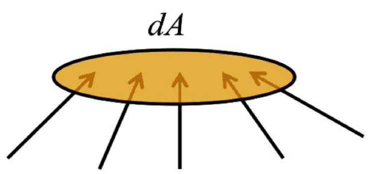
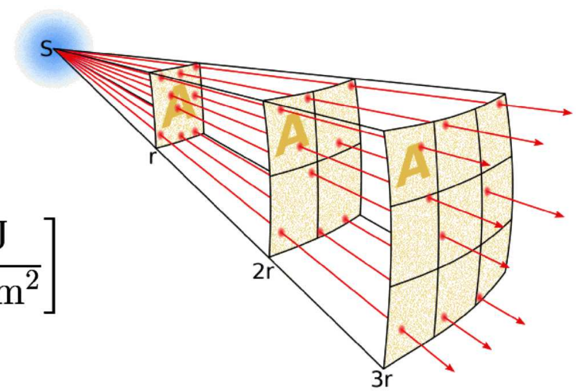
RADIATION PROCESSES

The final part of the course: photons & electromagnetic radiation.

Usually we'll be considering systems with spatial scales $\gg$ the *wavelengths* of radiation. Thus, the radiation travels along straight "rays" in vacuum.

A basic concept is the **energy flux** of radiation. Essentially the same quantity as the Poynting flux $\mathbf{S}$ of E&M waves. Consider area element dA. How much photon energy dE passes through dA in time interval dt?





$$F = \lim \frac{dE}{dA\ dt} \qquad \left[\frac{\mathrm{J}}{\mathrm{s\ m}^2}\right]$$

(in the limit of $dA \to 0$, $dt \to 0$)

Even though $\mathbf{F}$ sums over all rays that make it through dA (no matter where they come from), we'll see that it is "weighted" like the bulk/centroid gas velocity $\mathbf{u}$ is a weighted version of microscopic particle velocity $\mathbf{v}$.

Observationally, detectors have finite area, so actual fluxes must be computed by dividing photon "count rates" by the total (effective) area of the detector.

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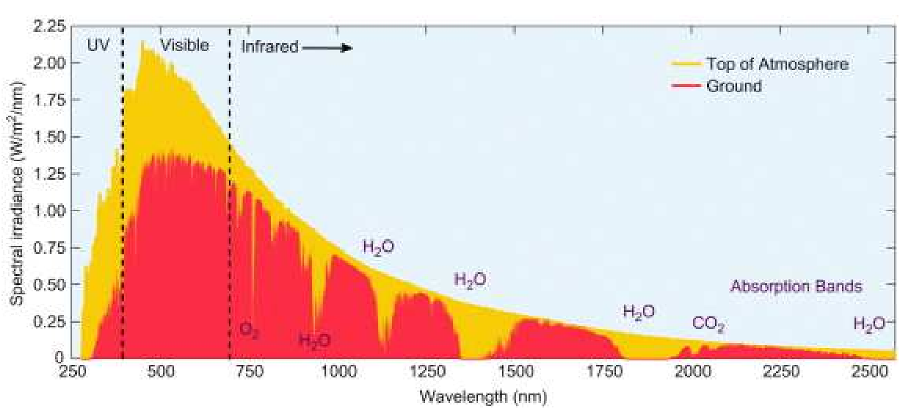
When we plot a spectrum/SED, we're showing how much flux is measured in different "bins" of λ or ν.   For example, the Sun's flux:



The integral under the curve gives the *total* ("bolometric") flux measured at Earth:

$$F \;=\; 1{,}361 \;\; \mathrm{W/m^2} \qquad \text{(“solar constant” at top of atmosphere)}$$

Thus, the spectral quantities are measured "per unit:"

$$F_\lambda \;=\; \{\text{flux per unit wavelength}\} \qquad \left[\frac{\mathrm{W}}{\mathrm{m^2\ nm}}\right]$$

$$F_\nu \;=\; \{\text{flux per unit frequency}\} \qquad \left[\frac{\mathrm{W}}{\mathrm{m^2\ Hz}}\right]$$

$$\text{and...} \qquad F \;=\; \int_0^\infty d\lambda\; F_\lambda \;=\; \int_0^\infty d\nu\; F_\nu \;\;.$$

These "per unit" fluxes are like continuous probability distributions; i.e., we can think of them like

$$F_\lambda \;=\; \lim_{d\lambda\to 0}\left[\frac{\text{flux between } \lambda \text{ and } \lambda + d\lambda}{d\lambda}\right] \;\;, \qquad \text{etc.}$$

Converting between F_λ and F_ν can be tricky. The limiting form given above means that we can think of them kind of as

$$F_\lambda \;=\; \left|\frac{\partial F}{\partial \lambda}\right| \qquad \text{and} \qquad F_\nu \;=\; \left|\frac{\partial F}{\partial \nu}\right|$$

so if we know one, the other comes from the chain rule.

Let's say we know F_λ. Then,

$$F_\nu \;=\; \left|\frac{\partial F}{\partial \nu}\right| \;=\; \left|\frac{\partial F}{\partial \lambda}\right| \left|\frac{\partial \lambda}{\partial \nu}\right| \;\;.$$

We know that $c = \lambda\nu$, so

$$\lambda \;=\; \frac{c}{\nu} \;=\; c\nu^{-1} \qquad \text{so} \qquad \frac{\partial \lambda}{\partial \nu} \;=\; -c\nu^{-2}$$

and

$$F_\nu \;=\; \left(\frac{c}{\nu^2}\right) F_\lambda \qquad \text{or, equivalently,} \qquad \left(\frac{\lambda^2}{c}\right) F_\lambda \;\;.$$

. . . .

Sometimes, we don't have a spectrometer to measure the high-resolution SED (either F_λ or F_ν). Often, astronomers rely on **photometry** to measure the integrated flux F (sometimes called f) in finite 'passbands' of the spectrum.

These fluxes are often called **apparent brightness** (same units: W/m^2 or erg/s/cm^2), and astronomers tend to convert them to dimensionless units called **MAGNITUDES...**

The human eye seems to be a logarithmic detector; i.e., it perceives equal flux ratios as $\sim$equal intervals.

Ancient Greeks devised a system where the brightest stars had magnitude 1, next brightest 2, and on down to 6 (the limit of naked-eye perception).

In the 1800s, Pogson formalized that system by noting that a 1st magnitude star is 100 times brighter than 6th magnitude.

So, if 5 magnitude steps correspond to a ratio of 100, then one magnitude step is a ratio of $(100)^{1/5} \approx 2.512$. In other words, to "step up" from 6th to 1st magnitude, you'd have to you'd have to multiply the flux by 5 successive factors of 2.512, to get the required factor of 100. Thus,

$$
\begin{aligned}
(m_1 - m_2) &= -5 \quad \text{corresponds to } \; f_1/f_2 = 100 \\
(m_1 - m_2) &= -2.5 \; \text{corresponds to } \; f_1/f_2 = 10 \\
(m_1 - m_2) &= -1 \quad \text{corresponds to } \; f_1/f_2 = 10^{2/5} \; = \; 2.511886 \; .
\end{aligned}
$$

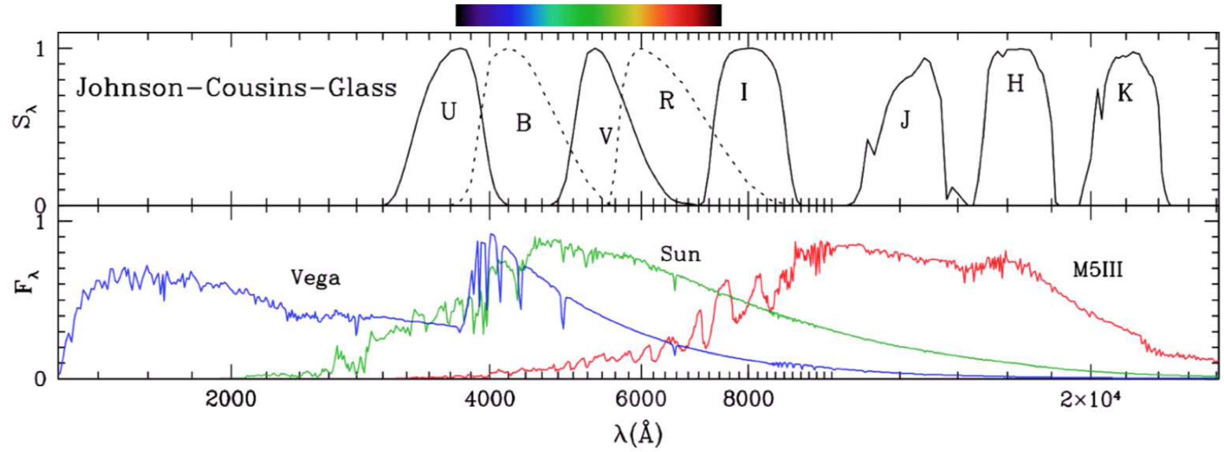
To be quantitative, we need absolute standards,

$$
m - m_{\rm std} \; = \; -2.5 \log_{10} \left(\frac{f}{f_{\rm std}} \right)
$$

There are 2 similar systems for benchmarking a known $f_{\rm std}$ with $\boxed{m_{\rm std} = 0}$:

- Vega's spectrum (archetypical 0th magnitude star)
- An ABsolutely defined spectral function that everyone agrees on (the "AB system") developed by John Oke in the 1970s: $f_\nu = \text{constant}, \quad f_\lambda \propto 1/\lambda^2$.

Apparent magnitudes measured with multiple filters provide "color" info:



Notation: $m_{\rm V}$ (similar to eye response) often called "V magnitude" or just V.

"Colors" = ratios of fluxes in different filters = differences in magnitude (e.g., $B - V$, $J - K$).

Sometimes we want to know how much of the total/bolometric flux *we're missing* by just using one filter. For specific types of stars, there are tables of **bolometric corrections:**

$$\text{i.e., for the } V \text{ band,} \qquad m_{\rm bol} \;=\; m_V \;+\; {\rm BC}_V$$

and because $m_{\rm bol} < m_V$, ${\rm BC}_V$ must always be ≤ 0.

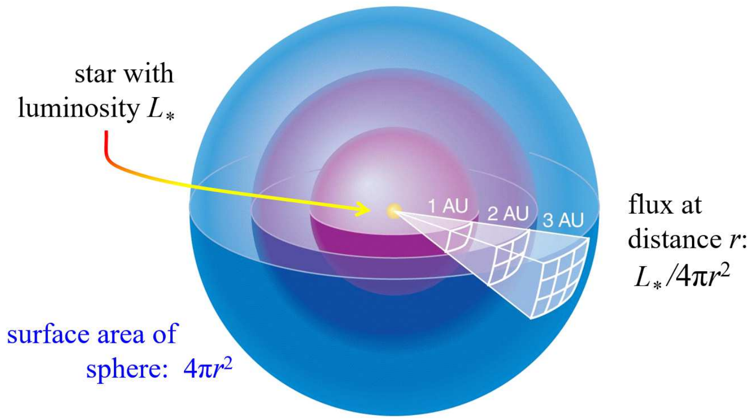
. . . .

The measured **flux** of a distant source is useful, but there are other associated quantities that sometimes are more useful.

One down-side to using flux is that it's not an "intrinsic" property of a source. The further away you go from a compact/point source, the flux drops as $1/r^2$.

Thus, if we integrate the flux over a closed surface that encompasses the source, we should "catch" all of the electromagnetic energy, and get the same answer no matter the distance r. Define the **luminosity** as the total power emitted (in erg/s, or Watts):

$$L \;=\; \oiint dA\; F \;=\; \oint d\Omega\, r^2\; F \;=\; \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, r^2\; F \;=\; (4\pi r^2) F \;\; .$$



Also, **absolute magnitudes** (capital M's) are a way of talking about the intrinsic luminosity of an object, independent of its distance, while still using the magnitude system.

Absolute = apparent for a benchmark distance of 10 parsecs:

$$M = m - 5\log_{10}\left(\frac{d}{10\text{ pc}}\right)$$

The *bolometric* (full-spectrum) absolute magnitude is equivalent to the total luminosity:

$$M_{\text{bol}} - M_{\text{bol},\odot} \;=\; -2.5\log_{10}\left(\frac{L_*}{L_\odot}\right) \qquad \text{IAU defined } M_{\text{bol},\odot} \equiv 4.74\ .$$

Because stars have luminosities between about 10^{-6} & $10^{+6}\,L_\odot$, they have M_{bol} between about –15 and +15.

. . . .

The concepts & units are proliferating... let's summarize:

Astronomy term	symbol	units	SI radiometry term
bolometric luminosity	L	W	power or radiant flux
bolometric flux	F or f	W/m^2	irradiance or flux density
flux	F_λ or f_λ	$\text{W/m}^2\text{/nm}$	spectral irradiance
flux	F_ν or f_ν	$\text{W/m}^2\text{/Hz}$	spectral irradiance
total intensity	I	$\text{W/m}^2\text{/sr}$	radiance
specific intensity	I_λ	$\text{W/m}^2\text{/sr/nm}$	spectral radiance
specific intensity	I_ν	$\text{W/m}^2\text{/sr/Hz}$	spectral radiance

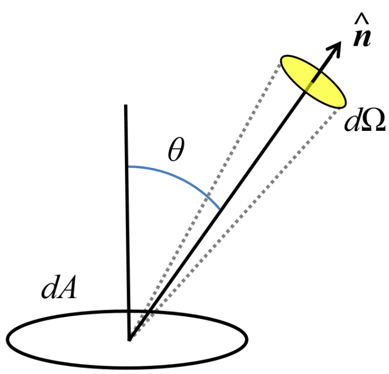
The last one we haven't talked about yet: **INTENSITY.**

Recall that the flux F sums over all rays that make it through dA, no matter where they come from. Often, we want to know more: what is the full 3D distribution of ray paths, and how energy is distributed as a function of **solid angle?**

Thus, we define the specific intensity (I_ν or I_λ) to describe all information contained in the flux, *plus* how the photon rays are arranged in direction.

I_ν describes how much photon energy is flowing

$\rightarrow$ through a particular area,
$\rightarrow$ in a particular direction (i.e., into a particular solid angle),
$\rightarrow$ per unit frequency (i.e., energy "bin"),
$\rightarrow$ per unit time.



$$I_\nu(\hat{\mathbf{n}}) = \lim \frac{dE}{(dA \cos\theta)\, d\Omega\, d\nu\, dt}$$

(in the limit of $dA \to 0$, $d\Omega \to 0$, $d\nu \to 0$, $dt \to 0$)

Standard units of I_ν: erg/s/cm^2/sterad/Hz

Alternate units:

- change ergs (or Joules) to photons; i.e., divide by $E = h\nu$.
- instead of "per unit frequency bin," use bins in λ instead (convert using chain rule, like $F_\nu \leftrightarrow F_\lambda$).

Soon, we will discuss how $I_\nu(\hat{\mathbf{n}})$ describes the same things as the photon distribution function $f(\mathbf{p})$. For now, I'll just give the conversion between the two:

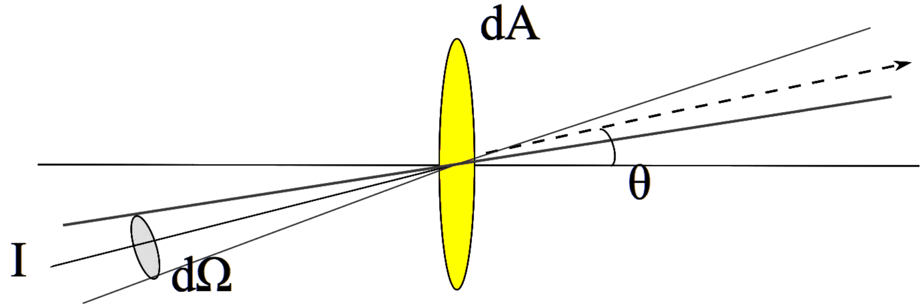
$$I_\nu = \frac{h^4\nu^3}{c^2} f \qquad \text{and of course} \quad |\mathbf{p}| = \frac{h\nu}{c} \ , \quad \hat{\mathbf{n}} = \frac{\mathbf{p}}{|\mathbf{p}|}$$

then later we will motivate it in more detail, since I_ν is so central to astrophysics.

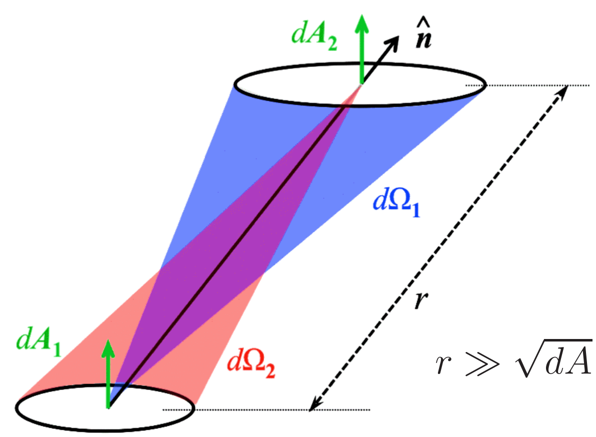
We'll also see how Boltzmann equation for f (with sources & sinks on the right-hand side) becomes the **equation of radiation transfer** for I_ν.

In vacuum,

- we're *not* considering light rays that bend (no GR!)
- I_ν is **constant** along a given ray (unlike flux; for proof see below)
- $d\Omega$ can mean either "into" or "out of" the projected area:



- In reality, I_ν describes the flux of energy flowing from one area dA_1 into another (dA_2).
- However, since we prefer to specify I_ν locally (all properties at one location), we convert one of the areas into solid angle measured from our location.
- Both descriptions are identical!



$$dE_1 = I_\nu(\hat{\mathbf{n}}, \mathbf{r}_1, t)\, dt\, d\nu\, d\Omega_1\, (d\mathbf{A}_1 \cdot \hat{\mathbf{n}})$$
$$dE_2 = I_\nu(\hat{\mathbf{n}}, \mathbf{r}_2, t)\, dt\, d\nu\, d\Omega_2\, (d\mathbf{A}_2 \cdot \hat{\mathbf{n}})$$

$$r^2\, d\Omega_1 = d\mathbf{A}_2 \cdot \hat{\mathbf{n}} \qquad r^2\, d\Omega_2 = d\mathbf{A}_1 \cdot \hat{\mathbf{n}}$$

In vacuum, $dE_1 = dE_2$, so $\quad I_\nu(\hat{\mathbf{n}}, \mathbf{r}_1, t) = I_\nu(\hat{\mathbf{n}}, \mathbf{r}_2, t)$

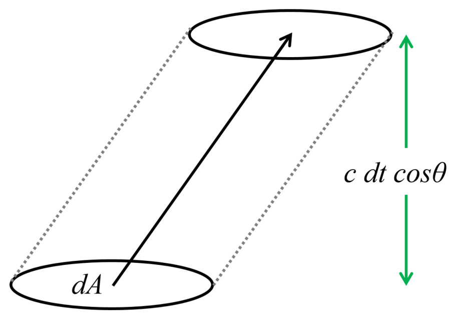
. . . .

Sometimes it's difficult to mentally reconcile $I_\nu \propto f(\mathbf{p})$, since:
I_ν counts the photons passing through a **surface,** but
f counts the particles per unit **volume** of 6D phase space.

Let's explore the connection by looking at the 6D volume element:

$$d^3\mathbf{r}\; d^3\mathbf{p} \;=\; [(dA\,\cos\theta)(c\,dt)] \;\times\; \left[p^2\,dp\;d\Omega\right]$$
$$\left[\frac{h^3\nu^2\,d\nu}{c^3}\,d\Omega\right] \qquad (\text{since } p = h\nu/c)$$

where earlier we often assumed an isotropic distribution and thus assumed $d\Omega$ integrates to 4π.



So, does our conversion between f and I make sense?

$$\text{If} \quad f \;=\; \frac{\#}{d^3\mathbf{r}\,d^3\mathbf{p}} \qquad \text{and} \qquad I_\nu \;=\; \frac{h^4\nu^3}{c^2}\,f\;,$$

$$\text{then} \qquad I_\nu \;=\; \frac{h^4\nu^3}{c^2}\,\frac{\#}{dA\,\cos\theta\,c\,dt}\,\frac{c^3}{h^3\,\nu^2\,d\nu\,d\Omega}$$

The factors of c cancel out completely, and what's left is

$$I_\nu \;=\; \frac{(\#\,h\nu)}{dA\,\cos\theta\,dt\,d\nu\,d\Omega} \qquad \text{where the numerator is essentially } dE.$$

. . . .

To make use of the radiation field in real astrophysical environments, sometimes $I_\nu(\hat{\mathbf{n}})$ contains too much information.

We can integrate over the full frequency/energy spectrum to obtain the *total* intensity I, which measures the directional flow of all photon energy,

$$I \;=\; \int_0^\infty d\nu\;I_\nu \;=\; \int_0^\infty d\lambda\;I_\lambda \qquad \text{where} \qquad I \sim \frac{E}{(dA\,\cos\theta)\,d\Omega\,dt}\;.$$

We can also take solid-angle moments over $d\Omega$, analogous to the $d^3\mathbf{p}$ moments of gas/plasma distribution functions.

Earlier, we defined the **energy density** that's present at any one time in a unit volume as $U \sim E/V \sim dE/dV$.

From the cartoon above, we know that photons flowing through a given area in dt time take up a "volume" $dV = dA \cos\theta \; c\,dt$.

Putting it all together, the above scalings give $I/c \sim U/d\Omega$. But if we realize that all angles "count" when summing up the total energy density present at a given location, we realize it's better to write

$$\boxed{U = \int d\Omega \, \frac{I}{c}}$$

If $I(\hat{\mathbf{n}})$ is **exactly** isotropic, then no net energy can flow from one point in space to another (i.e., zero flux).

This would be an equilibrium distribution, akin to our old friend the Maxwell-Boltzmann f. Of course, photons are quantum *bosons,* so they obey **Bose-Einstein statistics.** We'll come back to this in a bit.

However, the above integral over $d\Omega$ leads us to consider weighted moments over the solid-angle distribution of rays.

The energy density U is proportional to the 0th moment.

Let's define the **net flux** of energy as proportional to the 1st moment:

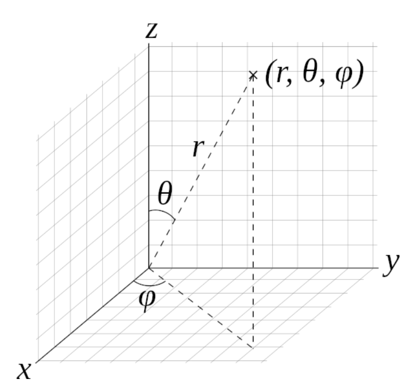
For a specific vector direction $\hat{\mathbf{n}}$, $$\boxed{\mathbf{F} \equiv \int d\Omega \; I \; \hat{\mathbf{n}}}$$

This is a generic way to write it. Let's be more specific...

For a spherical coordinate system, $\hat{\mathbf{n}}$ can vary over all θ and ϕ, but let's say we want the net flux in the z direction.

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$\hat{n}_x = \sin\theta\cos\phi\,,\;\; \hat{n}_y = \sin\theta\sin\phi\,,\;\; \hat{n}_z = \cos\theta\,.$



$$\begin{aligned} F_z &= \mathbf{F}\cdot\hat{\mathbf{e}}_z \\ &= \int d\Omega\ I(\hat{\mathbf{n}})\ \hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_z \qquad\qquad \text{and}\quad \hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_z = \cos\theta \end{aligned}$$

Thus,

$$F_z = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\ \sin\theta\ I(\theta,\phi)\ \cos\theta\ .$$

A completely isotropic $I(\theta,\phi)$ would give $F_z = 0$ because it's an odd function:

For $\mu = \cos\theta$, the θ integral is $\displaystyle\int_{-1}^{+1} d\mu\ \mu\ I(\mu)$ (note: $d\mu = -\sin\theta\, d\theta$) .

Note: this μ is different from the μ used for the mean atomic mass of a mixture of elements. Usage should be clear from context.

Thus, to get a non-trivial result for the net flux, let's assume a **nearly isotropic** intensity...

$$\boxed{I \approx I_0 + I_1\cos\theta} \quad \text{where}\quad |I_1| \ll |I_0|$$

In the integral to get the energy density, the I_1 term cancels out:

$$U = 2\pi\int_{-1}^{+1} d\mu\ \frac{I}{c} = \frac{4\pi I_0}{c}\ .$$

In the integral to get the net flux, the I_0 term cancels out:

$$F_z = 2\pi\int_{-1}^{+1} d\mu\ \mu\ (I_0 + I_1\mu) = \frac{4\pi I_1}{3}\ .$$

It's now possible to define the angle moments in the standard way that stellar atmosphere researchers developed a century ago...

$$\begin{aligned} J_\nu &= \frac{1}{4\pi}\int d\Omega\ I_\nu(\mu) = \frac{1}{2}\int_{-1}^{+1} d\mu\ I_\nu(\mu) \\ H_\nu &= \frac{1}{4\pi}\int d\Omega\ \mu\ I_\nu(\mu) = \frac{1}{2}\int_{-1}^{+1} d\mu\ \mu\ I_\nu(\mu) \\ K_\nu &= \frac{1}{4\pi}\int d\Omega\ \mu^2\ I_\nu(\mu) = \frac{1}{2}\int_{-1}^{+1} d\mu\ \mu^2\ I_\nu(\mu) \end{aligned}$$

J_ν is the **mean intensity,** i.e., just the mean value of I_ν averaged over all directions. Its connection to the energy density U was described above.

The 1st moment H_ν is Eddington's **flux** (sometimes the "Harvard flux"), and it's clear from the above that

$$\boxed{F_\nu \;=\; 4\pi H_\nu} \qquad \text{where } F_\nu \text{ is often called the "physical flux."}$$

The 2nd moment tells us about the overall anisotropy of the radiation field, with respect to the $\theta = 0$ ($\mu = 1$) axis. Note that μ^2 is highly peaked in both directions along the axis, so K_ν oversamples radiation in those directions, at the expense of radiation in the transverse directions.

Also, K_ν is related to the "radiation pressure" of a photon gas. Recall that the momentum associated with a single photon with $E = h\nu$ is

$$p = h\nu/c$$

and, with analogy to the gas/plasma pressure tensor ($\mathbb{P} \propto \langle \mathbf{vv} \rangle$), we could define a 3×3 **radiation pressure tensor,**

$$\mathbb{P} \;=\; \int d\Omega \; \hat{\mathbf{n}}\hat{\mathbf{n}} \; \frac{I_\nu(\hat{\mathbf{n}})}{c}$$

(i.e., proportional to the 2nd angle moment of f). The Cartesian component $(\mathbb{P})_{ij}$ gives the net rate of transport of the ith component of momentum flux (per dt, in frequency bin $d\nu$) through a surface (dA) with normal direction j.

In our standard coordinate system (where the z axis is $\theta = 0$), the <u>vertical</u> transport of the <u>vertical</u> component of radiation momentum corresponds to

$$(\mathbb{P})_{zz} \;=\; \frac{1}{c}\int d\Omega \;\; \cos^2\theta \;\, I_\nu(\hat{\mathbf{n}}) \;=\; \frac{4\pi}{c} K_\nu \; .$$

For our toy model of a nearly isotropic radiation field ($I = I_0 + \mu I_1$), we had

$$J \;=\; I_0 \qquad , \qquad H \;=\; \frac{I_1}{3} \ll I_0$$

and the frequency-integrated second moment is

$$K \;=\; \frac{1}{2}\int_{-1}^{+1} d\mu \; \mu^2 \; (I_0 + I_1\mu) \;=\; \frac{I_0}{3} \quad .$$

Later, we'll see that this result can be generalized into a useful **diffusion approximation** (also known as Eddington's first approximation) that tells us about the isotropy of the radiation field; i.e.,

$$J \approx 3K \qquad \text{tends to be valid}$$

when it's a dense, "optically thick" region close to isotropic equilibrium.

. . . .

What is the **EQUILIBRIUM** photon distribution $f(\mathbf{p})$?

Let's think more about particles at the quantum level. In Maxwell-Boltzmann statistics, there's no limit on the number of particles that can "fit" inside a 6D phase-space box $d^3\mathbf{r}\, d^3\mathbf{p}$, and that box can be as small as we want.

In quantum mechanics, there are two limitations:

(1) **Heisenberg's uncertainty principle:** The boxes can only be so small. In other words, the number of quantum "states" (i.e., particular wavefunction solutions of the Schrödinger eqn) that can exist is limited. We can't resolve any differences smaller than, say,

$$\Delta x\, \Delta p_x \geq h \ ,$$

so the phase-space volume of one indivisible quantum "cell" is roughly h^3. For a larger phase-space volume ($d^3\mathbf{r}\, d^3\mathbf{p}$), the number of unique quantum states that can exist is given by dividing it up into h^3 sized cells:

$$N_{\rm uqs} = \left[\int d^3\mathbf{r} \int d^3\mathbf{p} \ \frac{1}{h^3} \right] \times g_s$$

where g_s is the quantum *multiplicity* (or degeneracy) of a given type of particle that goes into the cells. $g_s =$ the number of possible states that a single particle can have. Often $g_s = 2s + 1$, where s is the quantum spin.

Free electrons, protons, neutrons: $s = 1/2$, $\quad g_s = 2$.
Photons have only 2 unique polarization states, so $g_s = 2$.

When $g_s > 1$, quantum mechanics allows > 1 "flavors" of state to coexist.

Note that $N_{\rm uqs}$ *isn't* a number of particles, it's the number of uniquely distinguishable quantum states in some large-scale region.

(2) **Pauli's exclusion principle:** For some types of particle, there can be only one particle occupying any given unique quantum state.
(Or, strictly speaking, only g_s particles per "h^3 box" in 6D phase space.)

Bose-Einstein statistics: particles that obey only Heisenberg, but can have an unlimited number stuffed into each quantum state ("bosons"): e.g., photons.

Fermi-Dirac statistics: particles that obey both Heisenberg and Pauli ("fermions"): e.g., matter.

On a quantum level, bosons kind of obey an "anti-Pauli" principle: the presence of a boson in a particular quantum state *enhances* the probability that other identical bosons will be found in that same state. Identical bosons attract one another; identical fermions repel!

Anyway:

	Min. volume of a 6D box	Max. # of particles in a box
M–B	0	∞
B–E	h^3	∞
F–D	h^3	g_s

There are several ways of deriving the general equilibrium distribution function:

$$f(\mathbf{p}) = \frac{C}{\exp[(E-\mu)/k_{\mathrm{B}}T] + \phi} \qquad \phi = \left\{ \begin{array}{cc} 0 & \text{M-B} \\ +1 & \text{F-D} \\ -1 & \text{B-E} \end{array} \right\}$$

where E is kinetic energy (depends on p^2) and μ is the chemical potential, which we'll discuss more below.

Collins § 1.1 gives a nice derivation of the three distributions (i.e., similar to our earlier look at random-walk diffusion, using the binomial theorem to count unique permutations of particles in quantum states).

There are other interesting derivations; see stat-mech textbooks. Later we'll explore one that Einstein used (**"detailed balance"** of quantum transitions), but we won't spend any more time deriving the above from first principles.

. . . .

Because f is the density of particles in phase space, for **fermions** we know that

$$\max(f) = \frac{g_s}{h^3} = C$$

which follows from the fact that the denominator of f (for $\phi = +1$) cannot be smaller than 1. In this case, it's the chemical potential μ that sets the normalization in *physical space* (i.e., the zeroth moment n).

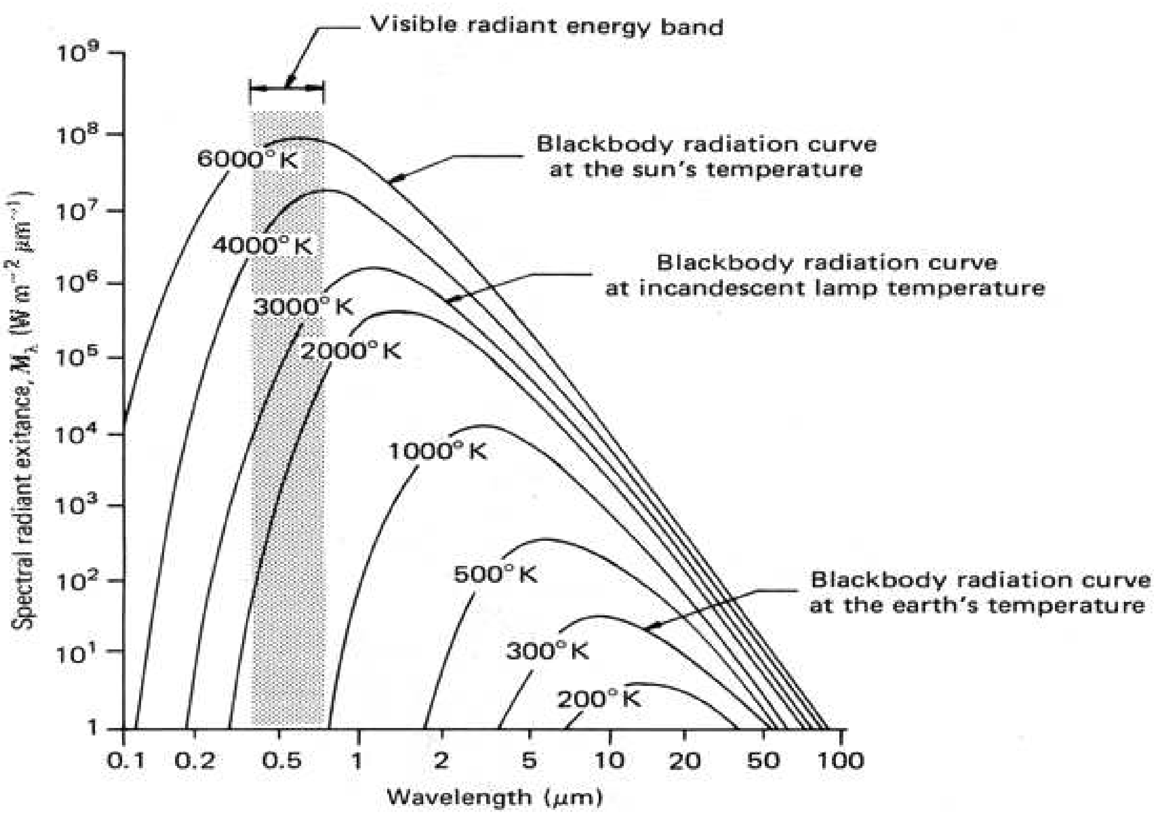
For **bosons,** the convention is to also use the above value of C, too. For photons, we set chemical potential $\mu = 0$ because there is no definite limit on their number density n in physical space.

In strict photon thermal equilibrium, the Bose-Einstein distribution corresponds to the local **Planck function,**

$$f_{\rm BE} = \frac{2}{h^3(e^{h\nu/kT} - 1)}\ , \quad \text{so} \quad I_\nu = \frac{h^4\nu^3}{c^2} f = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = B_\nu(T)$$

Also,

$$B = \int_0^\infty d\nu\ B_\nu(T) = \frac{\sigma T^4}{\pi} \quad \text{where } \sigma = \text{the Stefan-Boltzmann constant.}$$



Let's also work out some standard momentum-space moments of f. These are related to the moments of I_ν, as defined above.

Remember that $E = h\nu$, and $p = h\nu/c$, so

$$d^3\mathbf{p} \;=\; 4\pi p^2\,dp \;=\; \frac{4\pi h^3 \nu^2\,d\nu}{c^3}\;.$$

Thus,

$$n \;=\; \int d^3\mathbf{p}\; f \;=\; \frac{4\pi h^3}{c^3}\int_0^\infty d\nu\;\nu^2\;\frac{2/h^3}{e^{h\nu/kT}-1}$$

$$\text{i.e.,}\qquad n \;=\; \frac{8\pi}{c^3}\int_0^\infty \frac{d\nu\;\nu^2}{e^{h\nu/kT}-1} \;=\; bT^3$$

where the definite integral can be looked up, and b is expressed in terms of a Riemann ζ function. Plugging in the numbers, $b \approx 20.3\ \mathrm{cm}^{-3}\ \mathrm{K}^{-3}$.

More relevant is the photon **energy density,**

$$U \;=\; \int d^3\mathbf{p}\;h\nu\;f(\mathbf{p}) \;=\; n\langle h\nu\rangle \;=\; \frac{8\pi h}{c^3}\int_0^\infty \frac{d\nu\,\nu^3}{e^{h\nu/kT}-1} \;=\; \left(\frac{8\pi^5 k_\mathrm{B}^4}{15h^3c^3}\right)T^4 \;=\; aT^4$$

where a is the "radiation constant" $a = 4\sigma/c$.

Surprisingly, there's a decent summary of different ways of computing the "peak" energy of a Planck function on Wikipedia...

The distributions B_ν, B_ω, $B_{\tilde{\nu}}$ and B_k peak at a photon energy of

$$E = \left[3 + W\left(\frac{-3}{e^3}\right)\right] k_\mathrm{B}T \approx 2.821\;k_\mathrm{B}T,$$

where W is the Lambert W function and e is Euler's number.

The distributions B_λ and B_y however, peak at a different energy

$$E = \left[5 + W\left(\frac{-5}{e^5}\right)\right] k_\mathrm{B}T \approx 4.965\;k_\mathrm{B}T,$$

The reason for this is that, as mentioned above, one cannot go from (for example) B_ν to B_λ simply by substituting ν by λ. In addition, one must also multiply the result of the substitution by

$$\left|\frac{d\nu}{d\lambda}\right| = c/\lambda^2.$$

This $\frac{1}{\lambda^2}$ factor shifts the peak of the distribution to higher energies. These peaks are the *mode* energy of a photon, when binned using equal-size bins of frequency or wavelength, respectively. Meanwhile, the *average* energy of a photon from a blackbody is

$$E = \left[360\frac{\zeta(5)}{\pi^4}\right] k_\mathrm{B}T \approx 3.832\;k_\mathrm{B}T \;\;=\;\; \langle h\nu\rangle\;,$$

where ζ is the Riemann zeta function.

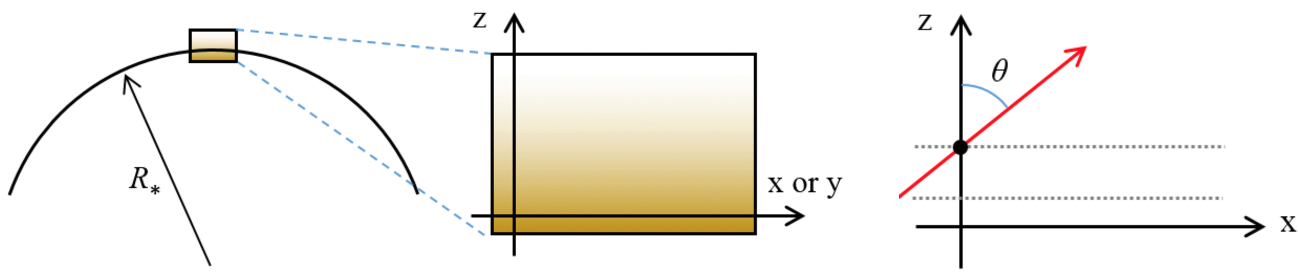
The Equation of Radiative Transfer (RT)

Although a complete description of the flow of radiation through matter (e.g., a stellar interior or stellar atmosphere) depends on multiple pieces of physics, there is one primary conservation equation: Boltzmann's equation!

Before writing it, though, let's simplify our description of the system.

The full form $I_\nu(\hat{\mathbf{n}}, \mathbf{r}, t)$ depends on 7 scalar quantities. ($\hat{\mathbf{n}}$ is a unit vector; only 2 angles.) It pays to simplify the problem...

- Ignore time-dependence (i.e., all $\partial/\partial t = 0$).
- Assume spherical symmetry (i.e., spatial properties depend on r only).
- This also reduces the $\hat{\mathbf{n}}$ vector down to just one angle. Also, if there's no preference for the x or y transverse dimensions, the only important angle is θ, and we assume everything's symmetric in ϕ.
- We often also assume **plane-parallel geometry...**



If the total extent of the atmosphere ($z_{\text{max}} - z_{\text{min}}$) is $\ll R_*$, then we can forget about spherical stellar curvature and use $r \approx z$.

Many equations will look nicer if we use $\boxed{\mu = \cos\theta}$

Thus, we want to solve for $I_\nu(\mu, z)$. [reduced 7 scalars to 3]

....

Thus, the Boltzmann equation for the photon $f(\mathbf{p})$ is given by

$$\cancel{\frac{\partial f}{\partial t}} + \mathbf{v} \cdot \nabla f + \cancel{\mathbf{F} \cdot \nabla_p f} = \{\text{sources, sinks}\}$$

where we note that the system is time-steady and **non-relativistic** (i.e., there are no external forces strong enough to *bend* light-rays or change a photon's $\mathbf{p}$).

However, we're keeping the RHS general: there *are* local processes that can create and/or destroy photons (or change their $\mathbf{p}$, like collisions).

Remembering that $I_\nu \propto f$ and that $\mathbf{v} = c\hat{\mathbf{n}}$, the equation simplifies to

$$\hat{\mathbf{n}} \cdot \nabla I_\nu = \mathcal{S}_{\text{net}}$$

and the RHS has been boiled down to a single "net" rate of creation and/or destruction. What gives rise to $\mathcal{S}_{\text{net}} \neq 0$?

- Photons can be **absorbed** by atoms & ions. Rate $\mathcal{S}$ depends on local properties of the gas (e.g., ρ, T) and on "incoming" I_ν.
- Photons can be **emitted** "spontaneously" by atoms & ions. Rate $\mathcal{S}$ just depends on local properties.
- Photons can also be lost by being **scattered out** of a given direction $\hat{\mathbf{n}}$ or up/down in frequency ν. Rates depend on both local properties and on incoming I_ν.
- There can also be **scattering into the beam** which is essentially a net "creation" of photons in the given direction $\hat{\mathbf{n}}$ and frequency ν. The photons came from some *other* bins in direction & frequency, but those are kept track of elsewhere. Rates depend on local properties and on the I_ν in all those "other" directions, but not on the $I_\nu(\hat{\mathbf{n}})$ that we're following.

The left-hand side above can be simplified in plane-parallel geometry.

$$\hat{\mathbf{n}} = (\cos\theta)\hat{\mathbf{e}}_z + (\sin\theta)\hat{\mathbf{e}}_x$$

Spatially, I_ν is a function of z only, so $\quad \nabla I_\nu = \left(\frac{\partial I_\nu}{\partial z}\right)\hat{\mathbf{e}}_z$.

Thus, the equation of radiative transfer is

$$\cos\theta \, \frac{\partial I_\nu}{\partial z} = \mathcal{S}_{\text{net}}$$

which we can rewrite by breaking out the creation/destruction terms on the right side:

$$\mu \frac{\partial I_\nu}{\partial z} = j_\nu - \chi_\nu I_\nu$$

where j_ν is the emission coefficient, and χ_ν is the opacity. We've baked in the assumption that loss rates are always proportional to the "incoming" I_ν.

The opacity χ_ν has units of 1/length. Most sources of opacity depend on at least one power of the local density ρ, so we often define

$$\chi_\nu \;=\; \kappa_\nu \rho \;\;=\;\; [\kappa_\nu(\text{abs}) \,+\, \kappa_\nu(\text{scat})]\;\rho$$

where the κ's are typically just called **absorption coefficients** (even if they arise from a combination of true absorption & scattering).

Units of κ: (cm^2/g); i.e., a cross section per unit mass of "stuff" that's doing the absorbing.

We'll occasionally refer to $\ell_{\rm mfp} \equiv 1/\chi_\nu$ as the mean free path of a photon (irrespective of the type of process that halts the path every so often).

The gain term j_ν is often written as a product of χ_ν times an intensity-like quantity called the **source function** (i.e., $S_\nu = j_\nu/\chi_\nu$),

$$\boxed{\mu \frac{\partial I_\nu}{\partial z} \;=\; \chi_\nu \,(S_\nu - I_\nu)}$$

but keep in mind that although I_ν is a function of both μ and z, S_ν depends only on z (and not angle). What is S_ν? We'll get to that soon.

. . . .

Summary: Sources of Opacity

You probably know that the complete frequency-dependent κ_ν has contributions from multiple physical processes. For example:

- **Electron (Thomson) scattering:** unbound charged particles can scatter E&M wave trains. Cross section for free electrons:

  $$\sigma_e \;=\; \frac{8}{3}\pi r_e^2 \qquad\qquad r_e = \frac{e^2}{m_e c^2} \quad \text{(classical electron radius)}$$

  and if the photons are non-relativistic ($h\nu \ll mc^2$), the electrons provide a constant "gray" Thomson-scattering opacity

  $$\chi_e \;=\; 1/\ell_{\rm mfp} \;=\; \sigma_e n_e \;=\; \kappa_e \rho \qquad \rightsquigarrow \rightsquigarrow \qquad \kappa_e \;\approx\; 0.2(1+X)\ \text{cm}^2\ \text{g}^{-1}\ .$$

  Ions can scatter too, but because $\sigma \propto 1/m^2$, their effect is many orders of magnitude weaker.

- **Rayleigh scattering:** Neutral atoms can scatter photons, too. If the photon isn't energetic enough to boost bound electrons to excited states, it may still be (temporarily) absorbed, and cause the bound electron to oscillate a bit around its unperturbed energy level. Then it re-radiates, and the atom's effective cross section is

  $$\sigma_{\rm R} \approx \frac{\sigma_e}{[1-(\lambda/\lambda_{\rm eff})^2]^2} \qquad \text{(For H}^0\text{, } \lambda_{\rm eff} = 1026\ \text{Å.)}$$

  Reduces to Thomson scattering for $\lambda \ll \lambda_{\rm eff}$.

- **Free-free (bremsstrahlung) absorption & emission:** when an unbound electron undergoes a "hyperbolic" Coulomb collision with a positive ion Z_i, it can gain or lose kinetic energy. Gains [losses] occur if an incoming photon is absorbed [emitted].

  $$\chi_{\nu,\rm ff} = 3.7\times10^8\ \frac{Z_i^2\, n_i n_e}{T^{1/2}\ \nu^3}\left(1-e^{-h\nu/kT}\right)\bar{g}_{\rm ff} \qquad \text{cm}^{-1}$$

  and the free-free **Gaunt factor** $\bar{g}_{\rm ff}$ is related to the Coulomb logarithm. In astronomy, we often use

  $$\bar{g}_{\rm ff} \sim \begin{cases} 1\ , & h\nu \gtrsim k_{\rm B}T \\ \ln\left(\frac{k_{\rm B}T}{h\nu}\right)\ , & h\nu \ll k_{\rm B}T \end{cases}$$

  In the Rayleigh-Jeans tail ($h\nu \ll k_{\rm B}T$, good for radio),

  $$\chi_{\nu,\rm ff} \approx 0.018\ \frac{Z_i^2\, n_i n_e}{T^{3/2}\ \nu^2}\ \bar{g}_{\rm ff} \qquad \text{cm}^{-1}$$

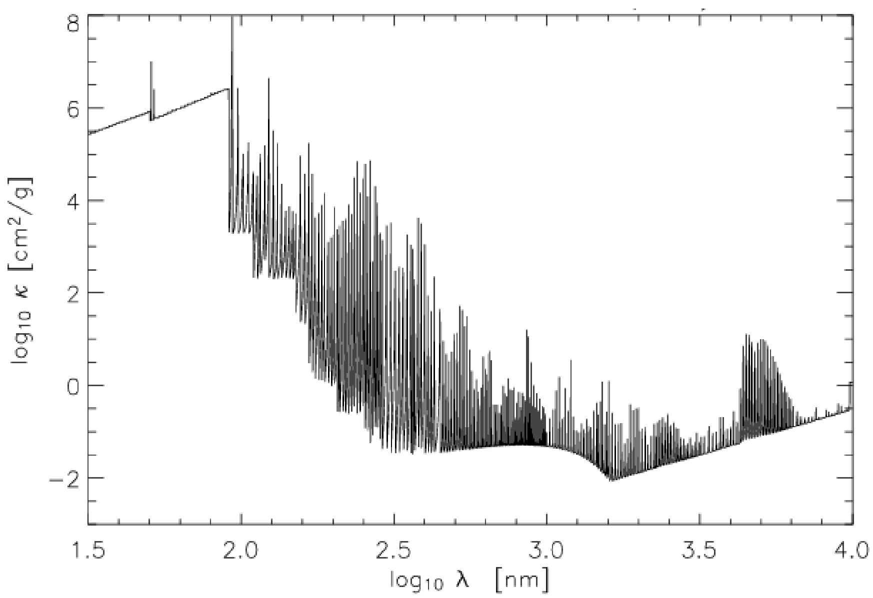
  and, in ionized plasmas, the Rosseland mean opacity is "Kramers:"

  $$\kappa_{\rm ff} \approx 3.7\times10^{22}(1-Z)(1+X)\rho T^{-7/2} \qquad \text{cm}^2\ \text{g}^{-1}.$$

- Bound electrons in an atom are good opacity sources. Energy from incoming photons may excite them to higher energy levels (**bound-bound abs. or scatt.**) or eject them via ionization (**bound-free abs.**). We'll examine each of these opacity sources in more detail later.

- An interesting hybrid of b-f & f-f is the **$\mathbf{H^-}$ (hydride) ion.** If an H^0 atom absorbs a photon in the presence of another free e^-, it can form a short-lived (unstable) H^- ion, which rapidly separates into $H^0 + e^-$.

Example of a sum over all sources (for one choice of ρ & T):



For many applications, all we need is the MEAN opacity, averaged over the whole ν spectrum. But how to do the weighting?

Planck mean? $$\langle\kappa\rangle = \frac{\int d\nu\ \kappa_\nu\ I_\nu}{\int d\nu\ I_\nu} \qquad \left(\begin{array}{c}\text{weights most highly } \textit{absorbed} \\ \text{parts of the spectrum}\end{array}\right)$$

That's sometimes useful, but for dense regions like stellar interiors, we want to know about the radiation that **gets out** (i.e., the bulk of the radiation that carries the energy out), so we want to weight it by "transparency..."

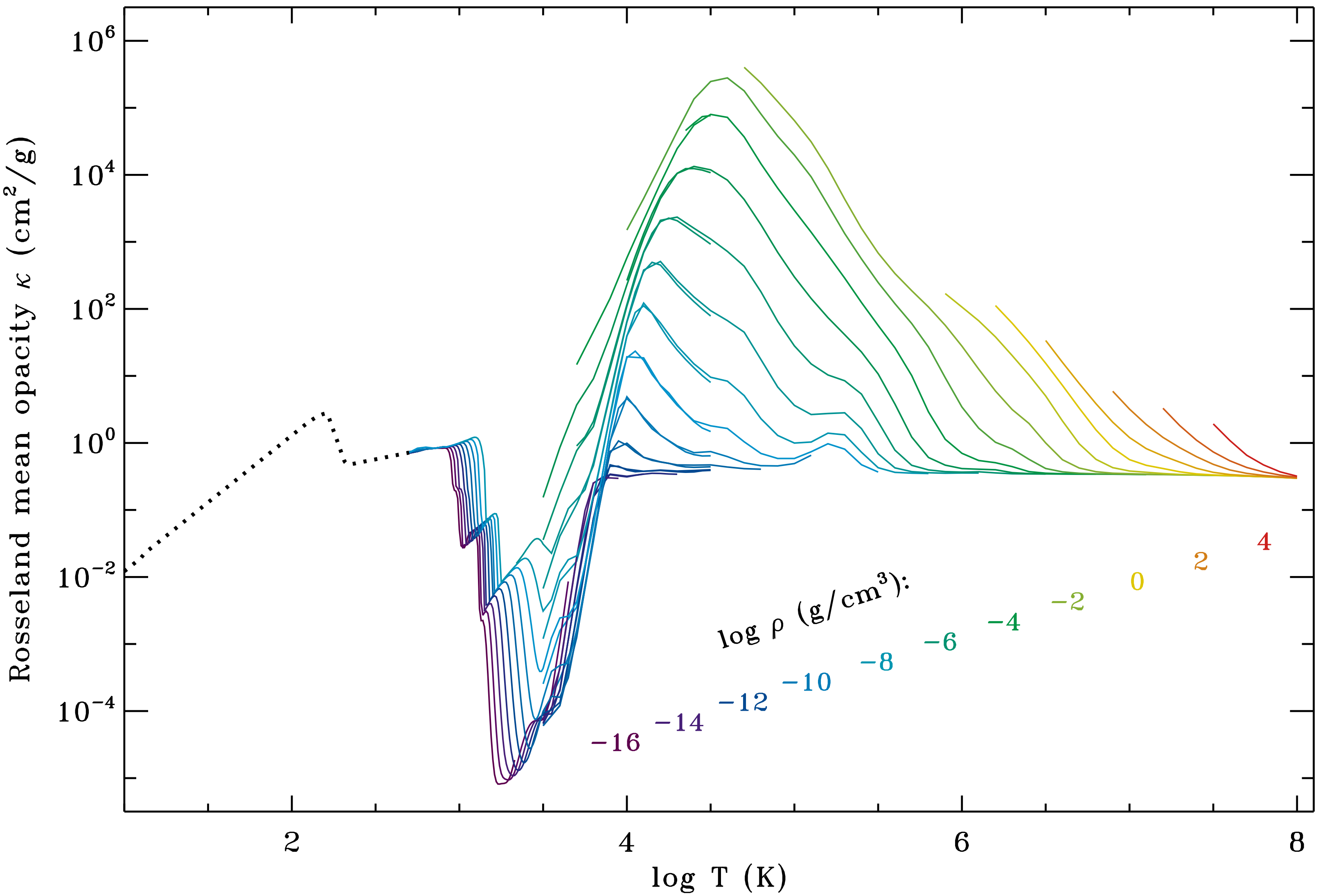
Rosseland mean: $$\frac{1}{\langle\kappa\rangle} = \frac{\int d\nu\ \frac{1}{\kappa_\nu}\ \frac{\partial I_\nu}{\partial r}}{\int d\nu\ \frac{\partial I_\nu}{\partial r}} \qquad \text{(we'll derive this later)}$$

Handout: Numerically computed Rosseland mean absorption coefficient versus T and ρ.

In stellar interiors ($T \approx 10^5$–10^7 K), $\kappa \propto \rho T^{-3.5}$ (Kramers' opacity).

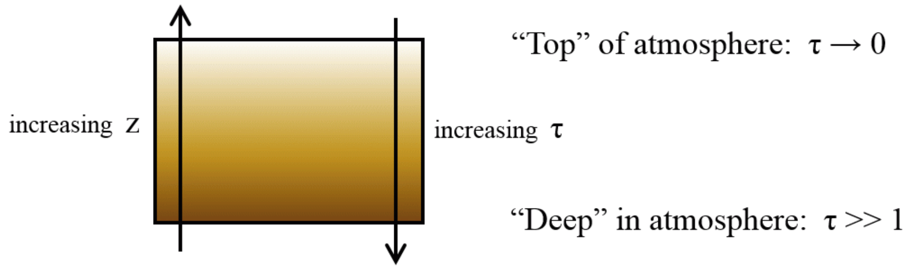
In stellar atmospheres ($T \approx 10^4$ K), it's more complicated, but often $\kappa \propto T^n$, where $n > 0$. Bound-free & H^- opacity.

In extremely hot plasmas ($T > 10^7$ K) with ∼no more atoms, opacity is dominated by electron scattering ($\kappa \approx$ constant).



Rosseland mean opacity κ, in units of cm^2 g^{-1}, shown versus temperature (x-axis) and density (multi-color curves, plotted once per decade), computed with $X = 0.7$ and $Z = 0.02$. Curves that extend from $\log T = 3.5$ to 8 are from the Opacity Project (opacities.osc.edu). Overlapping curves from $\log T = 2.7$ to 4.5 are from Ferguson et al. (2005, ApJ, 623, 585). The lowest-temperature region (black dotted curve) shows an estimate of ice-grain and metal-grain opacity from Stamatellos et al. (2007, A&A, 475, 37).

There's another thing we can do to simplify the *math* of the radiative transfer equation: define a new "depth" coordinate:



Optical depth τ_ν goes up as you go "deeper" into the atmosphere;

$$\boxed{d\tau_\nu \;=\; -\chi_\nu\, dz}$$

Note that it's frequency dependent. Some photons escape more easily than others.

- $\tau \ll 1$: "optically thin" regions where photons mainly flow freely; absorptions/scatterings are *rare.*
- $\tau \gg 1$: "optically thick" regions where photons are trapped. Absorptions/scatterings are so frequent that the distribution $I_\nu(\hat{\mathbf{n}})$ is rapidly randomized to be nearly **isotropic.**
- $\tau \sim 1$: this defines the **photosphere;** i.e., the layer where most of the photons <u>that we see</u> are created!

Since τ goes to zero as $z \to +\infty$ (high up), we can integrate to get the absolute optical depth "scale" as

$$\tau_\nu \;=\; \int_0^{\tau_\nu} d\tau'_\nu \;=\; -\int_{+\infty}^{z} dz'\; \chi_\nu(z') \;=\; \int_z^{+\infty} dz'\; \kappa_\nu(z')\,\rho(z')$$

and, lastly, the equation of radiative transfer simplifies to

$$\mu\,\frac{dI_\nu}{d\tau_\nu} \;=\; I_\nu - S_\nu$$

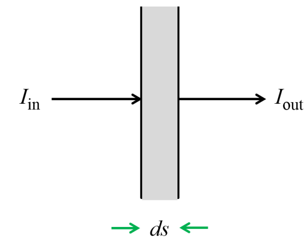
where partial deriv's were replaced by 'd' because we're following how the beams are processed vs. depth only (i.e., depth is the only coordinate that we'll differentiate with respect to) We'll just treat μ as a fixed parameter.

. . . .

Note that the RT equation is a 1st order ODE, which can be solved straightforwardly with the integrating factor method.

For completely arbitrary $S_\nu(\tau)$, there is a general **"formal solution"** that we'll examine later.

A very useful conceptual model is to assume S_ν is **constant** in a "slab" of gas/plasma, surrounded by vacuum. *Here,* it makes sense to define $d\tau_\nu = +\chi_\nu ds$ (where $ds = dz$ for $\mu = 1$).



For this sign definition, the RT equation is: $$\mu \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \ .$$

If we specify an incoming boundary condition $I_\nu = I_\nu^*$ at the left side of the slab (i.e., at $\tau_\nu = 0$) at normal incidence ($\mu = 1$), then the integrating-factor method gives the solution for the emergent intensity on the right:

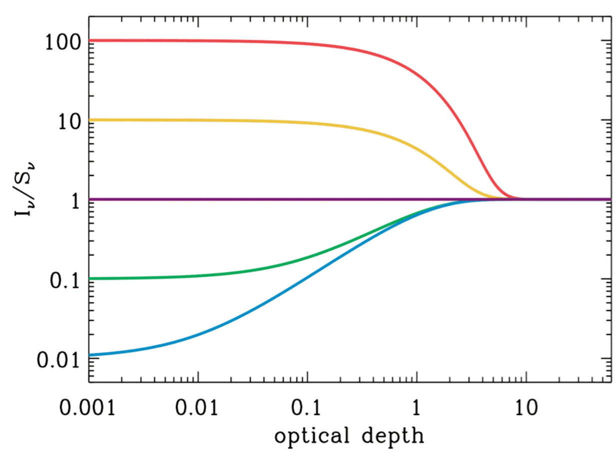
$$I_\nu \ = \ I_\nu^* \, e^{-\tau_\nu} \ + \ S_\nu \, (1 - e^{-\tau_\nu})$$

In the shallow layers ($\tau_\nu \ll 1$) we see $I_\nu \approx I_\nu^*$, which makes sense because nothing much has happened to the incident beam yet.

The deeper you go ($\tau_\nu \gg 1$), the more $I_\nu \to S_\nu$. The slab eventually "forgets" about the incident beam.

- The source function is like an "attractor" for the intensity.
- At every point along a ray, I_ν wants to approach S_ν and it will get there once the medium becomes optically thick.

Plot vs τ_ν for $I_\nu^* = \left\{ \begin{matrix} 100 \\ 10 \\ 1 \\ 0.1 \\ 0.01 \end{matrix} \right\} S_\nu$



We'll find this model illustrative for the so-called **Schuster reversing-layer** picture of spectral line formation.

In order to know how to evaluate S_ν, and thus how to compute $I_\nu(\mu)$, we need to know more about the thermodynamic state of the atmosphere. There are 3 traditional limiting cases:

STE: strict thermodynamic equilibrium:

- Particles obey Fermi-Dirac statistics, photons obey Bose-Einstein. All with a common T. All $\ell_{\rm mfp}$'s are short.
- i.e., all you need to specify the $f(\mathbf{p})$'s are: $\{n_{\rm particles}, n_{\rm photons}, T\}$.
- It's an idealized, isotropic *hohlraum* (blackbody cavity), with $I_\nu = B_\nu$.
- Because there are no spatial gradients in an isotropic homogeneous medium like this, the $dI_\nu/d\tau$ term in the RT equation is equal to zero. Thus, the RHS is zero, and $I_\nu = S_\nu$. Also, net flux is zero.
- Thus, $\boxed{I_\nu = S_\nu = B_\nu}$ in thermal equilibrium.

LTE: local thermodynamic equilibrium:

- For particles, conditions are STE within a collisional $\ell_{\rm mfp}$ (mean free path) of a given point.... i.e., everything depends on n & T only.
- In atmospheres, photons "see further:" photon $\ell_{\rm mfp} \gg$ particle $\ell_{\rm mfp}$. Thus, we *don't* assume photons obey equilibrium $f_{\rm B-E}(\mathbf{p})$.
- $I_\nu \neq B_\nu$, and I_ν is no longer isotropic.
- A nonzero *net flux* can thus flow through an atmosphere.
- If the distribution of atomic excited states (e^- energy level populations) is also STE, then emissions balance absorptions in such a way that **locally emitted photons** are sampled from a Planck function. Thus, $\boxed{S_\nu = B_\nu}$.

NLTE: non–{local thermodynamic equilibrium:}

- The most generally valid regime. All bets are off!
- Usually, NLTE matters in very low-ρ optically-thin regions.
- Atomic excited states evolve *slowly,* and can get "stuck" into weird, non-equilibrium states.

To sum up, we can ask the question: “Is knowing the local ρ and T sufficient for knowing the excitation & ionization (& emission) state of the gas?”

- In STE: Yes.
- In LTE: Yes.
- In NLTE: No! This means that **measured properties** of a given “pixel” aren’t guaranteed to be representative of the true plasma conditions at that location.

The remainder of lecture set 11 is devoted to:

- **Application 1:** Diagnosing radio sources from thermal bremsstrahlung spectra (1 page only).
- **Application 2:** Classical LTE stellar atmospheres (the rest).

Application 1: Thermal/Radio Diagnostics of Unknown Sources

Consider a distant cloud/star/galaxy. Many objects like this emit low-freq. radiation as if they are unilluminated, hot/ionized "slabs" in LTE:

$$\boxed{S_\nu \approx B_\nu} \quad \text{and our slab model gives:} \quad I_\nu = B_\nu(1-e^{-\tau_\nu}) \approx \begin{cases} \tau_\nu B_\nu\,, & \tau_\nu \ll 1 \\ B_\nu\,, & \tau_\nu \gg 1 \end{cases}$$

In the Rayleigh-Jeans tail ($h\nu \ll k_{\rm B}T$), $\quad B_\nu \approx 2k_{\rm B}T\,\nu^2/c^2$

and also, if the cloud is homogeneous with depth ℓ, then $\tau_\nu = \chi_\nu \ell$.

For this slab, thermal free-free emission gives

$$\tau_\nu = \chi_\nu \ell \propto \frac{n_e^2\,\ell}{T^{3/2}\,\nu^2} \qquad \text{(assuming fully ionized \& ignoring } \bar{g}_{\rm ff}\text{)}$$

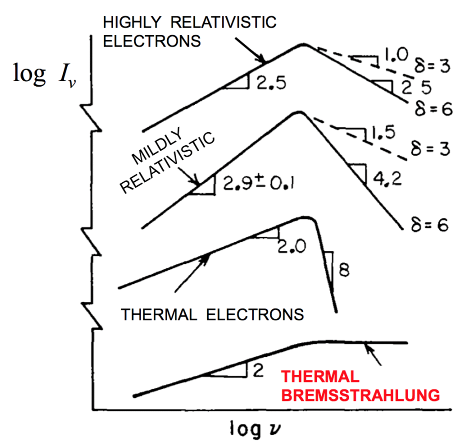
- The low-ν part of the spectrum corresponds to the most **optically thick** region. There, $I_\nu \propto T\nu^2$, so the spectrum allows T to be measured.
- At higher ν, it becomes **optically thin.** There, the ν^2 factors cancel out, and $I_\nu \propto$ constant (or $\sim \nu^{-0.1}$, if $\bar{g}_{\rm ff}$ is computed in gory detail).

  In this part of the spectrum, I_ν depends on both T and the so-called **emission measure,**

$$EM = \int dz\, n_e^2 \sim n_e^2\,\ell$$

  which is a diagnostic of "how much material is there" (along a presumably unresolved slab). *Warning:* some papers define $EM \sim n_e^2 \times$ volume.

- At even higher ν, either one starts to see the "Wien tail" of the Planck curve ($I_\nu \propto e^{-h\nu/kT}$), or if there are high-energy/relativistic particles, one may see signs of **synchrotron radiation.**



Adapted from Dulk (1985, *ARA&A,* 23, 169)

Application 2: Stellar Atmospheres

The rest of this document will be a derivation of some basic "laws" about the thin layer at a stellar surface (photosphere) that generates visible photons.

In this case, let us make one additional approximation: that the opacity κ_ν is **"gray"** (i.e., independent of frequency or wavelength).

We can then use un-subscripted intensity quantities (I, S, B) and identify those with **bolometric** values. Since one λ absorbs & scatters like any other, why not just integrate over the whole spectrum?

. . . .

Earlier I said that solving for $I(\mu, z)$ requires more than just the RT equation.

Another key constraint is **Energy Conservation.** Even though we've talked a lot about photons being created & destroyed, we can often assume that (in steady-state) the *total* energy emitted = *total* energy absorbed:

$$\int d\nu \int d\Omega \; \mathcal{S}_{\rm net} \; = \; \int d\nu \int d\Omega \; (j_\nu - \chi_\nu I_\nu) \; = \; 0 \; .$$

Exceptions:

- **Fusion** in stellar cores: there is truly "net creation" of photon energy from the nuclear reactions.
- **Convection** in stellar envelopes: if radiation can't transport the energy created by fusion, something else has to take over.

Outside of those regions, photon energy remains in that form. Even if photons at one ν are absorbed, it may heat up the plasma, which increases $B_\nu(T)$, and the total energy is conserved. Another way of writing energy conservation is

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} \; = \; 0 \qquad \text{or just} \quad \boxed{\nabla \cdot \mathbf{F} \; = \; 0} \quad \text{when time-steady.}$$

Think about analogies to the $\nabla \cdot \mathbf{B} = 0$ Maxwell's equation!

How is this derived? Integrate the RT equation over both $d\nu$ & $d\Omega$:

$$\int d\nu \int d\Omega \; \left(\mu \frac{dI_\nu}{dz} \right) \; = \; \int d\nu \int d\Omega \; (j_\nu - \chi_\nu I_\nu)$$

$$\text{i.e.,} \qquad dF/dz \; = \; 0 \; .$$

This means that in our tiny plane-parallel "box," we can be confident in assuming $F = \text{constant} \equiv \sigma T_{\text{eff}}^4$.

A natural consequence of plane-parallel symmetry is also that the vector $\mathbf{F}$ is pointing in the "vertical" direction.

You may scratch your head and wonder about **absorption.** Doesn't that eat away at the flux? No! Even if photons at one ν are absorbed, it may heat up the plasma (especially in LTE or STE), which increases $B_\nu(T)$, and more photons are emitted at other parts of the spectrum. As long as photons are just carrying the energy ($\epsilon = 0$), their total energy stays conserved.

Thus, for our gray (freq.-integrated) case, we know two things:

$$\boxed{\mu \frac{dI}{d\tau} = I - S} \quad \& \quad \boxed{\frac{dH}{d\tau} = 0} \qquad \text{(recall } F \propto H \text{ and } \tau \propto z\text{)}$$

We'll use the term **radiative equilibrium** for situations that satisfy both of the above equations.

. . . .

First, though, we should pause to think more about the **source function** S. There are 2 interesting limiting cases:

(1) In STE or LTE, emissions are balanced by absorptions in a way that depends only on the *thermalized* conditions of the particles. Thus, in most stellar atmospheres, it's safe to assume $\boxed{S \approx B}$ when the opacity is dominated by absorption/emission.

$$\text{Note:} \quad \text{the "gray" version:} \quad B = \int_0^\infty d\nu \; B_\nu(T) = \frac{\sigma T^4}{\pi} \; .$$

(2) In an NLTE region, what if the only source of opacity was scattering? Here, the source function S tells us how photons get scattered <u>into</u> the direction $\hat{\mathbf{n}}$ that we're following with the radiative transfer equation.

There are many physical mechanisms of scattering, and a lot of them are **isotropic;** i.e., the photon scatters into an essentially random angle, losing all "memory" of its incoming direction.

When scattering is isotropic, it doesn't really matter where it came from, so S should depend on the angle-average of I over *all directions.* In other words, for pure scattering, $\boxed{S \approx J}$.

In general, if the fraction of total κ_ν that comes from scattering is called a (sometimes called "albedo"), then

$$\boxed{S = (1-a)B + aJ \ .}$$

Sometimes you'll see $\epsilon = 1 - a$ (the fraction of total κ_ν that comes from true absorption/emission) defined as the "collisional destruction probability."

. . . .

The equation of radiative transfer is thus given by

$$\mu \frac{dI}{d\tau} = I - S = I - (1-a)B - aJ$$

and we can take the **0th moment** (again) by just multiplying each term by 1/2 and integrating over $d\mu$:

$$\frac{d}{d\tau}\left(\frac{1}{2}\int_{-1}^{+1} d\mu\,\mu\, I\right) = \left(\frac{1}{2}\int_{-1}^{+1} d\mu\, I\right) - (1-a)\left(\frac{1}{2}\int_{-1}^{+1} d\mu\, B\right) - a\left(\frac{1}{2}\int_{-1}^{+1} d\mu\, J\right)$$

In the 2 right-most terms, the factors of B and J are *not* functions of μ. They can be pulled out, and the integral is 2, which cancels out with the 1/2. Thus,

$$\begin{aligned} \frac{dH}{d\tau} &= J - (1-a)B - aJ \\ &= (1-a)(J - B) \end{aligned}$$

Remember energy conservation: $H =$ constant, so left-hand side $= 0$.

Thus, for a plane-parallel atmosphere in **radiative equilibrium,** we know that

- No flux is created or destroyed (i.e., $H =$ constant).
- $J = B$ (i.e., the mean radiation field is the Planck function), as long as $a \neq 1$.
- From definition of source function, this also means $\boxed{S = J = B}$. Same outcome as assuming LTE.

This doesn't mean that $I(\mu)$ is the Planck function.

How does this help us? Well, if we take the **first moment** of the equation of transfer, we multiply each term by $\mu/2$ and integrate as before.

Note that we can revert to earlier version

$$\mu \frac{dI}{d\tau} = I - S$$

because we know all about $S = B = J$ by now. Thus,

$$\frac{d}{d\tau}\left(\frac{1}{2}\int_{-1}^{+1} d\mu\,\mu^2\,I\right) = \left(\frac{1}{2}\int_{-1}^{+1} d\mu\,\mu\,I\right) - \left(\frac{1}{2}\int_{-1}^{+1} d\mu\,\mu\,S\right)$$

$$\frac{dK}{d\tau} = H - \{\text{zero!}\}$$

(noting that, after we pull out S, the rest is an <u>odd function</u>)

Because H is a constant with depth, this is a straightforward differential equation, which can be solved for

$$K(\tau) = H\tau + K_0$$

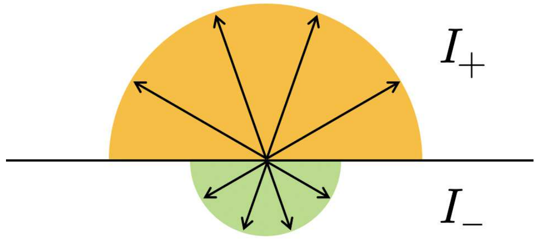
where K_0 is an integration constant that tells us the value of K at $\tau = 0$.

In the 1920s, Eddington thought about this problem, and he made 2 key approximations that allow us to move forward to something useful.

The "Eddington approximations" aren't perfect (often wrong by 10%–20%), but they're not terrible. They involve how the moments relate to one another:

- **1st approx** (diffusion approx): $\boxed{J = 3K}$. In radiative stellar interiors, we saw that this worked well for $I \approx I_0 + I_1 \cos\theta$. It's best for the deep, optically thick layers.
- **2nd approx** (surface approx): $\boxed{J_0 = 2H_0}$. This is true at the optically thin "top" of the atmosphere. If all radiation is going up and none is coming down, then "half of the sky" is filled, and half is empty. Thus, the flux is ~half of the mean intensity.

Thus, the 2nd approximation holds for $I_+ \neq 0$, but $I_- = 0$:



$$J = (I_+ + I_-)\ /\ 2$$

$$H = (I_+ - I_-)\ /\ 4$$

$$K = (I_+ + I_-)\ /\ 6$$

Let's plug in those approximations, one at a time...

$$\begin{aligned} J &= 3H\tau + 3K_0 && \text{(using the 1st)} \\ &= 3H\tau + J_0 \\ &= 3H\tau + 2H && \text{(using the 2nd)} \\ B &= 3H\left(\tau + \frac{2}{3}\right) && \text{(collecting terms \& recalling } S = B = J\text{)} \end{aligned}$$

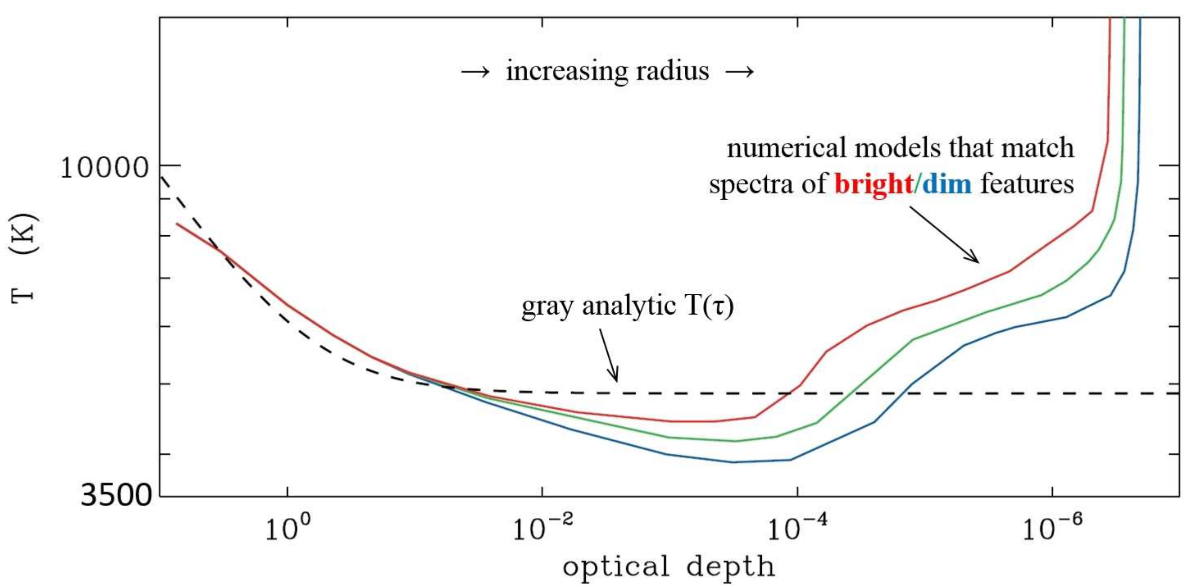
We've seen that $B = \sigma T^4/\pi$ and that

$$F = 4\pi H = \frac{L_*}{4\pi R_*^2} = \sigma T_{\text{eff}}^4 \quad \text{(which } \textit{defines } T_{\text{eff}}\text{)}$$

Plugging those in, we get the classical gray atmosphere temperature stratification law:

$$\boxed{T(\tau)^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right)}$$

Compare to empirically adjusted *solar* models (Vernazza et al. 1981):



Limiting values:

- As $\tau \to 0$, $T \to (1/2)^{1/4}\,T_{\rm eff} \approx 0.841\,T_{\rm eff}$.
  ... the real Sun can go down to $\sim$0.7 $T_{\rm eff}$
- When $\tau = 2/3$, $T = T_{\rm eff}$, and this height is traditionally called the optical "surface" or photosphere.
- For $\tau \gg 1$, T keeps growing. This is a close cousin to the *greenhouse effect,* in which more atmospheric absorption causes the atmosphere to heat up more and more. In fact, in the deep interior, this $T(\tau)$ gives the same answer as what we can get from our nearly-isotropic toy model ($I = I_0 + \mu I_1$). In that model the 1st moment of the RT equation is

$$\frac{dK}{dr} = \frac{1}{3}\frac{dJ}{dr} = -\chi H = -\frac{\chi F}{4\pi}$$

and we also have

$$J = I_0 = \frac{c}{4\pi}U = \frac{c}{4\pi}(aT^4) \ .$$

Thus,

$$\frac{dU}{dr} = 4aT^3\frac{dT}{dr} = -\frac{3\chi F}{c} \qquad \Longrightarrow \qquad \left(\frac{dT}{dr}\right)_{\rm rad} = -\frac{3\kappa\rho}{4acT^3}\left(\frac{L_r}{4\pi r^2}\right) \ .$$

defining the condition of **radiative diffusion** in stellar interiors.

A *higher luminosity* makes a stronger temperature gradient, because more flux = more anisotropy in $I(\mu)$, and thus more of a radial change in $U \propto T^4$.

Also, a *large opacity* makes a stronger temperature gradient, too... because large opacity = "good insulation" that allows the core to retain its heat while being surrounded by blackness of space.

How does this depth dependence help us understand the radiation emerging from the atmosphere? We'll now work out how $I(\mu)$ produces the so-called **limb darkening** effect.

To do that, I'll note that we've gotten pretty far without truly *solving* the RT equation for the "real" case of S varying with depth.

We've noted that if S is known, the RT equation is just an ordinary 1st order differential equation that can be solved with the integrating-factor method.

For an atmosphere with no "illumination from above," the so-called **formal solution** to the RT equation in a semi-infinite atmosphere is:

$$I(\mu,\tau) \;=\; \begin{cases} \int_\tau^\infty \frac{dt}{\mu} S(t) e^{(\tau-t)/\mu} & \mu > 0 \;\; \text{(upward rays)} \\ \int_\tau^0 \frac{dt}{\mu} S(t) e^{(\tau-t)/\mu} & \mu < 0 \;\; \text{(downward rays)} \end{cases}$$

This also illustrates why S is called the source function. I is "built up" by a weighted integral over S, over the regions that a photon will have traversed over its lifetime.

Aside: An even more general solution, which accounts for illumination from an outside source I_ν^* and has a slightly different definition for the direction in which τ_ν increases along the ray, is:

$$I_\nu(\tau_\nu) \;=\; \int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(\tau_\nu - t_\nu)} dt_\nu \;+\; I_\nu^* e^{-\tau_\nu} \;\;,$$

and the boundary condition $I_\nu = I_\nu^*$ is applied at the "top" of an atmospheric slab ($\tau_\nu = 0$). We saw earlier that if you take the limiting case of $S_\nu =$ constant, then this simplifies to

$$I_\nu \;=\; I_\nu^* \, e^{-\tau_\nu} \;+\; S_\nu \, (1 - e^{-\tau_\nu})$$

which should make sense, since the deeper you go ($\tau_\nu \gg 1$), the more $I_\nu \rightarrow S_\nu$.

For a 1D stellar atmosphere, we can work through the formal solution for:

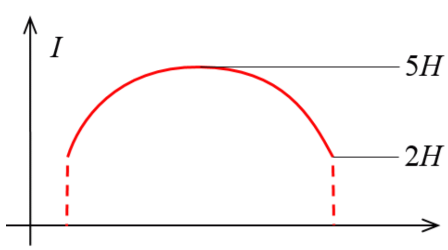
- The gray/Eddington solution for $S(\tau)$, and
- The specific value $\tau = 0$ (i.e., to solve for the "emergent intensity" coming from the top of the atmosphere).

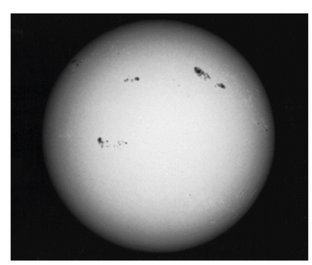
For $S(t) = 3H[t + (2/3)]$,

$$I(\mu, 0) \;=\; \frac{3H}{\mu}\int_0^\infty dt\,\left(t + \frac{2}{3}\right)e^{-t/\mu} \;=\; \rightsquigarrow\;\rightsquigarrow\; = \;\begin{cases} 3H\left(\mu + \frac{2}{3}\right) & \mu > 0\ , \\ 0 & \mu < 0\ . \end{cases}$$

This convenient "conversion" from τ-dependence to μ-dependence is called the **Eddington-Barbier relation.** (*Not* general for any $S(\tau)$.)

As we look across an image of the Sun, we see rays that go from
$\mu = 0$ (limb) to $\mu = 1$ (center) to $\mu = 0$ again (other limb):



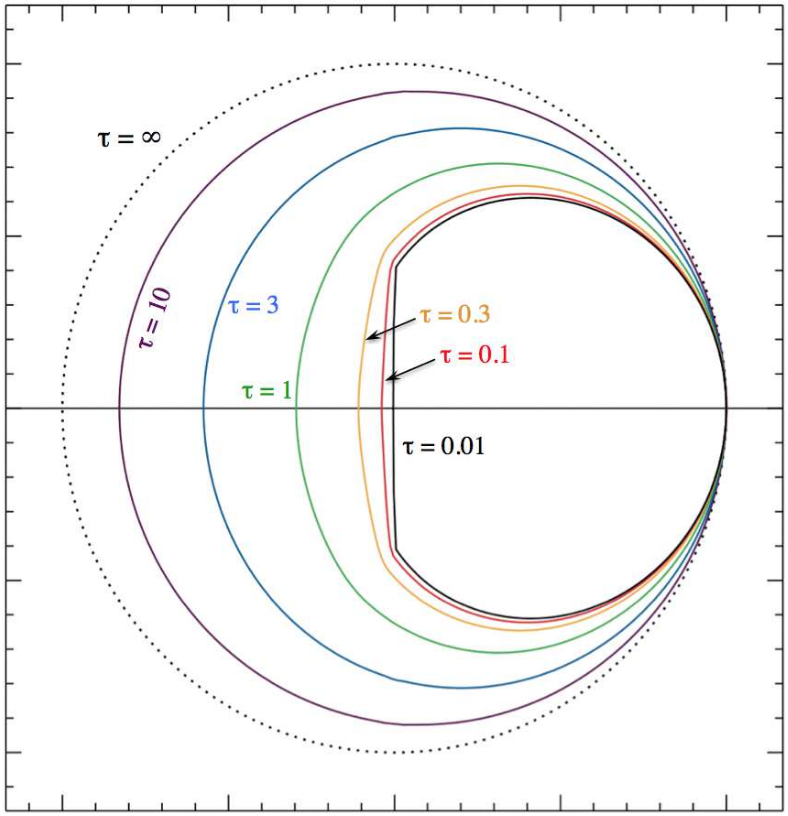


This relationship makes intuitive sense... the rays skimming the limb ($\mu \approx 0$) penetrate less deep (in the r direction) than the rays hitting disk-center.

Photons that come from "shallower" (higher up) regions emerge from *cooler* plasma... and $B_\nu(T)$ is lower for lower T.

Thus, limb darkening is a good (albeit indirect) way to probe the **depth dependence** of gas properties in a stellar atmosphere.

FYI, the full angle & depth dependence of the gray $I(\mu, \tau)$ can be plotted using polar coordinates...



We can use the derived $I(\mu, \tau)$ to check the validity (i.e., **self-consistentcy**) of the Eddington approximations.

The full gray/Eddington solution is

$$I(\mu, \tau) = \begin{cases} 3H\,(\tau + \mu + q) & \text{for } \mu > 0\ , \\ 3H\left[(\tau + \mu + q) - (\mu + q)\,e^{\tau/\mu}\right] & \text{for } \mu < 0\ . \end{cases}$$

where $q = 2/3$. We can use this to *re-compute* the J, H, K moments at the surface ($\tau = 0$):

$$J = \frac{7}{4}H \qquad H = H \qquad K = \frac{17}{24}H$$

Instead of $J = 2H$, we have $J = 1.75\,H$ (∼15% error).
Instead of $J = 3K$, we have $J = 2.471\,K$ (∼20% error).

Thus, the Eddington approximations are okay if we are willing to ignore inconsistencies at the 15–20% level.

($J = 3K$ gets better and better as $\tau \to \infty$.)

Another important question: How do we convert optical depth τ to **actual atmospheric height** z?

Recall the definition of optical depth:

$$\tau_\nu = \int_z^{+\infty} dz' \; \kappa_\nu(z')\,\rho(z')$$

One way we can get this into a form where we can solve for z is to approximate:

- In upper layers of atmosphere, $T \sim$ constant.
- Although the absorption coefficient is often approximated as $\kappa_\nu \approx \kappa_0 \rho^a T^b$, let's assume here that $a = b = 0$; i.e., that $\kappa_\nu \approx$ constant with depth, too.

For an isothermal (constant T) atmosphere, we derived earlier that

$$\rho(z) = \rho_0 \exp\left(-\frac{z}{h}\right) \qquad \text{where} \quad h = \frac{k_{\rm B}T}{\mu m_{\rm H} g} \approx \text{constant}\,.$$

The expression for $\tau(z)$ boils down to an integral over e^{-z}, and you can show that

$$\tau_\nu(z) = \kappa_\nu\,\rho(z)\,h\ .$$

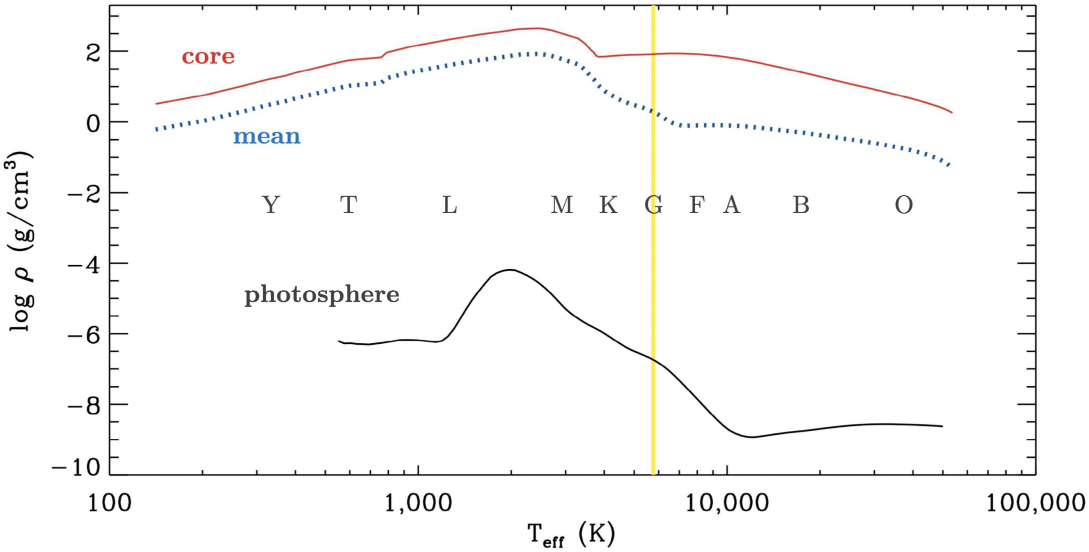
From the above, it's clear that τ_ν drops exponentially with scale height h, just like density.

κ_ν depends on photon frequency, but for a "mean" value of κ_0 we can solve for the **photospheric** density,

$$\kappa_0\rho_0 h \approx \frac{2}{3} \qquad \text{so} \qquad \rho_0 \approx \frac{2}{3\kappa_0 h}\ .$$

We can get a feel for the magnitude of the photospheric value of ρ by computing it for main-sequence stars & comparing with core and mean (mass over volume) values.

Densities for main-sequence stars:

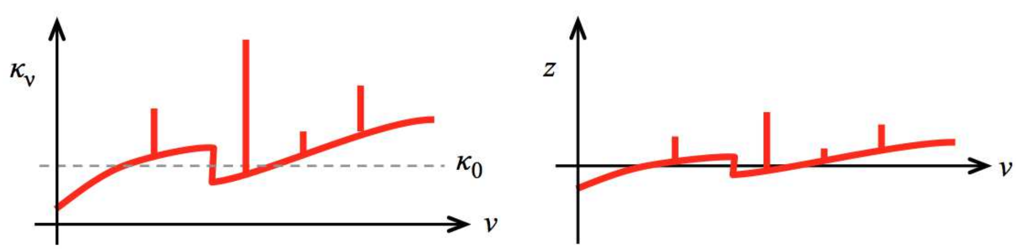


We shouldn't forget that κ_ν really varies as a function of ν, so different frequencies "see" the photophere at different <u>relative heights.</u>

$$\text{i.e.,}\quad \tau_\nu = \kappa_\nu \rho(z) h = \frac{2}{3} \qquad \text{means different things at different values of } \nu.$$

$$\rho(z) = \rho_0\, e^{-z/h} \approx \frac{2}{3\kappa_\nu h} \quad \rightsquigarrow \quad e^{-z/h} \approx \frac{2}{3\kappa_\nu \rho_0 h} \approx \frac{\kappa_0}{\kappa_\nu} \quad \rightsquigarrow \quad z \approx h \ln\left(\frac{\kappa_\nu}{\kappa_0}\right)$$

Thus, at some specific frequencies (with markedly HIGHER values of κ_ν), the photosphere occurs at a lower density & larger height:



There is one more interesting implication of seeing down to **different depths at different frequencies:**

Remember the Eddington-Barbier relation...

$$\text{If}\ \ S = S_0\left(\tau + \frac{2}{3}\right) \qquad \text{then} \qquad I(\mu, 0) = S_0\left(\mu + \frac{2}{3}\right)$$

and we take the 1st moment to find $H = S_0/3$.

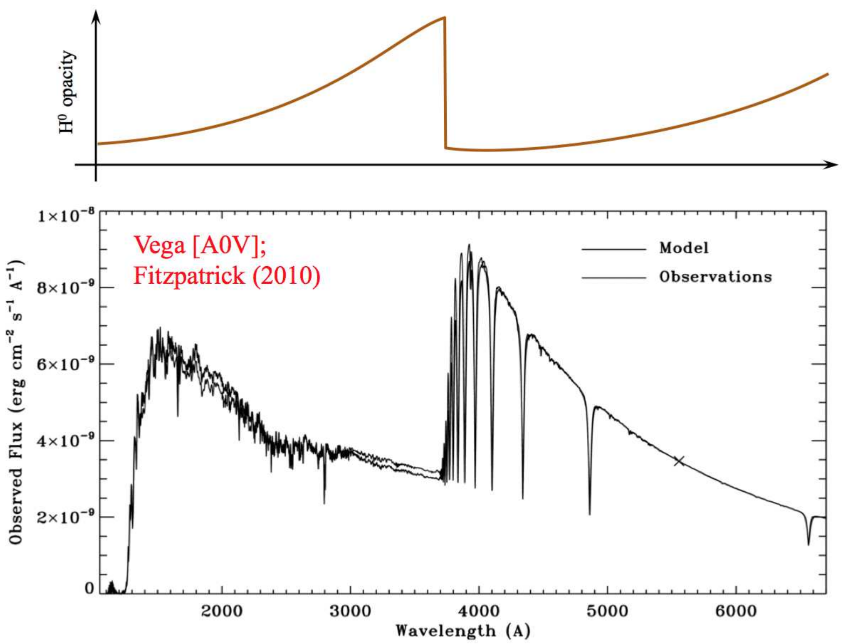
This gives the emergent flux:

$$\boxed{F_{\text{rad}} = \pi \left(\frac{4S_0}{3} \right) \; = \; \pi \left\{ S(\tau = 2/3) \right\} \;\; .}$$

This applies in a non-gray sense, too... for τ_ν and S_ν.

At wavelengths with low opacity, we see *deeper,* and that (usually) means we see a *higher* value of S_ν at $\tau_\nu = 2/3$.

An impressive example of this: H^0 edge **(Balmer jump)** at 3646 Å:



Thus, we have at least 2 good ways of probing the *depth dependence* of what's going on in a stellar atmosphere:

(1) Limb darkening
(2) Spectral features (b-b lines & b-f edges); more about those later.