m/s}$$

Significant Figures

You should already know how to count up the number of significant digits in a quantity. In scientific notation, it's usually assumed that every digit given prior to the exponential is significant (e.g., 4.1800×10^7 has five significant digits).

- When combining two quantities (e.g., adding, multiplying), the answer should be given with the *least* number of significant digits from the initial quantities.
- However, when working through multi-step calculations, it's useful to keep at least one more significant digit in the intermediate results than will be needed in the final answer.
- If asked to approximate, the final answer should only have (at most) two significant digits.

Misc. Special Units

Solar physics:

1 X/M/C flare $\approx 10^{33}/10^{32}/10^{31}$ erg ... 1 microflare $\approx 10^{27}$ erg ... 1 nanoflare $\approx 10^{24}$ erg.

Radiative transfer:

$$1 \text{ jansky (Jy)} = 10^{-26} \frac{\text{W}}{\text{m}^2 \text{ Hz}} = 10^{-23} \frac{\text{erg}}{\text{cm}^2 \text{ s Hz}} \qquad (\text{specific flux } F_\nu)$$

$$1 \text{ rayleigh} = \frac{10^6}{4\pi} \frac{\text{photons}}{\text{cm}^2 \text{ s sr}} \qquad (\text{total intensity } I)$$

Misc. Plasma Physics

The Langmuir plasma frequency of species j (Gaussian units) is defined as

$$\omega_{pj}^2 \equiv \frac{4\pi q_j^2 n_j}{m_j} .$$

For protons, $\dfrac{\omega_{pp}}{c} = \dfrac{\Omega_p}{V_A}$. For a pure proton-electron plasma, $\dfrac{\omega_{pp}^2}{\Omega_p} = \dfrac{\omega_{pe}^2}{|\Omega_e|}$.

The Electromagnetic Spectrum

Name of band	**Wavelength range**	**Freq. or Energy**
radio: ultra low frequency (ULF)	100–1000 km	0.3–3 kHz
radio: very low frequency (VLF)	10–100 km	3–30 kHz
radio: longwave / low frequency (LF)	1–10 km	30–300 kHz
radio: AM / medium frequency (MF)	0.1–1 km	0.3–3 MHz
radio: shortwave / high frequency (HF)	10–100 m	3–30 MHz
radio: FM / very high frequency (VHF)	1–10 m	30–300 MHz
radio: ultra high frequency (UHF)	0.1–1 m	0.3–3 GHz
radar: S-band	7.5–15 cm	2–4 GHz
radar: X-band	2.5–3.7 cm	8–12 GHz
microwave: super high frequency (SHF)	1–10 cm	3–30 GHz
millimeter: extremely high frequency (EHF)	1–10 mm	30–300 GHz
sub-millimeter / terahertz	0.1–1 mm	0.3–3 THz
far infrared (FIR)	30–500 μm	0.6–10 THz
mid infrared (MIR)	3–30 μm	0.04–0.4 eV
near infrared (NIR)	0.7–2 μm	0.6–1.8 eV
optical/visible light (ROY G. BIV)	400–700 nm	1.8–3.1 eV
ultraviolet A (UV-A, "black light")	315–400 nm	3.1–3.9 eV
ultraviolet B (UV-B)	280–315 nm	3.9–4.4 eV
ultraviolet C (UV-C)	100–280 nm	4.4–12.4 eV
near ultraviolet	200–400 nm	3.1–6.2 eV
rocket ultraviolet	120–300 nm	4.1–10 eV
far ultraviolet	90–200 nm	6.2–13.8 eV
vacuum ultraviolet	10–200 nm	6.2–124 eV
extreme ultraviolet (EUV)	9–100 nm	12.4–138 eV
soft X-rays	0.5–9 nm	0.14–2.5 keV
hard X-rays	0.01–0.5 nm	2.5–100 keV
gamma rays	< 0.01 nm	> 100 keV
low-energy gamma rays	2–10 pm	0.1–0.6 MeV
medium-energy gamma rays	0.1–1000 fm	1 MeV – 10 GeV
very high energy (VHE) gamma rays	0.01–10 am	0.1–100 TeV
ultra high energy (UHE) gamma rays	0.01–10 zm	0.1–100 PeV

Properties of the Chemical Elements

Z_i	Symbol & Name		A_i	B_i/A_i	$\log(n_i/n_{\rm H})$	$\log(m_i n_i/\rho)$	FIP (eV)
1	H	Hydrogen	1.008	0.000	0.00	–0.13	13.598
2	He	Helium	4.003	7.704	–1.09	–0.62	24.587
3	Li	Lithium	6.94	5.606	–11.04	–10.33	5.392
4	Be	Beryllium	9.012	6.463	–10.62	–9.80	9.323
5	B	Boron	10.81	6.928	–9.30	–8.40	8.298
6	C	Carbon	12.011	7.680	–3.54	–2.59	11.260
7	N	Nitrogen	14.007	7.476	–4.17	–3.16	14.534
8	O	Oxygen	15.999	7.976	–3.31	–2.24	13.618
9	F	Fluorine	18.998	7.779	–7.60	–6.45	17.423
10	Ne	Neon	20.180	8.032	–3.94	–2.77	21.565
11	Na	Sodium	22.990	8.111	–5.78	–4.55	5.139
12	Mg	Magnesium	24.305	8.261	–4.45	–3.20	7.646
13	Al	Aluminum	26.982	8.332	–5.57	–4.27	5.986
14	Si	Silicon	28.086	8.448	–4.49	–3.17	8.152
15	P	Phosphorus	30.974	8.481	–6.59	–5.23	10.487
16	S	Sulfur	32.06	8.493	–4.88	–3.51	10.360
17	Cl	Chlorine	35.45	8.520	–6.69	–5.27	12.968
18	Ar	Argon	39.948	8.595	–5.62	–4.15	15.760
19	K	Potassium	39.098	8.557	–6.93	–5.47	4.341
20	Ca	Calcium	40.078	8.551	–5.70	–4.23	6.113
21	Sc	Scandium	44.956	8.619	–8.86	–7.34	6.561
22	Ti	Titanium	47.867	8.723	–7.03	–5.48	6.828
23	V	Vanadium	50.942	8.742	–8.10	–6.52	6.746
24	Cr	Chromium	51.996	8.776	–6.38	–4.80	6.767
25	Mn	Manganese	54.938	8.765	–6.58	–4.97	7.434
26	Fe	Iron	55.845	8.790	–4.54	–2.93	7.902
27	Co	Cobalt	58.933	8.768	–7.06	–5.42	7.881
28	Ni	Nickel	58.693	8.781	–5.80	–4.16	7.640
29	Cu	Copper	63.546	8.752	–7.82	–6.15	7.726
30	Zn	Zinc	65.38	8.760	–7.44	–5.76	9.394
31	Ga	Gallium	69.723	8.724	–8.98	–7.27	5.999
32	Ge	Germanium	72.630	8.732	–8.38	–6.65	7.899
33	As	Arsenic	74.922	8.701	–9.70	–7.96	9.789
34	Se	Selenium	78.971	8.718	–8.66	–6.89	9.752
35	Br	Bromine	79.904	8.696	–9.46	–7.69	11.814
36	Kr	Krypton	83.798	8.717	–8.88	–7.09	14.000
37	Rb	Rubidium	85.468	8.697	–9.68	–7.88	4.177
38	Sr	Strontium	87.62	8.733	–9.17	–7.36	5.695
39	Y	Yttrium	88.906	8.714	–9.79	–7.97	6.217
40	Zr	Zirconium	91.224	8.710	–9.41	–7.58	6.634

Properties of the Chemical Elements (continued)

Z_i	Symbol & Name		A_i	B_i/A_i	$\log(n_i/n_{\rm H})$	$\log(m_i n_i/\rho)$	FIP (eV)
41	Nb	Niobium	92.906	8.671	–10.53	–8.69	6.759
42	Mo	Molybdenum	95.95	8.662	–10.12	–8.27	7.092
43	Tc	Technetium	[97]	8.624	—	—	7.119
44	Ru	Ruthenium	101.07	8.620	–10.25	–8.38	7.360
45	Rh	Rhodium	102.905	8.588	–11.22	–9.34	7.459
46	Pd	Palladium	106.42	8.585	–10.43	–8.53	8.337
47	Ag	Silver	107.868	8.554	–11.04	–9.14	7.576
48	Cd	Cadmium	112.414	8.551	–10.29	–8.37	8.994
49	In	Indium	114.818	8.523	–11.20	–9.27	5.786
50	Sn	Tin	118.710	8.523	–9.98	–8.04	7.344
51	Sb	Antimony	121.760	8.488	–10.99	–9.04	8.608
52	Te	Tellurium	127.60	8.478	–9.82	–7.85	9.010
53	I	Iodine	126.904	8.450	–10.45	–8.48	10.451
54	Xe	Xenon	131.293	8.444	–9.78	–7.79	12.130
55	Cs	Cesium	132.905	8.416	–10.92	–8.93	3.894
56	Ba	Barium	137.327	8.409	–9.73	–7.72	5.212
57	La	Lanthanum	138.905	8.383	–10.89	–8.88	5.577
58	Ce	Cerium	140.116	8.377	–10.42	–8.41	5.539
59	Pr	Praseodymium	140.908	8.354	–11.25	–9.23	5.470
60	Nd	Neodymium	144.242	8.346	–10.58	–8.55	5.525
61	Pm	Promethium	[145]	8.318	—	—	5.582
62	Sm	Samarium	150.36	8.304	–11.05	–9.00	5.644
63	Eu	Europium	151.964	8.269	–11.48	–9.43	5.670
64	Gd	Gadolinium	157.25	8.250	–10.92	–8.86	6.150
65	Tb	Terbium	158.925	8.210	–11.69	–9.62	5.864
66	Dy	Dysprosium	162.500	8.193	–10.90	–8.82	5.939
67	Ho	Holmium	164.930	8.165	–11.52	–9.43	6.022
68	Er	Erbium	167.259	8.152	–11.07	–8.98	6.108
69	Tm	Thulium	168.934	8.126	–11.89	–9.79	6.184
70	Yb	Ytterbium	173.045	8.112	–11.15	–9.04	6.254
71	Lu	Lutetium	174.967	8.086	–11.90	–9.79	5.426
72	Hf	Hafnium	178.486	8.072	–11.15	–9.03	6.825
73	Ta	Tantalum	180.948	8.045	–12.15	–10.02	7.550
74	W	Tungsten	183.84	8.030	–11.21	–9.08	7.864
75	Re	Rhenium	186.207	8.004	–11.74	–9.60	7.834
76	Os	Osmium	190.23	7.989	–10.65	–8.50	8.438
77	Ir	Iridium	192.217	7.964	–10.68	–8.53	8.967
78	Pt	Platinum	195.084	7.948	–10.39	–8.23	8.959
79	Au	Gold	196.967	7.925	–11.09	–8.93	9.226
80	Hg	Mercury	200.592	7.915	–10.83	–8.66	10.438

Properties of the Chemical Elements (continued)

Z_i	Symbol & Name		A_i	B_i/A_i	$\log(n_i/n_H)$	$\log(m_i n_i/\rho)$	FIP (eV)
81	Tl	Thallium	204.38	7.894	–11.08	–8.90	6.108
82	Pb	Lead	207.2	7.882	–10.05	–7.87	7.417
83	Bi	Bismuth	208.980	7.858	–11.35	–9.16	7.286
84	Po	Polonium	[209]	7.841	—	—	8.418
85	At	Astatine	[210]	7.815	—	—	9.318
86	Rn	Radon	[222]	7.797	—	—	10.748
87	Fr	Francium	[223]	7.768	—	—	4.073
88	Ra	Radium	[226]	7.749	—	—	5.278
89	Ac	Actinium	[227]	7.720	—	—	5.380
90	Th	Thorium	232.038	7.698	–11.97	–9.74	6.307
91	Pa	Protactinium	231.036	7.665	—	—	5.89
92	U	Uranium	238.029	7.641	–12.54	–10.30	6.194
93	Np	Neptunium	[237]	7.608	—	—	6.266
94	Pu	Plutonium	[244]	7.591	—	—	6.026
95	Am	Americium	[243]	7.567	—	—	5.974
96	Cm	Curium	[247]	7.550	—	—	5.991
97	Bk	Berkelium	[247]	7.527	—	—	6.198
98	Cf	Californium	[251]	7.510	—	—	6.282
99	Es	Einsteinium	[252]	7.486	—	—	6.368
100	Fm	Fermium	[257]	7.468	—	—	6.50
101	Md	Mendelevium	[258]	7.443	—	—	6.58
102	No	Nobelium	[259]	7.426	—	—	6.626
103	Lr	Lawrencium	[262]	7.402	—	—	4.96
104	Rf	Rutherfordium	[267]	7.385	—	—	6.02
105	Db	Dubnium	[270]	7.361	—	—	6.8
106	Sg	Seaborgium	[269]	7.342	—	—	7.8
107	Bh	Bohrium	[270]	7.316	—	—	7.7
108	Hs	Hassium	[270]	7.297	—	—	7.6
109	Mt	Meitnerium	[278]	7.268	—	—	[8.3]
110	Ds	Darmstadtium	[281]	—	—	—	[9.9]
111	Rg	Roentgenium	[281]	—	—	—	[10.6]
112	Cn	Copernicium	[285]	—	—	—	[12.0]
113	Nh	Nihonium	[286]	—	—	—	[7.3]
114	Fl	Flerovium	[289]	—	—	—	[8.6]
115	Mc	Moscovium	[289]	—	—	—	[5.6]
116	Lv	Livermorium	[293]	—	—	—	[6.9]
117	Ts	Tennessine	[293]	—	—	—	[7.7]
118	Og	Oganesson	[294]	—	—	—	[8.9]

Atomic mass A_i is usually given as a standard weighted mean of isotopes that occur on Earth, but numbers in brackets refer to the "atomic mass number" (total of the number of protons and neutrons) of the most prevalent isotope. Binding energies per nucleon (B_i/A_i) are from Audi & Wapstra (1993) and succeeding papers in the series. Solar abundances of naturally occurring elements are provided in two forms: by number and by mass (both as base-10 logarithms of dimensionless ratios), taken from Asplund et al. (2021). FIP = first ionization potential of the dominant isotope.