Radiative Processes II: Beyond Classical Stellar Atmospheres

We gleaned a lot of insight from the previous study of idealized stellar atmospheres. However, the wider world of astrophysics contains environments that are:

(1) Non-gray / Non-Eddington / Non-plane-parallel / Non-LTE
(2) Dominated by spectral lines (bound-bound transitions)
(3) Dominated by ionization & recombination (bound-free transitions)
(4) Irradiated from outside (e.g., planets & comets)
(5) Influenced by energy/momentum exchange between light & matter

We probably won't get through all of this material in class....

Non-gray effects

We discussed how κ_ν has lots of structure vs ν (e.g., sharp lines, bound-free edges, slow free-free variations).

We need to confront the reality that radiative equilibrium isn't so trivial any more. The gray 0th moment equation was essentially

$$\frac{dH}{dz} = -\kappa\rho(J - B)$$

and it's still valid for the monochromatic (subscript ν) quantities. However, this is really

$$\frac{d}{dz}\int_0^\infty d\nu\ H_\nu = -\rho\left[\int_0^\infty d\nu\ \kappa_\nu J_\nu - \int_0^\infty d\nu\ \kappa_\nu B_\nu\right] .$$

For a plane-parallel atmosphere, conservation of energy gives LHS = 0, and thus RHS = 0. That's still true, but it does **not** mean that

$$\frac{dH_\nu}{dz} = 0 \quad \text{or} \quad J_\nu = B_\nu \quad (\text{at all values of } \nu).$$

Photons can "migrate" around in frequency, maintaining the integrated radiative equilibrium while violating it at any given ν.

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Next, how do we relate "gray" results to the real non-gray world?

Examining the frequency dependence of the next higher moment equation helps us define the **Rosseland mean opacity.**

$$\text{Gray version:} \quad H = \frac{dK}{d\tau} = -\frac{1}{\rho\langle\kappa\rangle}\frac{dK}{dz} = -\frac{1}{\rho\langle\kappa\rangle}\int_0^\infty d\nu\, \frac{dK_\nu}{dz} \;.$$

$$\text{The more general non-gray version is:} \qquad H_\nu = -\frac{1}{\rho\kappa_\nu}\frac{dK_\nu}{dz} \;.$$

If we integrate the non-gray version over all frequencies, we get

$$H = \int_0^\infty d\nu\; H_\nu = -\frac{1}{\rho}\int_0^\infty d\nu\, \frac{1}{\kappa_\nu}\frac{dK_\nu}{dz}$$

However, we already have the gray version of H (above), so we can solve for

$$\frac{1}{\langle\kappa\rangle} = \frac{\int d\nu\, \frac{1}{\kappa_\nu}\frac{dK_\nu}{dz}}{\int d\nu\, \frac{dK_\nu}{dz}}$$

That's nice, but we don't know the weighting function (dK_ν/dz) a-priori.

Deep in the atmosphere (where it's close to LTE, and I_ν is close to isotropic), we can approximate

$$K_\nu \approx \frac{1}{3}J_\nu \approx \frac{1}{3}B_\nu \quad \text{so,} \quad \frac{dK_\nu}{dz} \approx \frac{1}{3}\frac{dB_\nu}{dz} \approx \frac{1}{3}\frac{dB_\nu}{dT}\frac{dT}{dz} \;.$$

The $(1/3)dT/dz$ term appears in both numerator & denominator of $\langle\kappa\rangle$, and it's independent of ν so can be taken out.

$$\text{Thus,} \quad \boxed{\frac{1}{\langle\kappa_{\rm Ross}\rangle} \equiv \frac{\int d\nu\, \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int d\nu\, \frac{dB_\nu}{dT}}}$$

More weight is given to regions where κ_ν is low (i.e., where the most radiation is transported out, and not blocked).

This type of weighting applies well in stellar interiors, too.

Non-Eddington effects

Even if we were to assume that other approximations (gray, plane-parallel, LTE) were okay, we saw that the Eddington approximations gave rise to a **non-self-consistent** solution for $I(\mu, \tau)$.

If the source function is known, we saw that

$$
\begin{aligned}
J(\tau) &= \frac{1}{2}\int_{-1}^{+1} d\mu \; I(\mu,\tau) \\
&= \frac{1}{2}\int_{0}^{+1} d\mu \int_{\tau}^{\infty} \frac{dt}{\mu}\, S(t)\, e^{(\tau-t)/\mu} \;-\; \frac{1}{2}\int_{-1}^{0} d\mu \int_{0}^{\tau} \frac{dt}{\mu}\, S(t)\, e^{(\tau-t)/\mu} \;.
\end{aligned}
$$

Exchanging the order of integration, we can use absolute values to combine both pieces into a single integral. It ends up looking like

$$J(\tau) = \frac{1}{2}\int_{0}^{\infty} dt\, S(t)\, E_1(|t-\tau|)$$

where the first exponential integral is used, with

$$E_1(x) = \int_{1}^{\infty} \frac{dy}{y} e^{-xy} \qquad \text{and we used} \quad y = \pm\frac{1}{\mu} \;.$$

This integral appears frequently enough to call it an “operator” on the source function:

$$\Lambda_\tau\left[S(t)\right] \equiv \frac{1}{2}\int_{0}^{\infty} dt\, S(t)\, E_1(|t-\tau|)$$

Thus, if we have already written

$$S = (1-a)B + aJ = \epsilon B + (1-\epsilon)J$$

($a =$ albedo ... $\epsilon = 1 - a =$ photon “destruction probability”)

then we can write the **Schwarzschild-Milne** Λ-iteration equation for S,

$$S(\tau) = \epsilon B(\tau) + (1-\epsilon)\Lambda_\tau\left[S(t)\right]$$

and one can refine S as many times as desired by repeating the iteration process, then substitute it back into the formal solution to get an improved $I(\mu, \tau)$.

Unfortunately, Λ iteration is hopelessly slow, especially deep in the atmosphere. Deep down, it can take $\sim e^{\tau}$ iteration steps to converge!

There are numerical tricks that can be used to obtain better "accelerated Lambda iteration" (ALI). Many stellar atmospheres codes use them.

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The { gray, LTE, plane-parallel } model *does* have an exact solution, that doesn't rely on the Eddington approximation.

Eberhard Hopf worked it out in the 1930s. In general, we can write

$$J(\tau) \propto \tau + q(\tau)$$

where $q = 2/3$ in the Eddington/gray model, but in general the self-consistent **Hopf function** $q(\tau)$ isn't constant.

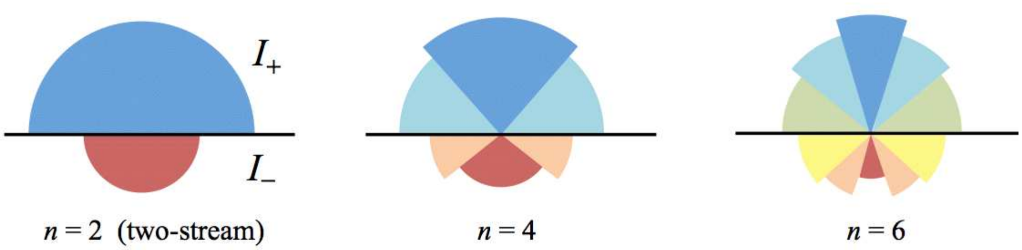
For pure scattering ($\epsilon = 0$), we know $J = S$, so the Λ-iteration process gives

$$\tau + q(\tau) = \frac{1}{2} \int_0^{\infty} dt \, [t + q(t)] \, E_1(|t - \tau|)$$

and if we solve this integral equation for $q(\tau)$, we've solved the non-Eddington gray atmosphere problem exactly.

In practice, a convenient way to solve for it is to use the **Wick–Chandrasekhar method of discrete ordinates.**

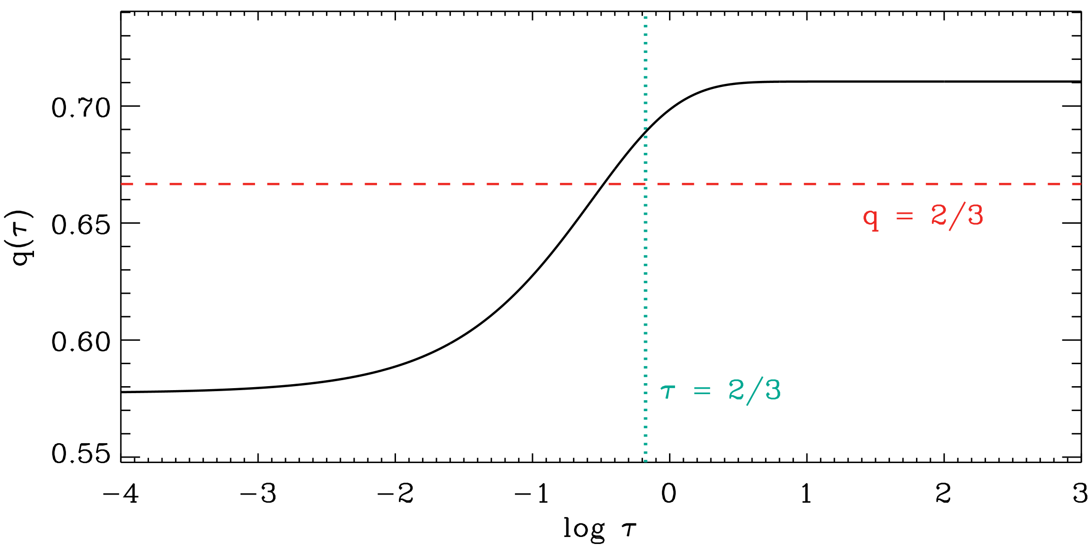
That's a mouthful to say it's a generalization of the 2-stream model to having multiple "wedges" of μ-space occupied by constant values of I.



Integrating over μ is optimized via "Gaussian quadrature" (see, e.g., *Numerical Recipes,* §4.5).

The upshot: E_1 functions go away, and in each wedge, the formal solution gives $I_i = k_i e^{-\sigma_i \tau}$, and there's a set of simultaneous eigenvalue equations for the (k_i, σ_i) coefficients.

The Hopf function can be expressed in the form of closed-form integrals (Mark 1947, *Phys. Rev.*, 72, 558). Numerical solution:



At the surface, there is an exact result: $q(0) = 1/\sqrt{3} \approx 0.57735...$

Thus, a better version of Eddington's 2nd approximation would have been

$$J = 3H[\tau + q(\tau)] \; = \; 3H\left(0 + \frac{1}{\sqrt{3}}\right) \; = \; \sqrt{3}\,H \; \approx \; \boxed{1.732H}$$

and this is closer to the iterated value of $1.75H$ than the original $2H$.

Non-plane-parallel effects

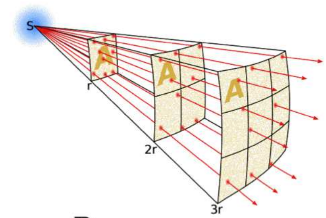
The plane-parallel version of radiative equilibrium ($F \approx$ constant) was justified for stars having scale heights $H \ll R_*$.

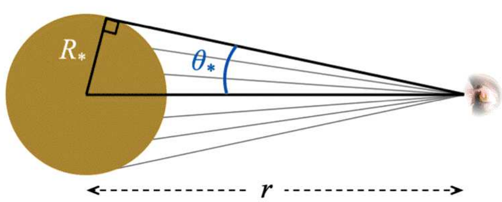
Some stars (giants & supergiants) have low enough surface gravity that this is no longer the case, and we must deal with the full spherical dependence of

$$\nabla \cdot \mathbf{F}_{\rm rad} = 0 \qquad \Longrightarrow \qquad F_{\rm rad} = L_*/(4\pi r^2) \ .$$

If one uses the two-stream approximation for the radiation field and models the moments of $I(\mu)$ by integrating over the "visible disk" of a star seen from $r > R_*$, we find $J = H = K$ in the limit of $r \to \infty$.

- The usual spherical cartoon focuses on the point of origin:
- However, if a central sphere is the source of radiation (assume two-stream, $I^+ \neq 0$, $I^- = 0$), the observer looks back to see a uniform-brightness "disk" on the sky:

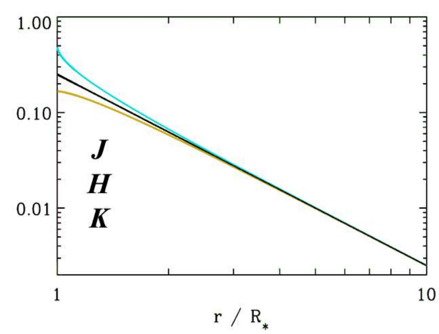




$$\sin\theta_* = \frac{R_*}{r}$$

$$\mu_* = \cos\theta_* = \sqrt{1 - \frac{R_*^2}{r^2}}$$

$$\left\{ \begin{array}{c} J \\ H \\ K \end{array} \right\} = \frac{1}{2}\int_{\mu_*}^{1} d\mu \; I^+ \left\{ \begin{array}{c} 1 \\ \mu \\ \mu^2 \end{array} \right\} = I^+ \left\{ \begin{array}{l} \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{R_*^2}{r^2}} \\ \frac{R_*^2}{4r^2} \\ \frac{1}{6} - \frac{1}{6}\left(1 - \frac{R_*^2}{r^2}\right)^{3/2} \end{array} \right.$$



Far from the source… $\quad F = \pi I^+ \left(\frac{R_*}{r}\right)^2$

Recall that $F = 4\pi H$. At surface, note that $J = 3K$ and $J = 2H$.

The geometrical factor in J is called the **dilution factor:**

$$W(r) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{R_*^2}{r^2}} \qquad\qquad \text{As } r \to \infty, \quad W \to \frac{R_*^2}{4r^2}$$

and is used often in models of H II regions & Strömgren spheres.

Non-LTE effects

This is an important one, since we'll see that our imposition of LTE *all the way* up to the surface ($\tau = 0$) was a very bad approximation.

Near the surface, escaping photons are truly "lost;" i.e., not replenished from a locally Planckian population (like they would be if LTE was true).

When there is scattering ($\epsilon < 1$ in source function), photons can interact with matter *without* being immediately absorbed & re-emitted in thermal equilibrium with the gas.

Thus, non-LTE can be a very non-linear and **non-local** problem.

In any case, $S \neq B \neq J$, and usually it's tackled numerically.

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However, there is a nice model that illustrates many non-LTE effects. Let's look at the **"Milne–Eddington" model**[1] for an atmosphere in which we make the following assumptions:

- Plane-parallel
- *Not* gray, but we'll leave off monochromatic ν subscripts
- Eddington's 1st approximation $(J = 3K)$
- Eddington's modified 2nd approximation $(J = \sqrt{3}\,H)$
- General source function with absorption & coherent/isotropic scattering:

  $$S \,=\, \epsilon B \,+\, (1-\epsilon)J$$

  where we assume $\epsilon =$ constant with depth
- Linear dependence of the Planck function with depth:

  $B(\tau) \,=\, a + b\tau$ (in LTE, $b/a = 3/2$ ... exact Hopf @ surface: $b/a = \sqrt{3}$)

If we're careful about our approximations, we can go back to the moment equations and derive something much more general than what we got before.

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[1] I use scare-quotes because this will be only a piece of it. The *complete* M–E model accounts for spectral line formation, but with the same foundational equations developed here; see also Dietz & House (1965, ApJ, 141, 1393).

$$\text{0th moment of RT:} \qquad \frac{dH}{d\tau} = \epsilon\,(J - B)$$

$$\text{1st moment of RT:} \qquad \frac{dK}{d\tau} = H \qquad \longrightarrow \qquad \text{Eddington's 1st:} \qquad \frac{dJ}{d\tau} = 3H$$

Combining these two 1st-order ODEs into one 2nd-order ODE...

$$\frac{d^2 J}{d\tau^2} = 3\frac{dH}{d\tau} = 3\epsilon\,(J - B)$$

Because $B(\tau)$ is "only" linear, then $d^2B/d\tau^2 = 0$, and

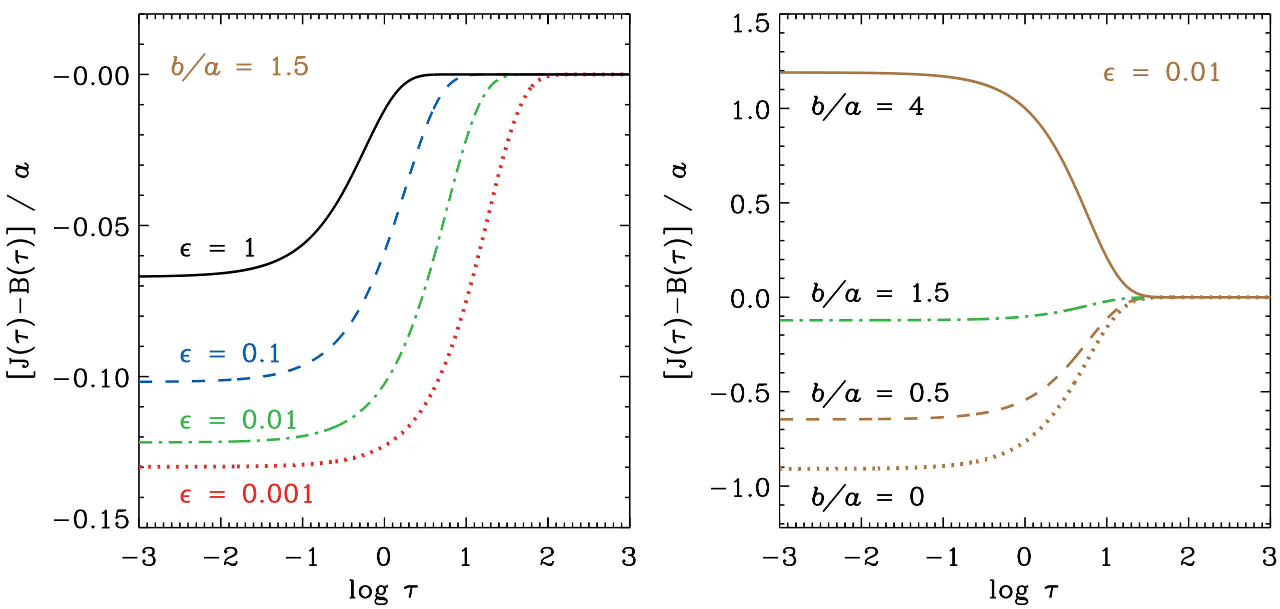
$$\frac{d^2}{d\tau^2}(J - B) = 3\epsilon\,(J - B)$$

and it's straightforward to integrate it, and apply the proper boundary conditions (Eddington's 2nd at surface; $J \to B$ at depth), to get

$$\boxed{J(\tau) = B(\tau) + \frac{(b/\sqrt{3}) - a}{1 + \sqrt{\epsilon}}\, e^{-\tau\sqrt{3\epsilon}}}$$

and we can compute $S(\tau)$, then get $I(\mu, \tau)$ from the formal solution.

Note: $J \neq B$ in the "upper" atmosphere! Example plots of $(J - B)$ vs. τ :



We get LTE back ($J = B$) when $b/a = \sqrt{3}$, but real non-LTE atmospheres can run the gamut from steeper ($b/a \gg 1$) to isothermal ($b/a = 0$) to "inverted" ($b/a < 0$).

Notes:

(1) $H \neq$ constant, but recall this is a monochromatic H_ν. Some regions of the spectrum may produce "radiative heating" ($J_\nu > S_\nu$) and some may produce "radiative cooling" ($J_\nu < S_\nu$), but **bolometric** flux needs to be conserved.

(2) The depth at which $J \to B$ is *not* the usual photosphere ($\tau \approx 1$).

For $\epsilon \ll 1$, the atmosphere is **effectively thick** (i.e., effectively LTE) only below a

$$\textbf{thermalization depth} \qquad \tau_{\rm th} \approx \frac{1}{\sqrt{\epsilon}} \ .$$

Think about what it means when the destruction probability ϵ is $\ll 1$.

A photon "created" thermally can scatter many times ($N \approx 1/\epsilon \gg 1$) before it is truly absorbed.

Between scatterings, a photon travels ~ 1 mean free path ($\ell_{\rm mfp} \approx 1/\chi$).

After scattering N times, undergoing random changes in direction each time, the *net* vertical distance traversed via **random walk** is $\Delta z \approx \ell_{\rm mfp}\sqrt{N}$.

Deep in the atmosphere, an approximate way to *define* the optical depth is $\tau \approx \chi\,\Delta z \approx \Delta z/\ell_{\rm mfp}$. Inserting the random-walk constraint gives $\boxed{\tau \approx \tau_{\rm th}}$.

Distinction between optical depth & thermalization depth:

- Looking down into an atmosphere, we see photons that had their last "encounter" with matter at $\tau \approx 1$.
- If $\epsilon \approx 1$, then that last encounter was likely a local thermal emission.
- If $\epsilon \ll 1$, then that last encounter was likely a scattering. The thermal layer that *generated* the photon was typically much deeper, at $\tau \approx \tau_{\rm th}$.

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This is important in **hot stars,** with photospheres dominated by Thomson scattering of free electrons ($\epsilon \sim 10^{-4}$), so LTE is only recovered at $\tau \gtrsim 100$.

For cool, solar-type stars, photospheric opacity is dominated by H^- absorption, so $\epsilon \approx 1$ and the photosphere is $\sim$LTE.

SPECTRAL LINES

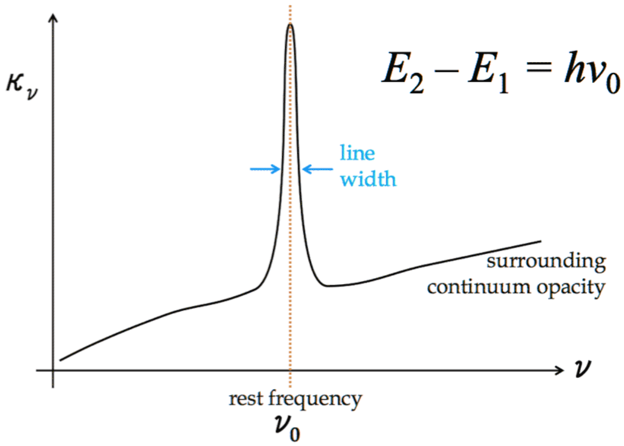
By far, lines are the most useful diagnostic of physical properties.

- Bound atoms produce much greater opacity than, say, free electrons.
- Discrete orbitals = **resonant oscillators** for photons in a narrow range of energies.
- Consider a given line as a transition between energy levels 1 (lower) & 2 (upper).

$$\kappa_\nu \, \rho \;=\; \chi_\nu \;=\; \chi_{\rm L} \, \phi(\nu)$$

line profile function:

$$\int d\nu \; \phi(\nu) \;=\; 1$$



$$E_2 - E_1 = h\nu_0$$

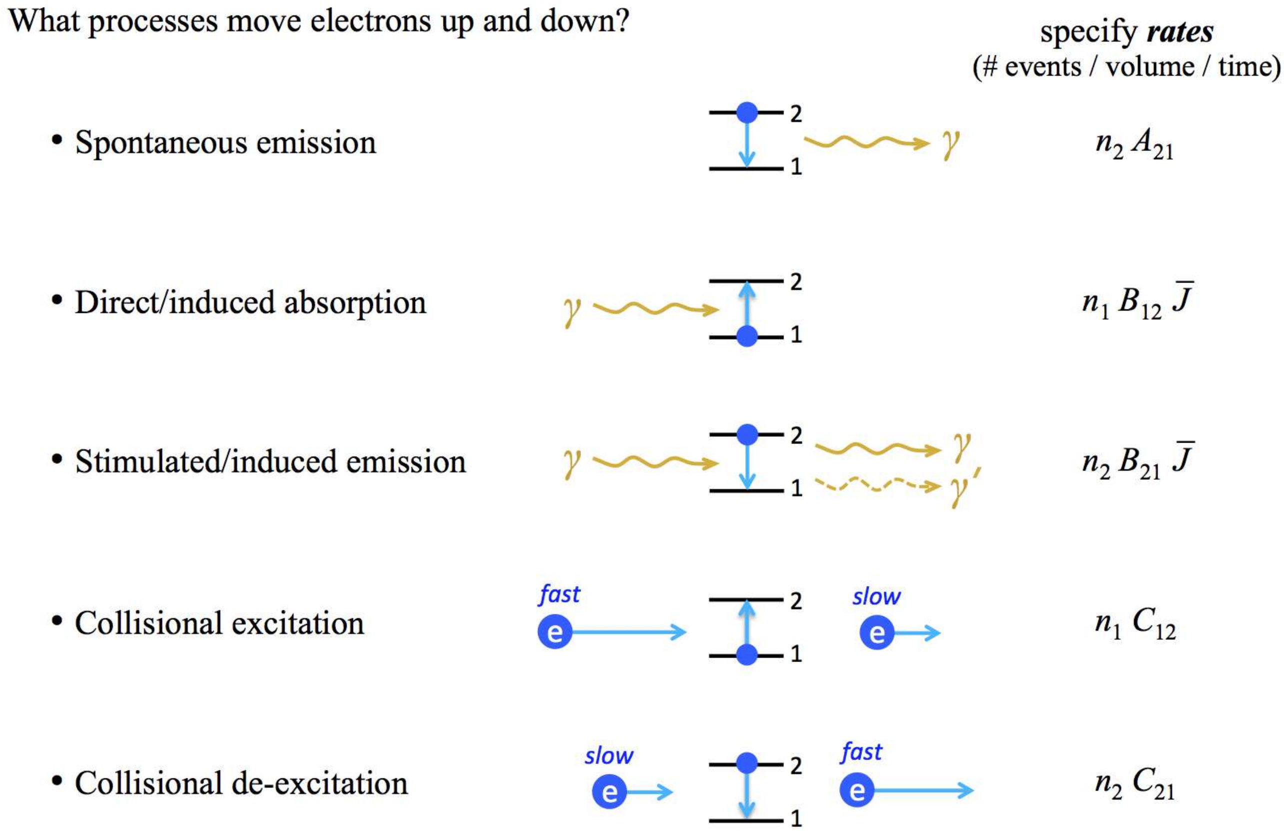
- Assume underlying continuum opacity is negligibly small.
- Assume all other transitions in the atom don't matter: effectively it's a ***two-level atom.***

We need to know more about:

- **Line formation:** How do we determine the "line strength" $\chi_{\rm L}$ for a specific transition between two bound e^- levels?
- **Line broadening:** How do we determine the shape of $\phi(\nu)$ for a given plasma environment?
- **Line radiative transfer:** How does that extra opacity affect the emergent I_ν from a stellar atmosphere?

We know that each unique *pair* of discrete energy levels in an atom corresponds to a given spectral line.

Consider lower level (1) and upper level (2).



A_{21}, B_{21}, and B_{12} are the Einstein coefficients for the three radiative processes.

The fundamental one is A_{21} (units of 1/s), and it's kind of like the inverse lifetime (or 'half-life') of the upper level.

The induced B rates are multiplied by the "local" radiation field, and they usually don't care about the direction; just **mean intensity.**

We define $$\bar{J} = \int d\nu\, \phi(\nu)\, J_\nu \qquad \text{(weighted by the line opacity)}\ .$$

Outgoing photons:

For A_{21}: the resulting radiation field I_ν is usually isotropic.

For B_{21}: the incoming photon is not absorbed. It's ejected along with its newly created "twin." That new outgoing photon has **same** energy, $\hat{\mathbf{n}}$, & polarization as incoming one. (Strictly, it's the incoming one that happens to have the same properties as the outgoing one that the atom "wants" to eject!)

The C_{12} and C_{21} collision rates are defined similarly to A_{21}. They depend on microscopic **cross sections** for electron impact excitation...

$$C_{12} = \frac{1}{\tau_{\rm coll}} \approx \langle n_e \sigma_{12} v \rangle = \int d^3\mathbf{v}\, f_e(\mathbf{v})\, \sigma_{12} v \; = \; n_e\, q_{12}(T_e)$$

and σ_{12} is measured (or computed) as a function of the kinetic energy in the collision reference frame ($\propto v^2$).

If we're in a time-steady equilibrium state (*not* necessarily STE or LTE), then the total number of 1→2 events should be balanced by the total number of 2→1 events. This is **statistical equilibrium:**

$$n_1(B_{12}\bar{J} + C_{12}) \; = \; n_2(A_{21} + B_{21}\bar{J} + C_{21})$$

In practice, when we know the rates (i.e., ρ & T) and the radiation field, then we can solve for the ratio of level populations n_2/n_1.

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In "thermalized" environments:

- If we are in STE or LTE, the gas and radiation field would be coupled together, and $\bar{J} = B_{\nu_0}(T)$. In LTE, we also know that

  $$n_i \sim g_i \exp\left[-\frac{E_i}{k_{\rm B}T}\right] \quad \Longrightarrow \quad \frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[-\frac{E_2 - E_1}{k_{\rm B}T}\right] = \frac{g_2}{g_1}\, e^{-h\nu_0/kT}$$

  where we must remember that n_i is the zeroth moment of the Maxwellian distribution of atoms in state i, with

  $$f \;\propto e^{-E_{\rm tot}/kT} \;, \qquad \text{and} \qquad E_{\rm tot} \;=\; \frac{1}{2}mv^2 \;+\; E_i \;\;,$$

  and g_i are the statistical weights (multiplicities) of each level i, i.e., the number of unresolved quantum sub-levels.

  The above is **Boltzmann's excitation formula.** In LTE, there is a Maxwellian distribution of energy levels. Higher-energy states are populated less frequently than lower ones.

- If the local electron $f(\mathbf{p})$ is Maxwellian, then collisions set up a Maxwellian-like distribution of *energy changes,* too. This implies a relationship between upward and downward collisions,

$$\frac{C_{12}}{C_{21}} = \frac{g_2}{g_1} e^{-h\nu_0/kT}$$

  In general, we have that $C_{21} > C_{12}$, which makes sense because it is easier for collisions to destabilize an upper state (make it decay) then it is for a collision to add just the right amount of energy to *boost* a bound electron up to the higher level.

  For $T \to 0$, $C_{12} <<< C_{21}$, since free electrons in such a cold environment would never hope to impart enough energy to excite the atom from 1 to 2.

  For $T \to \infty$, everything is in strong collisional balance, with $g_1 C_{12} = g_2 C_{21}$.

In LTE, both of the above conditions are true, and we see that

$$n_1 C_{12} = n_2 C_{21} \qquad \textit{and} \qquad n_1 B_{12} \bar{J} = n_2 (A_{21} + B_{21} \bar{J})$$

These 2 separate conditions are called **detailed balance.** In LTE, particles & photons obey *separate* balances (rate $\uparrow$ = rate $\downarrow$).

We can use this in the full detailed balance expression, along with Boltzmann's formula, to solve for

$$\bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) e^{h\nu_0/kT} - 1} \; .$$

This is the radiation field **needed** to maintain time-steady detailed balance while also in LTE. But if we're *really* in LTE, then we know

$$\bar{J} = B_{\nu_0}(T) = \frac{2h\nu_0^3/c^2}{e^{h\nu_0/kT} - 1} \qquad \text{at all temperatures,}$$

and thus the **Einstein relations** must hold:

$$\boxed{\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}} \qquad\qquad \boxed{A_{21} = \left(\frac{2h\nu_0^3}{c^2}\right) B_{21}} \; .$$

These relations do not depend on T or any other property of the environment; only on atomic physics. The Einstein relations are **valid even not in LTE!**

This can be a tough step to accept. If you want to see it proven rigorously—using quantum probabilities & Fermi's golden rule—see Hubeny & Mihalas (2015), *Theory of Stellar Atmospheres,* § 5.3–5.4.

Once we know the rates, we can compute the other things we need to solve the equation of radiative transfer...

Line formation: We want to compute both the opacity & emissivity due to a spectral line:

$$\chi_\nu = \chi_{\rm L}\,\phi(\nu) \qquad \phi(\nu) = \text{absorption profile}$$
$$j_\nu = j_{\rm L}\,\psi(\nu) \qquad \psi(\nu) = \text{emission profile}$$

Often, $\phi = \psi$, and these have units of 1/frequency. Thus, they appear on the RHS of the radiative transfer equation with the following units:

$$\frac{\partial I_\nu}{\partial z} = (j_{\rm L} - \chi_{\rm L} I_\nu)\phi \quad \rightsquigarrow \quad j_{\rm L} = \frac{dE}{dt\; d\Omega\; (dA\; dz)} \ .$$

But we already know the rate of creation of new photons (in this line):

$$n_2 A_{21} = \frac{\#}{dt\; dV}$$

and we can take this back to the units of $j_{\rm L}$ by noticing that $dE = \# h\nu_0$. Also, since the photons are emitted in all directions, we can also write $d\Omega = 4\pi$, and this gives

$$j_{\rm L} = \frac{h\nu_0}{4\pi}\,(n_2 A_{21}) \ .$$

Similarly, **opacity** involves processes that absorb incoming photons. Thus,

$$\chi_{\rm L} = \frac{h\nu_0}{4\pi}\,(n_1 B_{12} - n_2 B_{21})$$

Note that stimulated emission is treated as "negative absorption." It's counted in with opacity because it's $\propto$ the radiation field, just like direct absorption. (However, it's a *net* creator of photons.)

Thus, the line's **source function** is given by

$$S_\nu = \frac{j_\nu}{\chi_\nu} = \frac{j_{\rm L}}{\chi_{\rm L}} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

(and note that this is the value for the whole line).

Because it's based on "bookkeeping" (processes that generate photons & processes that destroy them), the above is always true.

In non-LTE, one can use all of the above definitions to write

$$S_\nu = (1 - a_\nu) B_\nu + a_\nu \bar{J} \qquad a_\nu \equiv \frac{A_{21}}{A_{21} + C_{21}(1 - e^{-h\nu_0/kT})}$$

and now we've *defined* the albedo a_ν (and thus the photon destruction probability $\epsilon_\nu = 1 - a_\nu$) in terms of the properties of each spectral line.

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In non-LTE, one often sees the level populations written using Donald Menzel's **departure coefficients,**

$$b_i = \frac{n_i\,(\text{actual})}{n_i\,(\text{LTE})}$$

and one can show that for a two-level atom, $$S_\nu = \frac{2h\nu_0^3/c^2}{(b_1/b_2)e^{h\nu_0/kT} - 1}$$

which means that $S_\nu = B_\nu$ for $b_1 = b_2 = 1$.

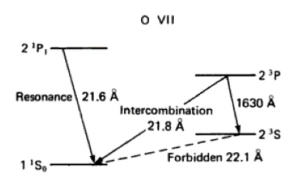
For strong resonance lines in non-LTE upper atmospheres, we often see $(b_1/b_2) \gg 1$. Some implications are:

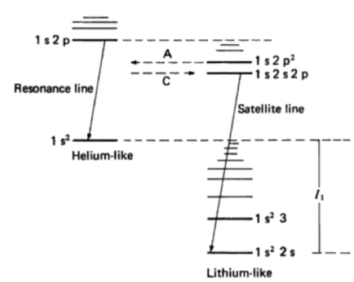
- Because photons escape so easily "at the top," we have $S_\nu < B_\nu$.
- n_1 is overpopulated and n_2 is underpopulated, because photons tend to get "lost" when they're more excited, so the upper levels get drained.

Which rates dominate?

We'll see lots of different relative values of… $\boldsymbol{A_{21}}$ vs. $\boldsymbol{B_{21} J}$ vs. $\boldsymbol{C_{21}}$

Allowed transitions:	$\gamma \rightarrow$ E dipole moment	strong ($A_{21} \sim 10^6 - 10^{10}\,\mathrm{s}^{-1}$)
Forbidden transitions:	$\gamma \rightarrow$ E quadrupole moment B dipole moment	weak ($A_{21} \sim 1 - 10^2\,\mathrm{s}^{-1}$)
Resonance lines:	1 = ground, 2 = excited (same multiplicity)	
Subordinate lines:	1 & 2 both excited	
Intercombination lines:	1 & 2 different multiplicities	
Satellite lines:	recombining transition (from X^{n+1} to X^n) associated with a resonance line of X^{n+1}	





Notes on Spectral Line Radiative Transfer

1. Simplified Line Formation

A simple, but instructive, way to think about the formation of spectral lines is the **Schuster reversing layer model** (Schuster 1905, *ApJ,* 21, 1; see also Mihalas 1999, *ApJ,* 525C, 25).

The spectral line (i.e., a particular transition from one bound electron energy level in an element to another energy level) is assumed to be formed in a "cloud" of gas sitting "above" the main source of radiation. That main part—which has traditionally been thought of as a dense stellar photosphere, but can be any other background radiation source—is assumed to produce a broad spectral continuum.

We also assume that the properties of the line-absorbing (or line-emitting) gas are constant throughout the cloud. If the thickness of the cloud is given by Δz, then the dimensionless optical depth in the cloud can be written

$$\tau_\nu \,=\, \kappa_\nu \rho \Delta z \,=\, \kappa_{\rm L} \phi(\nu) \rho \Delta z \ , \tag{1}$$

where ν is the photon frequency, κ_ν is the absorption coefficient in units of cross section per unit mass (e.g., cm^2 g^{-1}), and ρ is the mass density. For convenience, we write the absorption coefficient as $\kappa_{\rm L}\phi(\nu)$, which separates the total line opacity $\kappa_{\rm L}$ from the line broadening function $\phi(\nu)$.

The solution for the specific intensity I_ν that emerges from the Schuster cloud is given by

$$I_\nu \,=\, I_{0\nu} e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) \ , \tag{2}$$

where $I_{0\nu}$ is the intensity of the continuum that enters the cloud from below and S_ν is the so-called source function of the cloud; it takes account of the local properties of the cloud that would make it "glow" on its own, even in the absence of an external source of radiation.

We can continue to make simplifying approximations about these two intensity parameters, and in many cases it is useful to to think about them as blackbody Planck spectra. Thus, let us use the shorthand

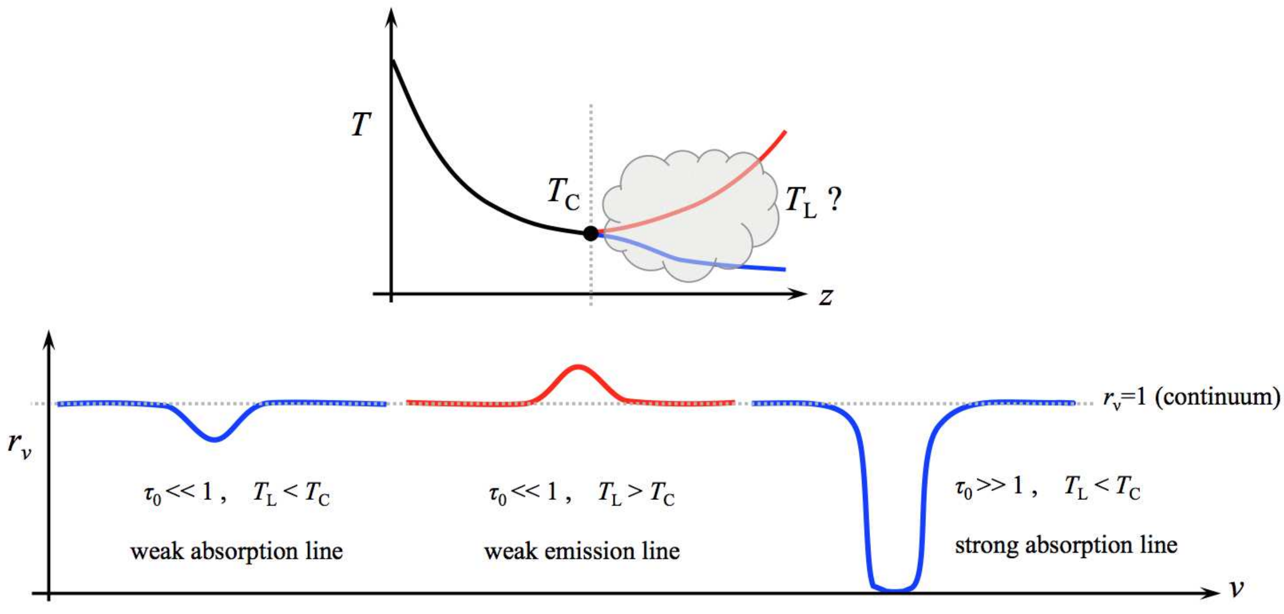
$$I_{0\nu} \approx B_{\rm C}(T_{\rm C}) \ , \quad S_\nu \approx B_{\rm L}(T_{\rm L}) \ , \tag{3}$$

where the subscript "C" denotes the conditions in the background medium where the continuum is formed, and "L" denotes the conditions in the Schuster cloud where the line is formed. In LTE, $B \propto T^4$, so we can rewrite equation (2) in terms of the dimensionless **residual intensity** r_ν as

$$\boxed{r_\nu \,=\, \frac{I_\nu}{I_{0\nu}} \,=\, e^{-\tau_\nu} + \left(\frac{T_{\rm L}}{T_{\rm C}}\right)^4 (1 - e^{-\tau_\nu}) \ .} \tag{4}$$

One can see that if the temperature in the cloud is lower than the temperature in the underlying continuum photosphere (i.e., $T_{\rm L}/T_{\rm C} < 1$) that $r_\nu < 1$ and we will have an absorption line. Conversely, if the cloud is hotter than the underlying continuum region, we will have an emission line with $r_\nu > 1$. Weak, optically thin lines, with $\tau_\nu \ll 1$ for all frequencies, have a residual intensity that simplifies to

$$r_\nu \,\approx\, 1 + \left[\left(\frac{T_{\rm L}}{T_{\rm C}}\right)^4 - 1 \right] \tau_\nu \ . \tag{5}$$



In the cores of strong lines, where $\tau_\nu \gg 1$, the line is saturated, with

$$r_\nu \approx \left(\frac{T_{\rm L}}{T_{\rm C}}\right)^4 , \tag{6}$$

and cold absorbing layers ($T_{\rm L}/T_{\rm C} \ll 1$) produce saturated absorption lines with nearly "black" cores.

If we look "above the limb" and our line-of-sight passes *only* through the cloud (and not through the underlying photosphere), we have to go back and not divide by $I_{0\nu}$ (since it's zero),

$$\text{With } I_{0\nu} = 0, \quad I_\nu = B_{\rm L}\left(1 - e^{-\tau_\nu}\right) \approx \tau_\nu B_{\rm L} \quad \text{(for weak lines with } \tau_0 \ll 1)$$

and thus there will be an *emission line* on top of "nothing" (instead of on top of a continuum with $r_\nu = 1$)

We characterize a line's total amount of absorption or emission by integrating over frequency. Thus, we define the **equivalent width**

$$W_\nu = \int d\nu \, (1 - r_\nu) \ . \tag{7}$$

Note that in this case the subscript ν does not mean that W_ν is a function of ν, but it is there to convey that this quantity has units of frequency (as opposed to the wavelength version of equivalent width, W_λ). For the standard case of a cold cloud that produces an absorption line, we see that as the number of atoms in the line-forming cloud increases, the line gets deeper, and the (positive definite) equivalent width grows in magnitude. The dependence of W_ν on the total line opacity (measured by some combination of $\kappa_{\rm L}$ and ρ) is known as the **curve of growth,** and its shape allows us to diagnose many useful details about the line-forming region.

. . . .

2. Line Broadening Processes

The shape of the curve of growth depends crucially on the line broadening function $\phi(\nu)$. The simplest possible picture of atomic transitions assumes that the two energy levels—let us label the lower level 1 and the upper level 2—have precisely known energies E_1 and E_2. Thus, the transition occurs only when photons of a single frequency ν_{12} are involved, with

$$E_2 - E_1 \;=\; h\nu_{12} \tag{8}$$

where h is Planck's constant. We know this is an idealization of the true quantum mechanical processes going on in bound atomic electron clouds, but for now let us assume it to be the case. In this case, for observers sitting in the rest frame of the atom that is emitting or absorbing a line photon, there is no line broadening at all. This observer will see only line photons at a single frequency that we will call $\nu_{\rm obs}$. The profile function is a Dirac delta function,

$$\phi(\nu_{\rm obs}) \;=\; \delta(\nu_{\rm obs} - \nu_{12}) \;. \tag{9}$$

that is equal to zero for all $\nu_{\rm obs} \neq \nu_{12}$. Note that, by convention, $\phi(\nu)$ has units of 1/frequency, or Hz^{-1}.

However, in a cloud of gas with a finite temperature, the individual atoms are all moving around with random velocities in 3D space. In kinetic theory we describe the microscopic motions of the atoms with a distribution function f written as a function of the particle vector momentum $\mathbf{p}$. In most "well-behaved" systems this function is a Maxwell-Boltzmann distribution,

$$f(\mathbf{p}) \;=\; \frac{n}{(2\pi mkT)^{3/2}} \exp\left(-\frac{p^2}{2mkT}\right) \tag{10}$$

where m is the particle mass, $n = \rho/m$ is the number density, k is Boltzmann's constant, T is the temperature, and for non-relativistic speeds, $p^2 = m^2(v_x^2 + v_y^2 + v_z^2)$. If we assume that this cloud of randomly moving atoms is being viewed by a stationary observer, along a line-of-sight that is parallel to, say, the x–axis, then whenever $v_x \neq 0$, each atom that emits a photon with frequency ν_{12} in its local rest frame will give rise to a photon that is slightly Doppler shifted in the observer's frame. This Doppler effect produces a shifted line profile function,

$$\phi(\nu_{\rm obs}) \;=\; \delta\left(\nu_{\rm obs} - \nu_{12} + \frac{\nu_{12} v_x}{c}\right) \tag{11}$$

where we define $v_x > 0$ as motion away from the observer and $v_x < 0$ as motion toward the observer.

Equation (11) is valid for a single atom that has a projected velocity of v_x in the observer's line-of-sight direction. However, we would like to average over the entire velocity distribution to obtain a **mean line profile function** for the absorbing medium. Thus, we define

$$\langle \phi(\nu_{\rm obs}) \rangle \;=\; \frac{\int d^3\mathbf{p}\, f(\mathbf{p})\, \phi(\nu_{\rm obs})}{\int d^3\mathbf{p}\, f(\mathbf{p})} \;. \tag{12}$$

For the Maxwellian distribution, the denominator in the above integral is just the number density n. The numerator is slightly more complicated to evaluate, but one can separate the exponential in $f(\mathbf{p})$ into multiplicative terms that are functions of only v_x, v_y, and v_z, so the integrals in all three dimensions of "momentum space" can be done separately. The v_y and v_z integrals can be reduced to standard definite integrals over a Gaussian function that can be looked up. The v_x integral involves the Dirac delta function, whose argument must be transformed from frequency units to momentum units. The details can be found in several radiative transfer textbooks, and the result is

$$\langle \phi(\nu_{\rm obs}) \rangle \;=\; \frac{1}{\Delta\nu_{\rm D}\sqrt{\pi}}\, e^{-x^2} \tag{13}$$

where we define the dimensionless line frequency x and the Doppler width $\Delta\nu_{\rm D}$ as

$$x = \frac{\nu_{\rm obs} - \nu_{12}}{\Delta\nu_{\rm D}} \;\;, \quad \Delta\nu_{\rm D} = \frac{\nu_{12}}{c}\sqrt{\frac{2kT}{m}} \;. \tag{14}$$

Equation (13) gives the line profile function for pure **Doppler broadening.**

Above, it was mentioned that the Dirac delta function form of the atomic rest-frame profile function was just an idealization. In reality, there are several effects that can make the atomic energy levels "fuzzy." First, there is just the Heisenberg uncertainty principle, which limits our knowledge of the energy of a state that has a finite lifetime. Recall that one form of the uncertainty principle is

$$\Delta E\,\Delta t \;\sim\; h \tag{15}$$

where for a photon, an uncertainty in its energy means that there is an uncertainty in its frequency, $\Delta E = h\Delta\nu$. If the excited state only lives for a finite lifetime $\Delta t = 1/\gamma$, then we know that we cannot know the photon's energy (i.e., frequency) to better than an uncertainty given by $\Delta\nu \approx \gamma$. One can show that the profile function in the atom's rest frame is then "fuzzed out" in frequency, and $\phi(\nu)$ must be interpreted as a probability distribution of finding the photon at any specific frequency.

The uncertainty principle provides a finite width to the profile, and quantum mechanical calculations show that it becomes a **naturally broadened** Lorentzian function,

$$\phi(\nu_{\rm obs}) \;=\; \frac{\gamma/8\pi^3}{(\nu_{\rm obs} - \nu_{12} + \nu_{12}v_x/c)^2 + (\gamma/4\pi)^2} \tag{16}$$

which reduces to equation (11) in the limit of $\gamma \to 0$.

For the simple two-level atom discussed above, $\gamma \approx A_{21}$. In most astrophysical cases, *pure* natural broadening produces a line width **too narrow to worry about** (i.e., $\gamma \ll \Delta\nu_{\rm D}$).

However, it should also be noted that there are several other physical processes, besides the natural broadening discussed above, that also can give rise to a Lorentzian line profile. One of these is **collisional broadening,** which takes account of the fact that the gas is not just filled with that one atomic species that is doing the emitting and absorbing. A higher total pressure in the system (especially from free electrons that zip around with high speeds) produces a higher rate of collisions between the emitting atoms and their neighbors. This can enhance the effective value of γ by several orders of magnitude above the natural broadening level.

For whatever non-zero value of γ is present in the atom's rest frame, we want to know how it combines with the Maxwell-Boltzmann Doppler broadening discussed above. We must insert the single-atom profile function given in equation (16) into the mean-profile integral of equation (12). Unfortunately, there is no simple closed-form solution to that integral, but astronomers have defined a new two-parameter special function, known as the **Voigt function,** that encapsulates the difficult integral. The profile is given by

$$\langle \phi(\nu_{\rm obs}) \rangle \;=\; \frac{\tilde{H}(a,x)}{\Delta\nu_{\rm D}} \tag{17}$$

and the normalized version of the Voigt function is defined as

$$\tilde{H}(a,x) \;=\; \frac{a}{\pi^{3/2}} \int_{-\infty}^{+\infty} \frac{dy\, e^{-y^2}}{(x-y)^2 + a^2} \;. \tag{18}$$

Algorithms for computing the Voigt profile have been described by, Armstrong (1967, *JQSRT,* 7, 61); Letchworth & Benner (2007, *JQSRT,* 107, 173), and others. Above, x and $\Delta\nu_{\rm D}$ have the same meanings they did for Doppler broadening, and the dimensionless "damping constant" $a = \gamma/(4\pi\Delta\nu_{\rm D})$ is a ratio that conveys the relative importance of the two sources of line broadening. When $a \ll 1$, the profile is mostly Doppler broadened and Gaussian. When $a \gg 1$, the profile is mostly Lorentzian. Note that $\tilde{H}(0,x) = e^{-x^2}/\sqrt{\pi}$, so that equation (17) agrees with equation (13) in the limit of $a \to 0$.

Figure 1 (top) shows several Voigt profiles for a set of example values of a, and compares them to the Gaussian function limit of $a = 0$. These plots show the dimensionless profiles given by $\Delta\nu_{\rm D}\langle\phi\rangle$. Note that these profiles are defined such that

$$\int d\nu \, \langle \phi \rangle \;=\; 1 \;. \tag{19}$$

Also, the middle panel of Figure 1 shows a set of residual line profiles r_ν as a function of x, all having Voigt profiles with $a = 0.1$ and computed for a range of optical depth normalization parameters τ_0 (see below).

. . . .

3. Evaluating the Curve of Growth

Finally, we want to evaluate the shape of the curve of growth W_ν as a function of total line opacity. Let us redefine the total opacity by writing the optical depth as

$$\tau_\nu \;=\; (\kappa_{\rm L}\, \rho\, \Delta z)\, \phi(\nu) \;=\; \left(\tau_0\, \Delta\nu_{\rm D}\, \sqrt{\pi}\right) \phi(\nu) \tag{20}$$

where τ_0 is a properly dimensionless optical depth that characterizes the overall line strength, and we have simplified the notation by writing $\langle\phi(\nu_{\rm obs})\rangle$ as $\phi(\nu)$.

Let us evaluate the equivalent width W_ν using equations (5) and (7). In general,

$$W_\nu \;=\; W_0 \int d\nu \, \left(1 - e^{-\tau_\nu}\right) \tag{21}$$

where we also define

$$W_0 = 1 - \left(\frac{T_{\rm L}}{T_{\rm C}}\right)^4 \tag{22}$$

to scale out the environmental effects of the layer and background. In the limit of a very weak line ($\tau_\nu \ll 1$), the integral becomes

$$W_\nu \;=\; (\tau_0 \Delta\nu_{\rm D} \sqrt{\pi})\, W_0 \int d\nu\, \phi(\nu) \;. \tag{23}$$

As was noted above, the profile functions are defined to be normalized over frequency, so the integral of $\phi(\nu)$ over $d\nu$ is equal to 1. Thus, weak lines exhibit $W_\nu \propto \tau_0$ and are said to be in the **linear part** of the curve of growth.

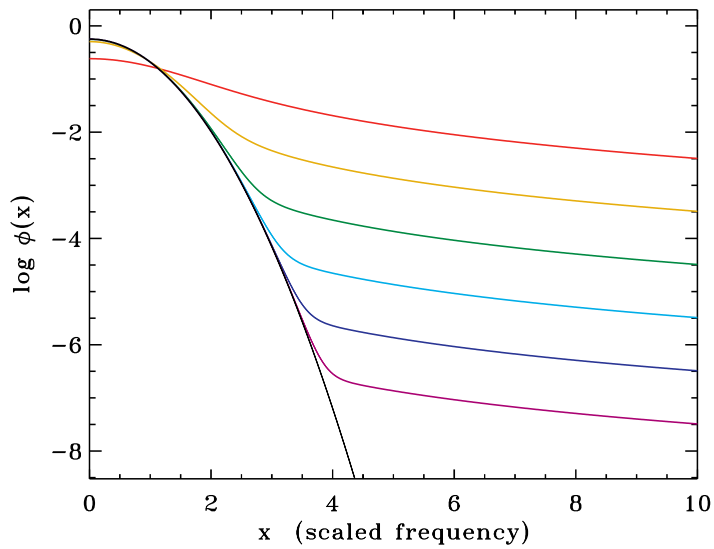
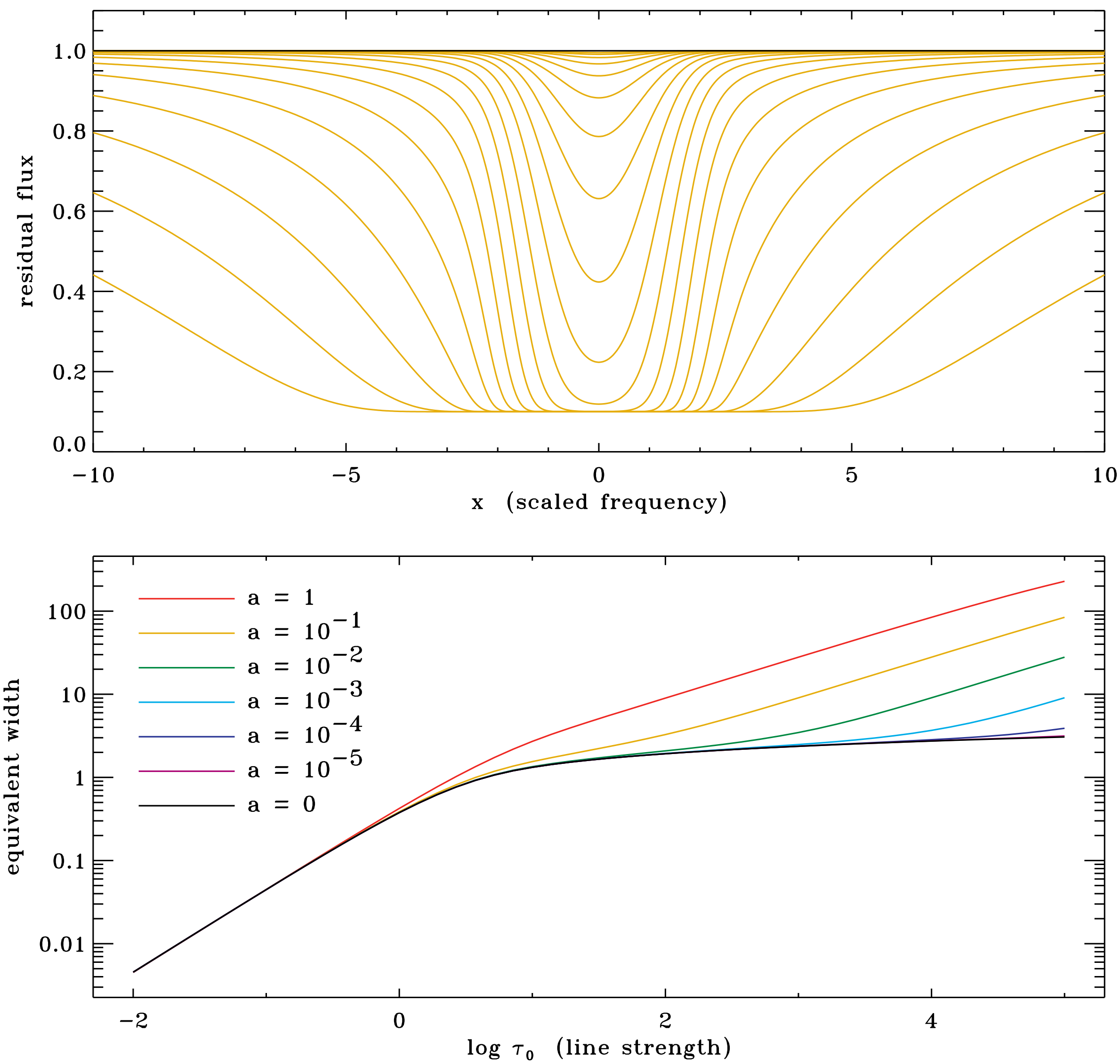


Figure 1:

(TOP) Dimensionless Voigt profile function $\phi(x)$ shown for a range of Lorentzian wing parameters a (see key in bottom panel).

(MIDDLE) Example absorption profiles for Schuster reversing layer model, with $a = 0.1$, $B_{\rm L}/B_{\rm C} = 0.1$, and a range of optical depth parameters τ_0.

(BOTTOM) Curve of growth integrated over profiles shown in middle panel for the wing parameters given in the top panel.



For a line that is reasonably strong (i.e., optically thick) in its Doppler core, but still not showing much absorption in its Voigt wings, we can use equation (13) for the profile function and write the relevant integral as

$$\int d\nu (1 - e^{-\tau_\nu}) \;=\; \Delta\nu_{\rm D} \int dx \, \left[1 - \exp\left(-\tau_0 e^{-x^2} \right) \right] \; . \tag{24}$$

This expression is difficult to evaluate, but when $\tau_0 \gg 1$ the core of the line is sufficiently saturated that one can approximate the integrand as

$$1 - \exp(-\tau_0 e^{-x^2}) \;\approx\; \begin{cases} 1 \, , & |x| \leq x_{\rm half} \; , \\ 0 \, , & |x| > x_{\rm half} \end{cases} \tag{25}$$

where

$$x_{\rm half} = \sqrt{\ln \left(\frac{\tau_0}{\ln 2} \right)} \tag{26}$$

is the dimensionless frequency at which the above function is equal to 0.5. This value effectively defines the boundary between the saturated core and the exponentially damped wings. With this approximation, it becomes much easier to evaluate the integral, and

$$W_\nu \;\approx\; 2 W_0 \, \Delta\nu_{\rm D} \; x_{\rm half} \tag{27}$$

and, since τ_0 is assumed to be much larger than $\ln 2$ in this limiting case, we have $W_\nu \propto \sqrt{\ln \tau_0}$, which is a slowly varying function that is often called the **flat, or saturated part** of the curve of growth.

As τ_0 grows even larger, the optical depth becomes large even in the Voigt wings. Far from the line core, the Voigt function can be approximated as $\phi(\nu) \approx a/x^2$. When that is plugged into the definitions of τ_ν and W_ν, the curve of growth scaling can be shown to become $W_\nu \propto \sqrt{\tau_0}$, which increases more slowly than linearly, but faster than in the flat part. This is called the **square-root part, or damping part** of the curve of growth.

The bottom panel of Figure 1 shows the curve of growth for the same set of Voigt parameters a as were shown in the top panel. For $a \ll 1$ there is a definite flattening with increasing τ_0, followed by the onset of the square-root behavior due to the Voigt wings. For $a \gtrsim 1$, however, the curve transitions directly from the weak linear behavior to the strong Voigt-wing square-root behavior.

. . . .

Historically, the curve of growth was very useful in analyzing stellar atmospheres. Comparing empirical to theoretical curves yields constraints on the T structure of the atmosphere and elemental abundances. Curves of growth were used to determine the difference between Population I and II stars.

In real life, nobody really does curve of growth analysis any more. It's more straightforward to do full-on **spectrum synthesis** and compare it to observed spectra.

Final topics of the course:

1. **Ionization & recombination in astrophysics**

   Examples: limiting cases of Saha, coronal, & nebular equilibrium
   H II regions ("Stromgren spheres")

2. **Irradiated (planetary) atmospheres**

   Examples: mean temperatures of Earth & Venus
   sublimation of ices from comets

3. **Energy & momentum exchange between light & matter**

   Examples: radiation pressure & the Eddington limit
   forces on dust grains in the solar system.

Ionization & Recombination in Astrophysics

There are many environments for which we don't really know *anything* until we know how the elements are distributed among their ionization states.

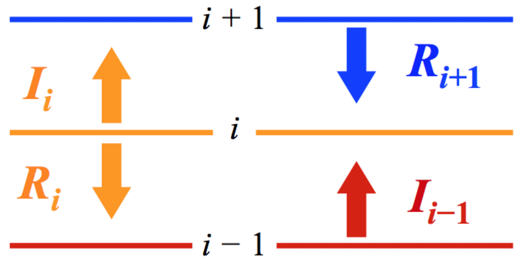
Notation: H I = H^0, H II = H^{+1}, . . . , Fe XIV = Fe^{+13}, . . .
but the chemical notation (X^{+n}) refers to ions themselves, while the Roman-numeral notation refers to their levels & lines.

If we're writing the **mass conservation equation** for a given stage i, then ionizations/recombinations act as source/sink terms:

$$\begin{aligned}\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) &= 0 \quad ? \\ &= n_{i-1} I_{i-1} - n_i I_i - n_i R_i + n_{i+1} R_{i+1}\end{aligned}$$

$$\left\{ \begin{array}{c} I_i = \text{ionization rate} \\ R_i = \text{recombination rate} \end{array} \right\} \text{out of stage } i \text{, into stage} \left\{ \begin{array}{c} i+1 \\ i-1 \end{array} \right\} \quad \text{(units: 1/s)}$$

For element with atomic number Z, the index i goes from 0 (neutral) to Z (fully stripped). There are $Z+1$ coupled mass conservation equations.



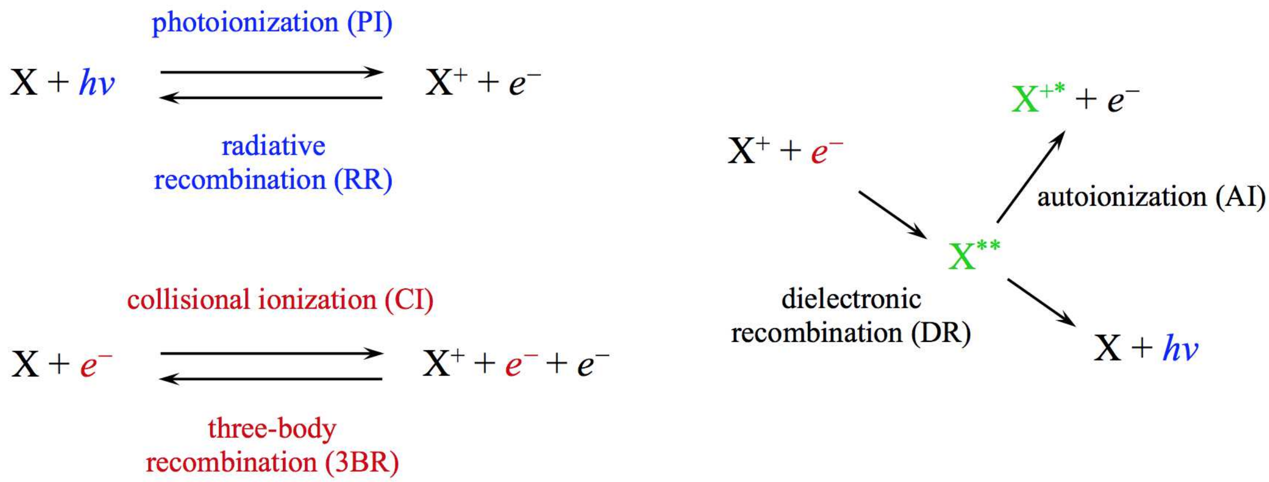
Usually, $R_0 = 0$ and $I_Z = 0$ (unless there are weird states like H^-).

For time-steady static equilibrium,

$$\frac{\partial n_i}{\partial t} = 0 \quad \text{and} \quad \mathbf{u}_i = 0 \ , \qquad \text{so we can show that} \quad \boxed{\frac{n_{i+1}}{n_i} = \frac{I_i}{R_{i+1}}}$$

What processes contribute to ionization & recombination?

$$\begin{aligned} I_i &= CI_i + PI_i + AI_i \\ R_i &= 3BR_i + RR_i + DR_i \end{aligned}$$



- PI and CI are kind of like the B_{12} and C_{12} types of excitation, but the energy of the "bullet" now exceeds the ionization potential.
- RR is similar to A_{21} emission, but it's not as "spontaneous." It depends on there being free electrons around, so the rate is $\propto n_e$.
- 3BR is similar to C_{21} de-excitation, but its rate is $\propto n_e^2$. Only important in high-density (LTE?) regions.
- DR is a two-stage process: (1) Free electron is captured into an excited state, and some leftover energy excites *another* of the bound electrons. Atom is in a doubly-excited state. (2) Then the excited state decays, emitting a photon.
- AI shares the first stage of DR, but instead of decaying, the highly/doubly-excited state may spontaneously kick out the newly bound electron.

Processes that depend on the presence of electrons are often written, e.g.,

$$CI_i + AI_i \;=\; n_e\, q_i(T) \qquad\qquad RR_i + DR_i \;=\; n_e\, \alpha_{i-1}(T)$$

where q and α (in cm^3/s) are the "rates" you may often find listed in books and papers.

(Why α_{i-1}? Convention... astronomers sometimes enumerate the recombination rate by the stage it's going *into.*)

In many astrophysical environments, only a subset of the terms dominate the total ionization/recombination balance:

	High ρ / LTE ("Saha")		Medium ρ ("coronal")	Low ρ ("nebular")
Dominant terms in I_i	PI	CI	$CI + AI$	PI
Dominant terms in R_i	RR	$3BR$	$RR + DR$	$RR + DR$

In **LTE,** there's a similar **detailed balance** as with lines: particle collisions balance themselves, and so do photon absorptions & emissions.

Doing the bookkeeping for *either* particles or photons (i.e., $CI = 3BR$, or $PI = RR$, as long as $S_\nu = J_\nu = B_\nu$) gives the **Saha** ionization-balance equation. The actual rates cancel out...

$$\frac{n_{i+1}}{n_i} = \frac{CI_i}{3BR_{i+1}} = \frac{2}{n_e\,\lambda_e^3}\frac{U_{i+1}}{U_i}\exp\left(-\frac{E_{i+1}-E_i}{k_{\rm B}T}\right)$$

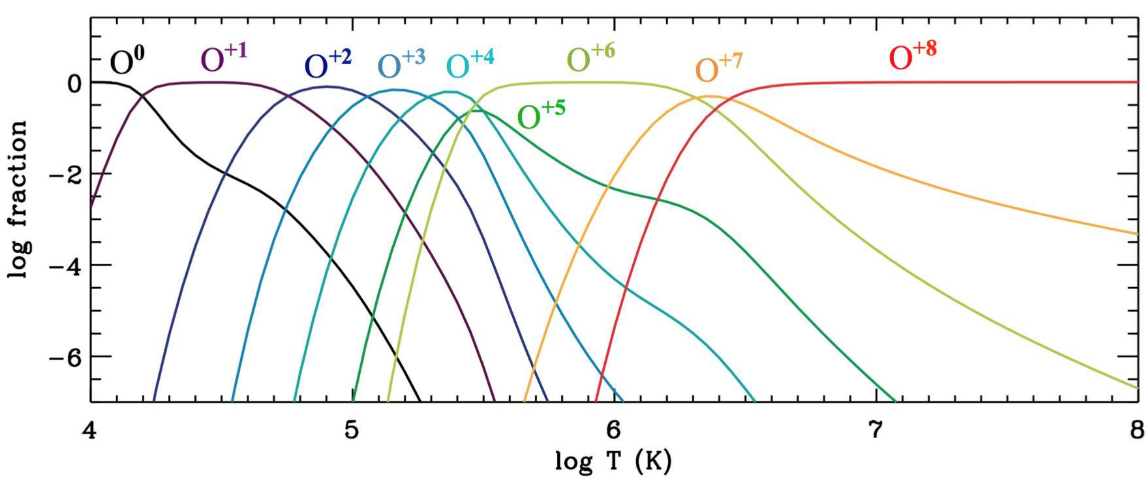
where $\lambda_e = h/\sqrt{2\pi m_e k_{\rm B} T}$ (thermal de Broglie wavelength of a free electron), E_i is ground-state energy for stage i, and U_i is the *partition function* of stage i (essentially a weighted mean of the bound g's).

. . . .

In hot ($T \gg 10^4$ K) **coronal plasmas,** density is too low for LTE, and there's usually *not* a strong source of ionizing photons (so ignore PI). Thus,

$$\frac{n_{i+1}}{n_i} = \frac{CI_i + AI_i}{RR_{i+1} + DR_{i+1}} = \frac{q_i(T)}{\alpha_i(T)} = \text{a function of } T \text{ only!}$$

and the whole set of absolute ratios $\{n_i/n_{\rm element}\}$ can be built up by knowing all of the pairwise ratios. For example, oxygen:

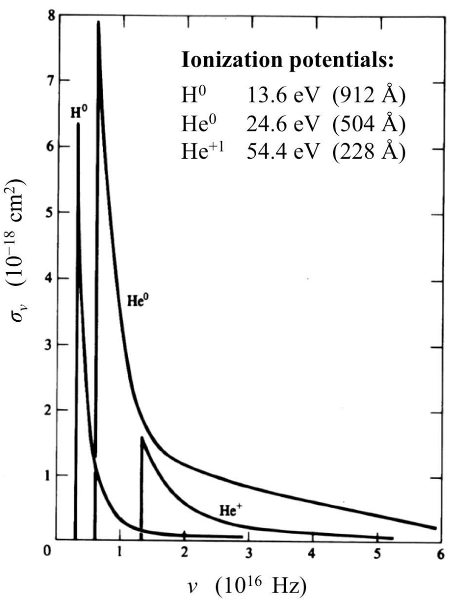


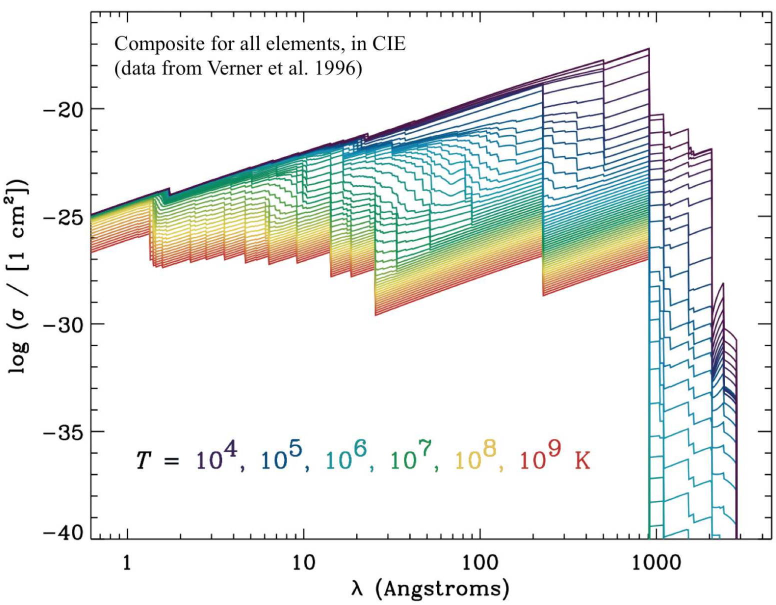
In very-low density regions (e.g., **nebulae** in the ISM) collisions are infrequent, and it's also far from LTE. PI is the main source of ionizations, and it's driven by any nearby source of "ionizing photons:"

$$PI_i \;=\; \int d\nu \, \frac{4\pi J_\nu}{h\nu} \, \sigma_\nu \qquad\qquad \text{(verify units: `events/s')}$$

which depends on the photoionization **cross-section** σ_ν for the ion in question. Like its terrestrial cousin (the photoelectric effect), there's an on/off threshold:

$$\text{Typically,} \qquad \sigma_\nu \approx \begin{cases} 0 \ , & h\nu < (E_{i+1} - E_i) \\ 1/\nu^3 \ , & h\nu \geq (E_{i+1} - E_i) \end{cases}$$





The (mostly neutral) ISM absorbs *a lot* in the EUV ($\lambda \approx 100$–900 Å).

The other part of PI_i ($4\pi J_\nu / h\nu$, integrated over ν) is the number of incident photons per unit time, per unit area.

Consider diffuse ISM hydrogen illuminated by a hot star.

$$\textit{Notation:} \qquad n_{\rm H} = \frac{\rho}{m_{\rm H}} = n_n + n_p \quad \text{(neutrals + protons)} \qquad\qquad n_p = n_e$$

In nebular ionization equilibrium,

$$\frac{n_p}{n_n} = \frac{PI_n}{n_e \, \alpha_n(T)} \qquad \rightsquigarrow\rightsquigarrow \qquad n_n \int d\nu \, \frac{4\pi J_\nu}{h\nu} \, \sigma_\nu \;=\; n_e^2 \, \alpha_n$$

If the region around the central star was a vacuum, then $J_\nu \approx H_\nu$, and

$$4\pi J_\nu \;=\; F_\nu \;=\; \frac{L_\nu}{4\pi r^2}$$

but the neutral gas absorbs photons! Thus, really,

$$4\pi J_\nu \;=\; \frac{L_\nu}{4\pi r^2}\, e^{-\tau_\nu(r)} \qquad \text{where} \qquad \tau_\nu(r) \;=\; \int_{R_*}^{r} dr'\; n_n\, \sigma_\nu$$

Close to the star, $\tau_\nu \ll 1$, and much of the hydrogen will be ionized. But once we reach a given distance away from the star (call it R_0), *ALL* of the star's ionizing photons will have gotten absorbed (with $\tau_\nu \gg 1$), and the gas will be neutral for all $r > R_0$. The ionized bubble of radius R_0 is called an H II region, or a **Strömgren sphere.**

The ionization balance can be written:

$$\int d\nu\; \frac{L_\nu}{h\nu}\; e^{-\tau_\nu(r)}\; n_n\, \sigma_\nu \;=\; 4\pi r^2\; n_e^2\, \alpha_n \;\; .$$

To derive how big the sphere will be, integrate over r (out to R_0):

$$\int_{R_*}^{R_0} dr \int d\nu\; n_n\, \sigma_\nu \frac{L_\nu}{h\nu}\; e^{-\tau_\nu} \;=\; \int_{R_*}^{R_0} dr\; 4\pi r^2\; n_e^2\, \alpha_n$$

- LHS: Note that $d\tau_\nu = n_n \sigma_\nu dr$, and we can change integration variables.
- RHS: Assume n_e & T are constant in the sphere, and that $R_0 \gg R_*$.

$$\underbrace{\int_0^\infty d\tau_\nu\; e^{-\tau_\nu}}_{=1}\; \underbrace{\int_{\nu_0}^\infty d\nu\; \frac{L_\nu}{h\nu}}_{\equiv Q_*} \;\approx\; \left(\frac{4\pi}{3} R_0^3\right) n_e^2\, \alpha_n$$

Taking note that the integral over $d\nu$ is zero for frequencies below the ionization threshold ν_0, the quantity Q_* is the star's "ionizing photon luminosity" (in units of photons s^{-1}).

$$\text{Thus,} \qquad R_0 \;=\; \left(\frac{3Q_*}{4\pi\, \alpha_n\, n_e^2}\right)^{1/3}$$

which, for typical parameters of OB stars & the surrounding ISM, gives $R_0 \approx 50$ pc, close to what is observed!

Irradiated (Planetary) Atmospheres

The Eddington/gray atmosphere had 2 boundary conditions, though we didn't tend to think about them as such:

- From beneath: $F = \sigma T_{\rm eff}^4 = \text{constant} > 0$.
- From above: $I(\mu < 0) = I_{\rm down} = 0$ (essentially $J = 2H$ surface approx).

If we move on to thinking about a **planetary atmosphere,** both of those boundary conditions change.

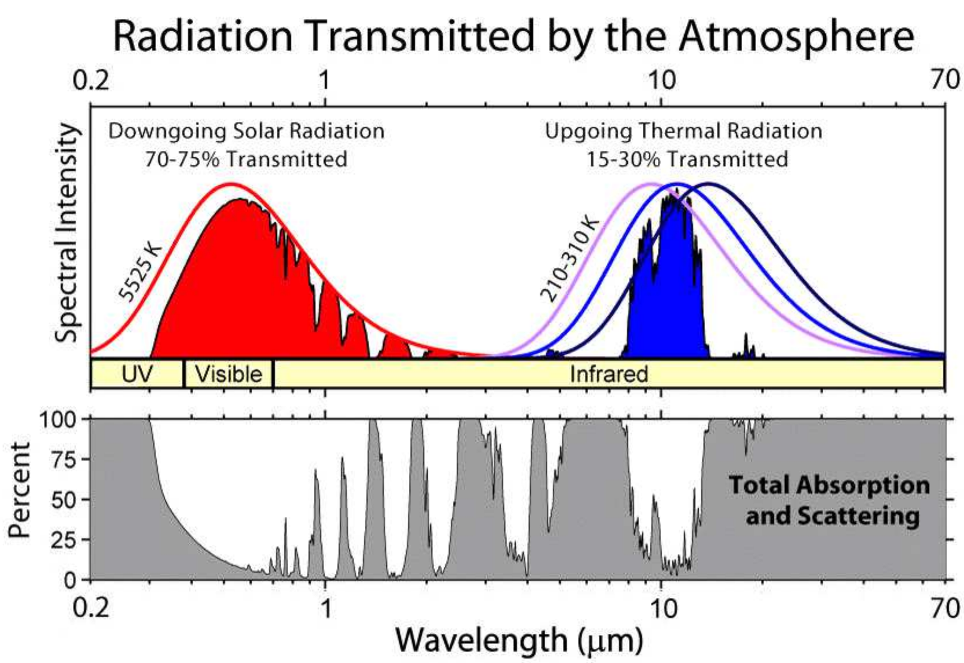
Ignoring internal heat generation (tides or radioactive decay!),
the total F can be zero, but $I_{\rm down} \neq 0$.

Recall, of course, that these quantities are integrated over ν.
In isolated parts of the spectrum, $F_\nu \neq 0$.

Spectrally, we often see that $F = 0$ is maintained as a balance between:

$$\left\{ \begin{array}{lll} \text{Inward flux:} & \text{from stellar irradiation} & (\text{“short-wave”}) \\ \text{Outward flux:} & \text{thermal re-radiation} & (\text{“long-wave”}) \end{array} \right\}$$

For the Earth, these two sources of radiation occupy (essentially) separate parts of the spectrum:



When the short-wave & long-wave pieces of the flux cancel each other out, we have $\boxed{|F_{\rm SW}| = |F_{\rm LW}|}$. We use this flux balance to determine the atmospheric temperature structure of terrestrial planets.

(When $I_{\rm up}$ and $I_{\rm down}$ occupy the *same* part of the spectrum, the problem of "irradiated atmospheres" gets more complicated. For a detailed pedagogical sequence of useful toy models, see Hansen [2008, *ApJ Suppl.,* 179, 484]).

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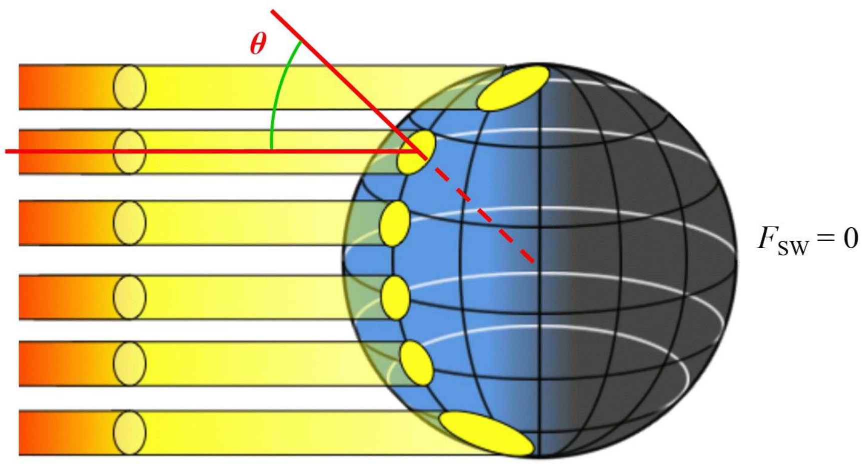
For terrestrial planets, there are several ways to estimate $F_{\rm SW}$.

First, examine *instantaneous* insolation on a spherical planet:



At "noon" ($\theta = 0$),

$$F_{\rm SW} \;=\; \frac{L_*}{4\pi r^2}\,(1-A) \;\equiv\; F_{\rm max}$$

where r is the star–planet distance D. (Strictly, $r = D - R_p$.)

For us, the insolation flux $(L_\odot/4\pi r^2_{\rm 1\,AU})$ is called the "solar constant."

A = the short-wave **Bond albedo** of the planet: the fraction of total incident flux that is immediately reflected back into space (at "top of atmosphere").

(Earth: $A = 0.30$, Venus: $A = 0.77$) Depends on cloud cover!

At other locations, $$F_{\rm SW} \;=\; \begin{cases} F_{\rm max}\cos\theta\ , & \text{on the day side} \\ 0\ , & \text{on the night side.} \end{cases}$$

For a tidally locked planet, that's all one needs. For a rotating planet, the incoming flux is "spread around" in longitude [and a bit in latitude, if there's axis obliquity, but usually $F_{\rm SW}(\text{poles}) \ll F_{\rm SW}(\text{equator})$].

If we assume rapid redistribution of the incoming flux, we can compute the average over the globe:

$$\langle F_{\rm SW} \rangle \;=\; \frac{1}{4\pi} \int d\Omega \;\, F_{\rm SW}(\theta,\phi) \;=\; \frac{F_{\rm max}}{4\pi} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi/2} d\theta \;\, \sin\theta \;\; \cos\theta}_{1/2} \;=\; \frac{F_{\rm max}}{4} \;.$$

This may not correspond exactly to any one point on the planet, but it lets us estimate surface-averaged quantities.

If the planet re-emits the absorbed radiation as something close to a blackbody, the outgoing long-wave flux can be written as

$$F_{\rm LW} \;=\; \eta\, \sigma T_{\rm eff}^4$$

where η is a fractional emissivity ($\eta = 1$ for a perfect blackbody; $\eta < 1$: "graybody." Earth's land/water surface has mean $\eta \approx 0.96$).

We can thus compute the planet-averaged **equilibrium temperature**

$$\langle F_{\rm SW} \rangle \;=\; F_{\rm LW} \qquad \Longrightarrow \qquad T_{\rm eff} \;=\; \left[\frac{L_* (1-A)}{16\pi\eta\sigma r^2} \right]^{1/4} \;\propto\; r^{-1/2} \;.$$

Earth: $T_{\rm eff} \approx 257\,{\rm K} \;\approx -16^\circ\,{\rm C}$.

Venus: $T_{\rm eff} \approx 229$ K. Lower because of higher albedo!

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Note: some books take an "energy in = energy out" approach:

$$\mathcal{E}_{\rm in} \;=\; \left[\frac{L_* \, (1-A)}{4\pi r^2} \right] \pi R_p^2 \;\;, \qquad \mathcal{E}_{\rm out} \;=\; \left(\eta\sigma T_{\rm eff}^4 \right) 4\pi R_p^2$$

which gives the same answer as above.

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This isn't the end of the story because terrestrial planets can have atmospheres. The deeper you go, the more opacity can "trap" photons... **the greenhouse effect.**

When computing height dependence of "air temperature," our standard gray/Eddington result is still more or less valid:

$$T_{\rm air}(\tau) \;=\; T_{\rm eff} \left[\frac{3}{4} \left(\tau + \frac{2}{3} \right) \right]^{1/4}$$

and at the infrared λ's of interest to the outgoing long-wave Planck function, the near-surface values are

$$\begin{array}{lll} \text{Earth:} & \tau \approx 1.4 & \longrightarrow \;\; T_{\rm air} \approx 288\,\text{K} \;\approx\; +15^\circ\,\text{C} \\ \text{Venus:} & \tau \approx 140 & \longrightarrow \;\; T_{\rm air} \approx 735\,\text{K} \end{array}$$

which makes sense, if we know that Venus' surface pressure is roughly 100 times that of Earth.

There are many other complications:

- Real atmospheres aren't gray or Eddington. For CO_2, strong-opacity bands are surrounded by transparent spectral "windows."
- Even for a very rapidly rotating planet, computing $T_{\rm air}$ as a function of position demands including a **latitudinal diffusion** term (due mainly to atmospheric & ocean dynamics). Gerald North's classic (1975, *JAtmSci,* 32, 2033) energy-balance model reproduces lots of what we see on Earth... including the possibility of "snowball" ice-age phases.
- Computing the "ground temperature" of the solid surface takes extra care. It's often $> T_{\rm air}$, but simple models overestimate the discontinuity.

Lastly, when there's an energy flux hitting a solid planet/asteroid/comet, there are *other things* that can happen, besides re-radiating...

Irradiated Phase Changes

Sublimation (solid $\rightarrow$ gas) or condensation/deposition (gas $\rightarrow$ solid).
Thus, there are 2 terms on the RHS...

$$F_{\mathrm{SW}} \;=\; \eta\,\sigma T^4 \;+\; \frac{\dot{M}}{4\pi R_p^2}\,\mathcal{L}$$

where $\mathcal{L}$ is the latent heat of sublimation (erg/g), which essentially converts energy flux into mass flux.

Generally, $\mathcal{L}$ is a known constant for a given type of solid surface material.

$$\text{Two unknowns on RHS:} \qquad \left\{ \begin{array}{c} \text{equilibrium } T \\ \text{subliming mass-loss rate } \dot{M} \end{array} \right\}$$

Chemistry to the rescue... we can combine two laws:

- Clausius-Clapeyron relation (surface vapor pressure as func. of T)
- Hertz-Knudsen relation ($\dot{M}$ per unit area as func. of vapor pressure)

to obtain

$$\frac{\dot{M}}{4\pi R_p^2} \;\approx\; P_\infty \sqrt{\frac{m_s}{k_{\mathrm{B}}T}}\,\exp\left(-\frac{m_s\mathcal{L}}{k_{\mathrm{B}}T}\right)$$

where P_∞ is vapor pressure in $T \rightarrow \infty$ limit, and m_s is the mass of 1 molecule of surface material (both known).

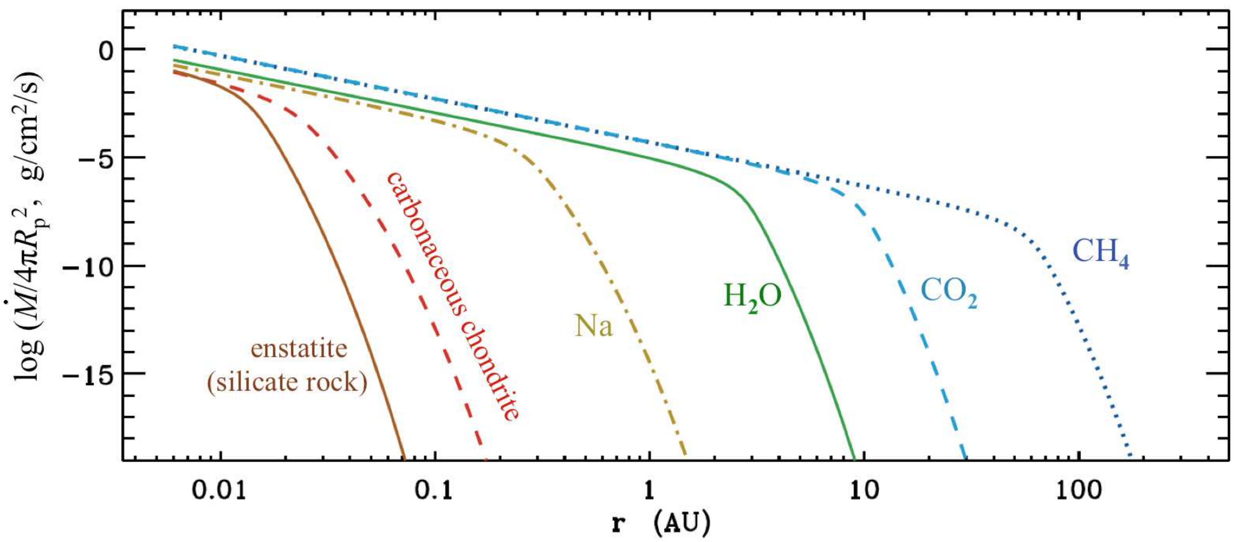
Solving the energy balance for $T(r)$ usually isn't possible analytically.
Limiting cases:

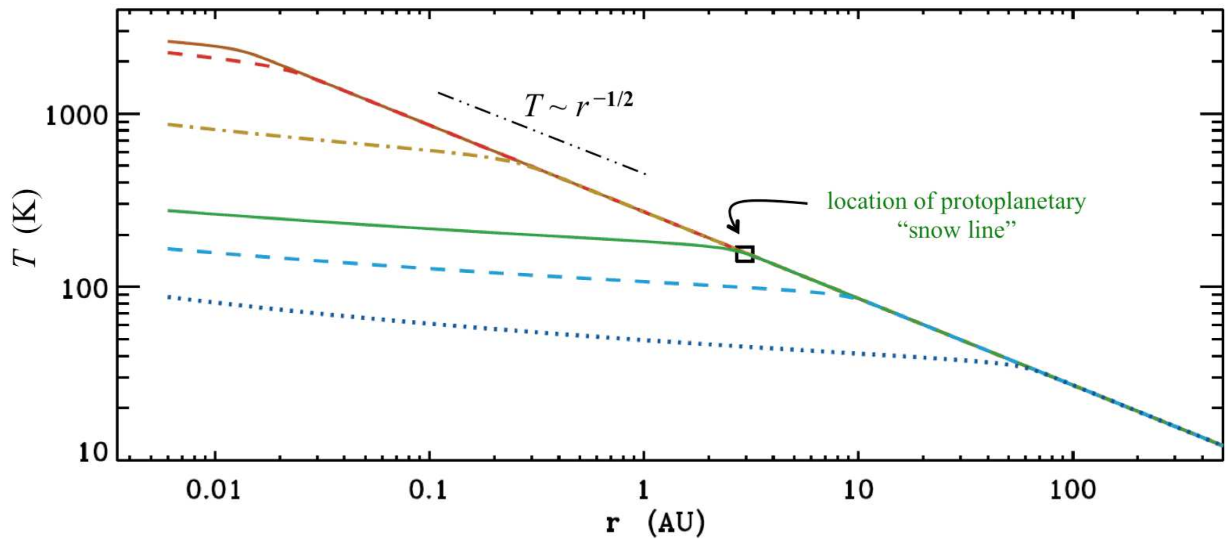
- "Cold" outer solar system ($k_{\mathrm{B}}T \ll m_s\mathcal{L}$): essentially too quiet for sublimation ($\dot{M} \rightarrow 0$), so the long-wave re-radiation dominates, and $\boxed{T \sim r^{-1/2}}$ as before.
- "Hot" inner solar system ($k_{\mathrm{B}}T \gtrsim m_s\mathcal{L}$): the energy goes into the phase-change (sublimation), not into heating it up. Thus,

$$\dot{M} \sim 1/r^2 \qquad\qquad T \text{ flattens to } \;\sim\; 1/(c_1 + c_2 \ln r)\;\; .$$

- The transition point ($k_{\mathrm{B}}T \approx m_s\mathcal{L}$) for water-ice is called the "snow line."

Example plot, for typical solids in our solar system, shown on next page:





Solar-system comets undergo **outgassing** (formation of an extended coma & tail) only once they reach the inner solar system ($r \lesssim$ a few AU) and their surface H_2O and CO_2 starts sublimating.

Exoplanets that orbit very close to their host stars may be continually undergoing **regolith ablation** from their surfaces, akin to how meteors are disintegrated once they enter the atmosphere.

Energy & Momentum Exchange Between Light & Matter

Earlier, we discussed the "radiation pressure" moment of the specific intensity. Another way to derive it (if $f(\mathbf{p})$ is known) is to take the second moment:

$$P = \frac{1}{3} n \langle pv \rangle \qquad \text{(which gives the right answer for } P_{\rm gas}\text{, too).}$$

Working it out for the Bose-Einstein (Planck) function,

$$P_{\rm rad} = \frac{1}{3} U_{\rm rad} = \frac{1}{3} a T^4$$

and, when this is important, there's an extra bulk acceleration term in the momentum conservation equation: $\mathbf{a}_{\rm rad} = -\nabla P_{\rm rad}/\rho$.

In many cases, the above formulation isn't the most useful version of $\mathbf{a}_{\rm rad}$.

A more robust version of the **radiative acceleration** (due to photons hitting particles and transfering some of their momentum) is:

$$\mathbf{a}_{\rm rad} = \frac{1}{c} \int d\nu \ \kappa_\nu \mathbf{F}_\nu$$

and, for optically thick stellar interiors (close to STE), one can show how this is consistent with $-\nabla P_{\rm rad}/\rho$ by using the **radiative diffusion** version of $(dT/dr)_{\rm rad}$ discussed earlier.

This version of $\mathbf{a}_{\rm rad}$ makes sense because you can only have a momentum exchange between light and matter when there's a nonzero **flux,** and when there's a nonzero **opacity.**

For gas/plasma above the surface of a star, the radiative acceleration can partially cancel out gravity:

$$a_{\rm tot} = -\frac{GM_*}{r^2} + \frac{\kappa L_*}{4\pi c r^2} \qquad \rightsquigarrow \rightsquigarrow \qquad |a_{\rm tot}| = \frac{GM_*}{r^2}(1 - \Gamma)$$

where $\Gamma = \kappa L_*/(4\pi c G M_*)$ is the "Eddington factor."

Γ must be < 1 for the outer layers of a star to remain hydrostatic & bound together. If $\Gamma > 1$, an object has exceeded the "Eddington limit."

Consider radiation impacting **macroscopic** objects, like dust grains. Recall:

$$\kappa_\nu \rho = \sigma_\nu n \qquad \rightsquigarrow \rightsquigarrow \qquad \kappa_\nu \sim \frac{\sigma_\nu}{m_{\rm grain}}$$

and for a big spherical grain, with radius a, we can define

$$\sigma_\nu \approx \pi a^2 Q$$

where here Q is an absorption efficiency (fraction of light actually absorbed; i.e., similar to $1 - A$ for planets).

Thus, the radiative force on a grain is given by

$$\mathcal{F}_{\rm rad} = m_{\rm grain}\, a_{\rm rad} = \frac{L_*}{4\pi r^2 c} \pi a^2 Q$$

but we also know that

$$\mathcal{F}_{\rm grav} = m_{\rm grain}\, g = \left(\rho_{\rm grain} \frac{4\pi a^3}{3} \right) \frac{G M_*}{r^2}$$

so planetary scientists often discuss the ratio of these forces,

$$\beta \equiv \frac{\mathcal{F}_{\rm rad}}{\mathcal{F}_{\rm grav}} = \rightsquigarrow = \frac{3 L_* Q}{16\pi G M_* c\, \rho_{\rm grain}\, a}$$

This ratio is independent of r. It's also $\propto 1/a$, so radiation pressure is more important for small grains, and not large planets!

If $\beta > 1$ anywhere, then the grain will be eventually **ejected** from the system. The smallest grains will be ejected more rapidly.

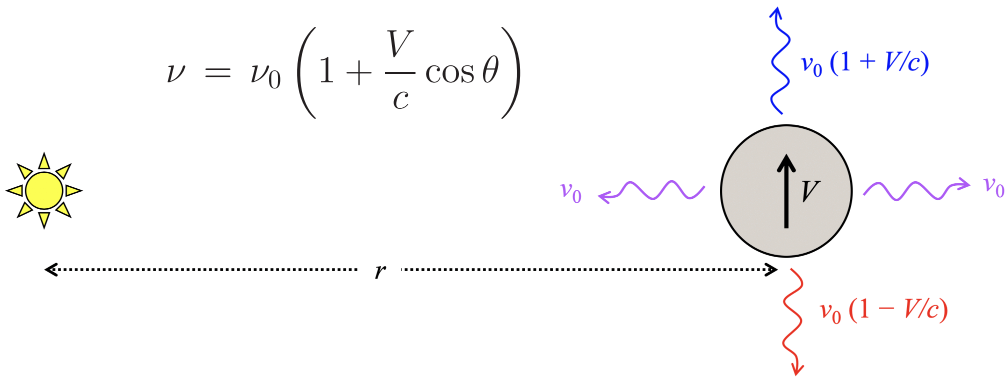
Large grains with $\beta < 1$ are still bound into closed orbits. However, some of them may spiral *into* the star because of the **Poynting-Robertson effect.**

P-R is a relatively weak effect involving absorption & re-emission from a *moving* dust grain. Let's look at a grain in the star's inertial frame:

- An absorbed photon (ν_0) is coming radially from the star.
- Re-emission is isotropic in the grain's frame, but an observer in an inertial frame sees different frequencies in different directions:

For $\theta = 0$ along the grain's velocity vector,

$$\nu \;=\; \nu_0 \left(1 + \frac{V}{c} \cos\theta \right)$$



More photon momentum $(h\nu/c)$ is lost in the forward direction than in the backward direction. A simple way to estimate the net loss is to just take a "forward minus backward" momentum difference:

$$\Delta p \;\approx\; p(\theta = 0) - p(\theta = \pi) \;=\; \frac{2\,V\,h\nu_0}{c^2} \quad .$$

Another way to think about it is to take a flux-weighted "first moment" of the angular dependence of the photon momentum:

$$\int d\Omega\;\hat{\mathbf{n}} \left[\frac{h\nu_0}{c} \left(1 + \frac{V}{c} \cos\theta \right) \right] \;=\; \rightsquigarrow \;=\; \frac{4\pi}{3}\,\frac{V\,h\nu_0}{c^2} \quad .$$

However, neither of these coefficients are exactly right.

We need **special relativity** to compute it exactly. Larmor (1913) & Robertson (1937) showed that the factor isn't 2 or $4\pi/3$, but just 1.

Thus, via Newton's 3rd law, when photons run away with a net momentum difference in one direction, the particle feels an equal and opposite force (i.e., a slowdown!) with magnitude

$$\mathcal{F}_{\rm PR} \;=\; \frac{\Delta p}{\Delta t} \;=\; \frac{h\nu_0}{\Delta t}\,\frac{V}{c^2}$$

and the trick is to realize that the "photon power" absorbed by the grain can be written as

$$\frac{h\nu_0}{\Delta t} \;\approx\; \frac{L_*}{4\pi r^2}\,\pi a^2 \quad .$$

This gives, for $Q = 1$,

$$\mathcal{F}_{\rm PR} \;=\; \frac{V}{c}\,\mathcal{F}_{\rm rad} \;=\; \frac{V}{c}\,\beta\mathcal{F}_{\rm grav}$$

and, for a dust grain in a circular orbit, $V = \sqrt{GM_*/r}$, so $\mathcal{F}_{\rm PR} \propto r^{-2.5}$.

As one approaches the star, $\mathcal{F}_{\rm PR}$ gets **stronger** than both $\mathcal{F}_{\rm grav}$ and $\mathcal{F}_{\rm rad}$.

For $\beta \sim 1$, $\mathcal{F}_{\rm PR}$ is a factor of V/c weaker than $\mathcal{F}_{\rm grav}$. Thus, the inspiral timescale is roughly c/V times LONGER than the orbital period.

(Example: for a 1 year orbital period, the inspiral time is $\sim 10^4$ years. This is SHORT compared to the age of the solar system! Thus, to account for the dust we see now, there must be continual sources making more dust all the time.)

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For an interesting connection with the gravitational dynamics part of RDP, see Wyatt & Whipple (1950, ApJ, 111, 134), who derived equations for how the P-R effect can influence a large object like an asteroid or moon. They derived equations that show how its semimajor axis and eccentricity evolve under the influence of radiation pressure, and they discovered some new *constants of integration.*
