Gravitational Dynamics: The $N \gg 1$ Body Problem

To do:

(1) Re-examination of collisional physics & the Boltzmann equation
(2) Collisionless particle orbits in "known" $\Phi(\mathbf{r})$, due to $N >>> 1$
(3) Collisionless equilibrium statistics & the virial theorem

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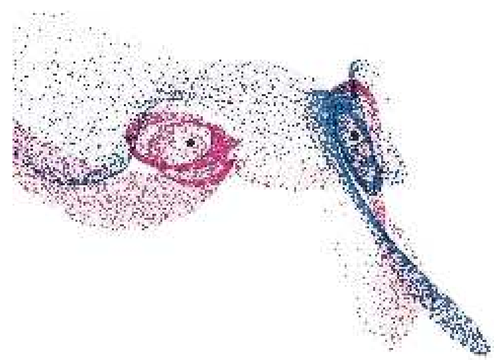
(1) Let's start by thinking about how collisions affect dynamics in things like galaxies & clusters.

Of course, since *"space is big,"* we're usually talking about gravitational **deflections** (hyperbolic trajectories due to multiple pairwise encounters). But sometimes there are direct collisions!

These days, treating this process is often the purview of numerical N-body simulation codes.



Recent records:

2010: Millennium XXL, $N = 3 \times 10^{11}$
2017: PKDGRAV3, $N = 2 \times 10^{12}$
2019: Quijote, $N = 8.5 \times 10^{12}$ **!!!**

Challenges:

- **Speed:** Computing mutual $\mathbf{g}$ forces scales as $\sim N^2$. However, the Barnes & Hut TREE method divides up the volume into nested sets of cells. *Nearby* pairs are computed in full. *Distant* regions are grouped as large quasi-particles. CPU time scales as $N \ln N$ ("linearithmic") instead of N^2.

- **Accuracy:** If integrating over long times, small errors can accumulate. Some celestial mechanics codes use *symplectic integration* $\longrightarrow$ reformulates equations using Hamiltonian/canonical coordinates ($\mathbf{p}$ & $\mathbf{q}$) that more automatically conserve energy & momentum when discretized.

  (Or try the 15th-order-accurate Rein & Spiegel (2015, *MNRAS,* 446, 1424) method?!)

For large numbers of stars, we can follow the evolution of their (smooth) phase-space distribution function $f(\mathbf{r}, \mathbf{p}, t)$ via the full Boltzmann equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}}$$

How do we treat the summed gravitational effect of "other stars" on the stars we're following with $f(\mathbf{r}, \mathbf{p}, t)$?

- In some situations, we can ignore collisions & assume the effects are due to a known $\boxed{\text{smooth}}$ potential,

$$\mathbf{F} = -\nabla U = -m\nabla\Phi \qquad \Longrightarrow \qquad \Phi(\mathbf{r}) = -G \int d^3\mathbf{r}' \, \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}$$

  where ρ is the spatial mass density of... everything that gravitates (e.g., stars, gas, dark matter). ρ is a zeroth moment of f.

- However, in some situations the effects of individual, pairwise, $\boxed{\text{grainy}}$ collisions have an impact on the overall evolution of the system, and we must deal with the infamous collision term on the RHS.

(Gravitational collision theory is a bit **nastier** than Coulomb collisions in a plasma. For the latter, at least we had *Debye screening* to cancel out the forces at large separations!)

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We will be investigating the **timescales** over which various collisional & collisionless processes affect an "average" star moving through the system.

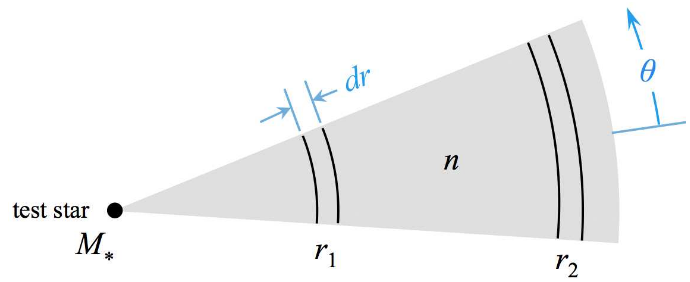
But first, we should figure out which interactions matter most: close ones, or far-away ones.

For collisions, sometimes it's temping to pay the most attention to the "nearest neighbors." However, gravity is a **long-range** force...

Consider a uniform-density distribution of stars (all with M_*) in some cone-shaped wedge of space with constant density of stars n.

The nearby stars exert a stronger force, but there are *more* stars in "wedges" that are further away ($dr \ll r_i$):

What is the magnitude of the gravitational force on the test star (at the origin) due to the various segments at r_1, r_2, etc.?



$$F(r_i) \approx N_{\rm stars} \frac{GM_*^2}{r_i^2} \qquad \text{where} \qquad N_{\rm stars}(r_i) = nV = n\,dr\,2\pi r_i^2 (1 - \cos\theta)$$

Thus,

$$F(r_i) \approx n\,dr\,2\pi\,(1 - \cos\theta)\,GM_*^2 \qquad \text{independent of } r_i\text{ !}$$

and we conclude that distant stars contribute **comparably** as nearby stars.

Of course, if n is *completely* uniform ($\theta = \pi$; full sphere around the test star), then net $\mathbf{F} = 0$. But most galaxies/clusters have large-scale asymmetries.

Moral: We need to worry about weak, small-angle interactions, just like in plasma Coulomb collisions... and we can't even cut it off at $b_{\rm max}$.

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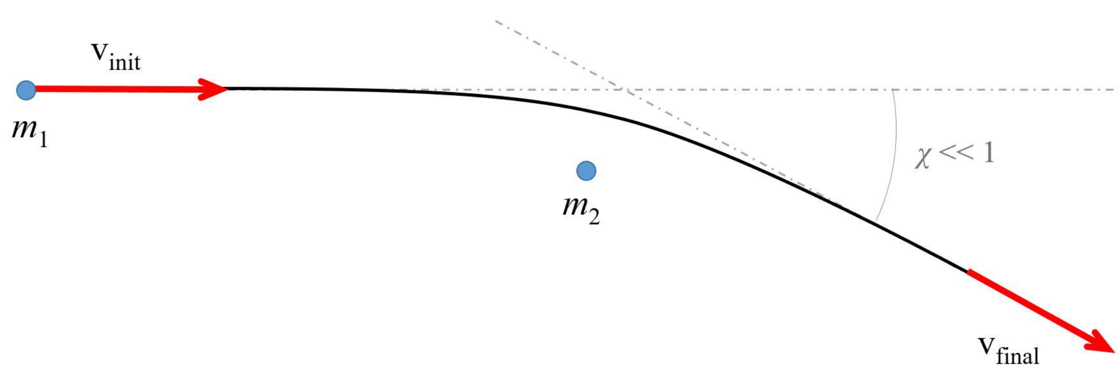
Consider 2 point masses (m_1 & m_2) undergoing a hyperbolic scattering event. In a frame centered on m_2,



For a single small-angle scattering event ($b \gg b_{\rm min}$), we saw that

$$|\Delta \mathbf{v}_\perp| \approx v_{\rm init} \frac{b_{\rm min}}{b} \qquad \text{where here,} \qquad b_{\rm min} = \frac{2Gm_1m_2}{m_{12}v_{\rm init}^2} = \frac{2G(m_1+m_2)}{v_{\rm init}^2}$$

noting that we must replace q_1q_2 by Gm_1m_2.

Binney & Tremaine (and others) tend to work with the 90° scattering impact parameter

$$b_{90} = \frac{b_{\rm min}}{2} = \frac{2GM_*}{v^2} \qquad (\text{for } m_1 = m_2 = M_*)$$

where $v \approx$ a typical speed in the system.

Using this as our effective $b_{\rm min}$, we can sum up multiple small-angle scattering events for a star of mass M_*, in a field of other identical stars with number density n, accumulating over a time interval Δt, as before...

$$\langle \Delta v_\perp^2 \rangle = \frac{8\pi G^2 M_*^2 \, n \, \Delta t}{v} \ln \Lambda$$

where the Coulomb logarithm $\Lambda = b_{\rm max}/b_{\rm min}$, and we saw above that it's prudent to assume $b_{\rm max} \approx R_{\rm gal}$, the overall size of the galactic system.

For larger elapsed times Δt, the dispersion $\langle \Delta v_\perp^2 \rangle$ grows.

How long will it take for the test star to undergo a sufficient number of scatterings such that $\langle \Delta v_\perp^2 \rangle$ becomes $\approx v^2$? (i.e., how long will it take for the star's orbit to fully randomize/thermalize & "forget" its initial condition?)

We obtain the **relaxation time** by setting $\langle \Delta v_\perp^2 \rangle = v^2$ and solving for Δt,

$$\Delta t \equiv t_{\rm relax} = \frac{v^3}{8\pi G^2 M_*^2 \, n \, \ln \Lambda} \; .$$

We would also like to know how $t_{\rm relax}$ compares with:

- $t_{\rm life}$ (the lifetime or age of the system)
- $t_{\rm cross}$ (the dynamical time or "crossing time" $\sim R_{\rm gal}/v$)

If $t_{\rm relax}$ is small, then **collisions** are important to the system's evolution.

Let's try to simplify the expression for $t_{\rm relax}$. Assume a large, spherical, self-gravitating system like a galaxy, filled with N identical stars.

It's not terrible to estimate that the star is orbiting ~circularly on the outskirts of a ~centralized mass distribution, with

$$v^2 \approx \frac{GM_{\rm gal}}{R_{\rm gal}} \qquad \text{(i.e., vis-viva)}$$

and that for a ~spherical system containing N stars of mass M_*,

$$M_{\rm gal} = NM_* \qquad \text{and} \qquad n = \frac{N}{\text{volume}} \approx \frac{3N}{4\pi R_{\rm gal}^3} \ .$$

$$\text{Thus,} \quad \Lambda = \frac{b_{\rm max}}{b_{\rm min}} \approx \frac{R_{\rm gal}}{b_{90}} = \frac{R_{\rm gal}}{2GM_*/v^2} = \frac{M_{\rm gal}}{2M_*} \sim \frac{N}{2} \qquad (\text{so } \ln\Lambda \sim \ln N)$$

and we can use all of the above to express the ratio

$$\frac{t_{\rm relax}}{t_{\rm cross}} = \frac{v^4/R_{\rm gal}}{8\pi G^2 M_*^2 n \ln\Lambda} \approx \rightsquigarrow \rightsquigarrow \approx \frac{N}{6 \ln N}$$

and this ratio often is $\gg 1$. More precise numbers are given by Meiron & Kocsis (arXiv:1801.01123). Let's examine some example environments:

	$R_{\rm gal}$	rms v	$t_{\rm cross}$	$t_{\rm life}$
Milky Way (MW) Galaxy	10 kpc	100 km/s	100 Myr	~10 Gyr
Globular cluster (GC)	5–10 pc	10 km/s	0.5–1 Myr	~10 Gyr
Young open cluster (OC)	2 pc	0.3 km/s	5 Myr	50–100 Myr

- **MW:** $\boxed{N \approx 10^{11}}$ so $t_{\rm relax}/t_{\rm cross} \approx 6 \times 10^8$. However, given the value of $t_{\rm cross}$, we estimate $t_{\rm relax} \approx 10^{17}$ years $\gg$ age of the universe! Thus, collisions are irrelevant in our Galaxy; it's safe to use collisionless Φ.
- **GC:** $\boxed{N \approx 10^5}$ so $t_{\rm relax}/t_{\rm cross} \approx 1400$. Globulars are small, so $t_{\rm cross}$ isn't that big. Thus, $t_{\rm relax} \approx$ 0.1–1 Gyr, shorter than its lifetime. Collisions probably have influenced the system over its lifetime. (That's probably why it's *spherical*... it has relaxed!)
- **OC:** $\boxed{N \approx 10^2}$ so $t_{\rm relax}/t_{\rm cross} \approx 3$. Collisions must play a key role in the dynamics of a young, star-forming cluster! They're smaller than GCs, but their velocity dispersions are tiny. Thus, $t_{\rm relax} \sim 15$ Myr, a few times shorter than its lifetime. Maybe collisional processes are partially responsible for *dissolving* clusters like this?

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Okay, so what happens when star–star collisions are important?

In general, the evolution of one object through a field of neighbors obeys much of the same physics as charged particles undergoing Coulomb collisions.

Recall that a "fast" object will be slowed down and thermalized via Chandrasekhar's **dynamical friction** (cumulative effect of many deflections).

Examples:

- **Galactic cannibalism:** Satellite galaxies (GCs, LMC, SMC) that orbit a massive central galaxy lose bulk kinetic energy to thermal energy (i.e., $v_{\rm th} \uparrow$). Their orbits decay, and they spiral in toward the central galaxy.
- **BH consolidation:** If one finds a supermassive black hole in the outskirts of a large galaxy (say, after a merger event), it will quickly ($t \lesssim 3$ Gyr) inspiral to the galactic center. The same goes for stellar-mass BH or NS anywhere in the central $\sim$10 pc of the galaxy.

Let's examine the first idea (satellite inspiral) in a bit more detail, following Binney & Tremaine, 2nd ed., § 8.1.1.

Consider an object of mass $M_{\rm obj}$ orbiting a galaxy of mass $M_{\rm gal}$ at a distance r. Recall estimates...

$$v^2 \approx \frac{GM_{\rm gal}}{r} \qquad t_{\rm cross} \approx \frac{r}{v} \, .$$

As the object plows through, it collides with stars, resuling in:
spread in $v_\perp$ ... slowing down in $v_\parallel$. For an orbiting object, $dv_\parallel/dt < 0$ means that it loses angular momentum. B&T solved for the **friction time** over which it will spiral into $r \approx 0$,

$$t_{\rm fric} \approx \frac{1.17}{\ln\Lambda} \frac{r^2 v}{GM_{\rm obj}} \, .$$

Makes sense, because if we use above assumptions of a circular orbit, we get

$$\frac{t_{\rm fric}}{t_{\rm cross}} \approx 1.17 \frac{\tilde{N}}{\ln \tilde{N}} \qquad \text{where} \qquad \tilde{N} = \frac{M_{\rm gal}}{M_{\rm obj}}$$

which is similar to the $t_{\rm relax}/t_{\rm cross}$ result we obtained above.

For our neigbors (LMC, SMC), $t_{\rm fric} \sim 10$ Gyr. They'll spiral in eventually.

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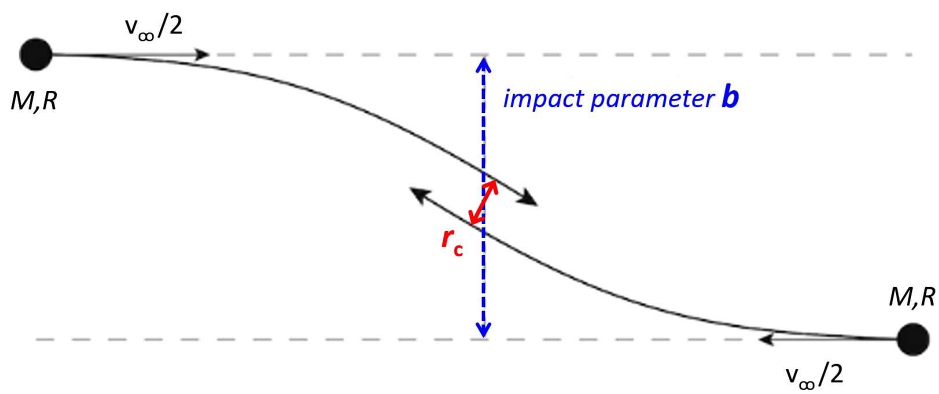
Gravitational Focusing & Dynamical Friction

Lastly, let's examine the **rare** cases of large deflections and actual star–star impacts, which can happen in the *dense cores* of GC/OC systems.

(Interestingly, this theory also applies to how planetesimals gobble up debris to grow into planets!)

Above, we estimated $t_{\rm relax}$ as a sum of $\gg 1$ weak, small-angle collisions. However, even *one* large-angle collision can be hugely important to a given star's trajectory. Let's figure out how frequent those are.

Consider 2 stars of equal masses M and radii R, approaching one another with relative velocity v_∞ and impact parameter b.   In the CM frame:



This is a hyperbolic Keplerian orbit, but we can use some basic physics to figure out whether or not they will physically collide.

The actual criterion for "will they collide?" is $\boxed{r_c \leq 2R.}$

But the outcome depends on their masses:

- For small enough M, gravity may be ignored, and $r_c \approx b$. Thus, they will collide only if $b \leq 2R$.
- For large M, there will be mutual **gravitational focusing** that decreases r_c (i.e., increasing the cross section for direct collisions).

In the two-body problem, E & ℓ are constants of motion. First consider total energy,

$$E = \tfrac{1}{2}mv^2 - \frac{GM^2}{r} = \text{constant}, \qquad \text{where reduced mass is} \quad m = M/2 \ .$$

and v is the relative velocity between the two objects.

Evaluate E both at $r \to \infty$ ($U \to 0$) and at closest approach ($r = r_c$ and $v = v_{\rm max}$),

$$\tfrac{1}{4} M v_\infty^2 \;=\; \tfrac{1}{4} M v_{\rm max}^2 \;-\; \frac{GM^2}{r_c} \;.$$

Second, consider the total angular momentum of the system (measured about the center of mass)

$$\ell \;=\; \left| \sum_{i=1}^{2} (\mathbf{r}_i \times \mathbf{p}_i) \right| \;=\; 2M |\mathbf{r}_1 \times \mathbf{v}_1| \;=\; 2M\, r_{\perp 1}\, v_1$$

where the last step is due to symmetry about the center, and $r_{\perp 1}$ is the projected distance (from the CM) perpendicular to the star's velocity vector.

Thus, for ℓ being constant at $r \to \infty$ and at closest approach,

$$2M \left(\frac{b}{2} \right) \left(\frac{v_\infty}{2} \right) \;=\; 2M \left(\frac{r_c}{2} \right) \left(\frac{v_{\rm max}}{2} \right) \qquad \Longrightarrow \qquad \frac{v_{\rm max}}{v_\infty} \;=\; \frac{b}{r_c} \;.$$

If gravity plays a major role, then $b \gg r_c$, so that $v_{\rm max} \gg v_\infty$.

Thus, inserting $v_{\rm max}$ into energy conservation, $\quad \tfrac{1}{4} M v_\infty^2 = \tfrac{1}{4} M \left(\dfrac{b^2 v_\infty^2}{r_c^2} \right) - \dfrac{GM^2}{r_c} \;.$

How much focusing actually occurs? We could solve for r_c in terms of the known initial conditions b and v_∞, but it's also insightful to solve for b,

$$b \;=\; r_c \sqrt{1 + \frac{4GM}{r_c\, v_\infty^2}}$$

and it makes sense that for small masses (weak deflection), $b \approx r_c$.

If we want to study actual physical impacts, this tells us that a stellar collision will occur only when $r_c \le 2R$, and

$$\text{when the initial } b \text{ is} \quad \le \quad b_{\rm coll} \;\equiv\; 2R \sqrt{1 + \frac{2GM}{R\, v_\infty^2}} \;\; .$$

The collision criterion is $b \le b_{\rm coll}$, and $b_{\rm coll}$ is **larger** than in the weak-gravity limit (in which $b \le 2R$ ensured a collision).

$$\text{For a spherical star,} \quad V_{\rm esc}^2 = \frac{2GM}{R} \;, \quad \text{so} \quad b_{\rm coll} \,=\, 2R\sqrt{1 + \frac{V_{\rm esc}^2}{v_\infty^2}}$$

and the effective **cross section** for direct collisions is $\boxed{\sigma_{\rm coll} = \pi b_{\rm coll}^2}$.

For a solar-type star ($V_{\rm esc} \approx 600$ km/s) in a typical GC with r.m.s. $v_\infty \approx 10$ km/s, the gravitational focusing enhancement in the cross section is a factor of $(1 + 60^2) \approx 3600$. A huge factor!

We've seen that the mean collision rate is given by

$$\nu_{\rm coll} \,=\, t_{\rm coll}^{-1} \,\approx\, n\, v_\infty\, \sigma_{\rm coll}$$

where we can think about v_∞ as the most-probable relative speed between any two members of the cluster.

Binney & Tremaine (§ 7.5.8) work this out more exactly for a Maxwellian distribution of speeds (with thermal speed $v_{\rm th} = v_\sigma \sqrt{2}$).

The numerical factors are slightly different, but overall it's similar to what we'd get from the above estimates. They get:

$$t_{\rm coll}^{-1} \,=\, 16\sqrt{\frac{\pi}{2}}\, n\, v_{\rm th}\, R^2 \left(1 \,+\, \underbrace{\frac{V_{\rm esc}^2}{v_{\rm th}^2}}_{\Theta} \right)$$

where $\Theta =$ the Safronov number.

For the dense central regions of a GC, $n \sim 10^5$ stars/pc^3, and for similar parameters as above, $t_{\rm coll} \sim 10^{11}$ years ($>$ age of universe). However, *without* focusing, it would be $\Theta \sim 3600$ times longer ($> 10^{14}$ years)!

Recall that for a GC, $t_{\rm relax} \approx 10^9$ years. One can show that

$$\frac{t_{\rm coll}}{t_{\rm relax}} \,\approx\, 0.4\, \ln \Lambda\, \frac{\Theta^2}{1 + \Theta}$$

which is about 100 in our example. This means large-angle collisions are "less important" than the summed effect of small-angle collisions, just like in plasmas.

There are several interesting implications of large Safronov number...

- Many galaxies and clusters undergo **core collapse** into a supermassive black hole. This is also a gravitationally dominated process. As $n \uparrow$, collisional relaxation speeds up, too.

  Binney & Tremaine (§ 7.5.3) show that core collapse occurs over a time that scales ∼universally like

  $$t_{\text{collapse}} \approx 300\, t_{\text{relax}} \quad .$$

  Thus, for young systems (not yet collapsed), t_{coll} is always $\gg$ both $\{t_{\text{relax}}, t_{\text{collapse}}\}$, so direct collisions are negligible.

  However, for older systems (in late stages of collapse), $v_{\text{th}} \uparrow$ and Θ drops to values of only 5–50. t_{coll} is now only slightly bigger than t_{relax}. We can see various results of stellar collisions actually occurring:

  – 2 main-sequence stars → **"blue stragglers"** (seen in old GCs)

  – Red giant + NS → spins up the NS; **"recycled pulsars"**

  – Close binary + 3rd star → a huge momentum transfer ejects **"hypervelocity stars"** (see homework)

- Let's switch gears to **protoplanetary disks** full of planetesimals. How do planets grow? In the cold, icy outer solar system, each direct collision is likely to "stick" as an agglomeration. Thus, we can write

  $$\frac{dM}{dt} \approx \frac{\Delta M}{\Delta t} \approx \frac{M}{t_{\text{coll}}}$$

  Note that the mass on the left side is the one body that we're tracking in time (as it grows), but on the right side it's the mass of each "target" planetesimal that it grabs every t_{coll}. Thus,

  $$\frac{dM}{dt} \approx M\, n\, v_{\text{th}}\, \sigma_{\text{coll}} \approx \rho\, v_{\text{th}}\, R^2 \left(1 + \underbrace{\frac{2GM}{R v_{\text{th}}^2}}_{\Theta} \right) \qquad \text{(Safronov's equation).}$$

  We assume ρ & v_{th} are constants, and $M \propto R^3$ for icy/rocky material.

The low-mass limit ($\Theta \ll 1$) is valid in the early stages of agglomeration. There, we have

$$\frac{dM}{dt} \propto R^2 \propto M^{2/3} \qquad \Longrightarrow \qquad M^{1/3} \propto R \propto t - t_0 \qquad \text{(slow growth)}$$

However, at later stages, the high-mass limit ($\Theta \gg 1$) is valid, with

$$\frac{dM}{dt} \propto R^4 \propto M^{4/3} \qquad \Longrightarrow \qquad -M^{-1/3} \propto t - t_0 \qquad \text{i.e.,} \qquad R \propto \frac{1}{t_0 - t}$$

i.e., a rapid increase that blows up at a finite time. **Runaway growth** like this can take a 1 km planetesimal up to a 1000 km protoplanet.

Agglomeration stops when either:

(a) M grows to the point of really being able to *perturb* orbits of nearby planetesimals. "Eccentricity pumping" scatters them away, like in an unstable MMR.

(b) The new protoplanet has cleared out its local orbit of other planetesimals – i.e., eventually $\rho \to 0$.

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(2) Collisionless Potential Theory

Let's now put aside collisions and think about how stars move in large systems like the Milky Way ($N \sim 10^{11}$) for which the net gravitational effect of "all other" objects (stars, gas, dark matter) can be treated via a known potential $\Phi(\mathbf{r})$.

Earlier, we reviewed the basics of the Poisson equation...

$$\nabla^2 \Phi \;=\; 4\pi G \rho \qquad \Longleftrightarrow \qquad \Phi(\mathbf{r}) \;=\; -G \int d^3\mathbf{r}' \, \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}$$

i.e., if we know *either* ρ or Φ, in principle we can solve for the other one. For full derivation, using Gauss' divergence theorem, see Binney & Tremaine § 2.1.

Spherically symmetric distributions of mass can be treated by summing up the mass in concentric shells,

$$dM_r \;=\; 4\pi r^2 \, \rho(r) \, dr \quad , \qquad \text{and} \qquad M_r(r) \;=\; \int_0^r dM_{r'}(r')$$

where the integrated $M_r(r)$ is the total mass interior to r.

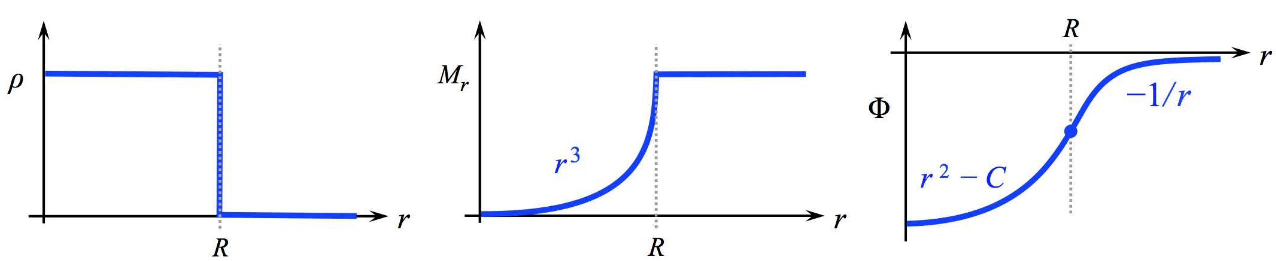
In the spherical case, the gravitational acceleration depends only on mass interior to r,

$$\mathbf{g} \;=\; -\frac{\partial \Phi}{\partial r} \hat{\mathbf{e}}_r \;=\; -\frac{G M_r(r)}{r^2} \hat{\mathbf{e}}_r$$

but the full potential formally depends on the mass distribution both inside and outside,

$$\Phi(r) \;=\; -\frac{G M_r(r)}{r} \;-\; G \int_r^\infty \frac{dM_{r'}(r')}{r'} \;\; .$$

Example: **constant-density sphere** (used a lot in ASTR–5700):



In many realistic/extended/centrally-condensed spherical systems, we might expect:

- Innermost regions would behave like the constant–ρ model.
- Outer regions would behave as if "all" interior mass dominates the gravitational force; i.e., $\Phi \sim -GM/r$.

Thus, people have searched for functions that bridge both limits:

(a) **Plummer's (1911)** model:

$$\Phi = -\frac{GM}{\sqrt{r^2+a^2}} \qquad \longrightarrow \qquad \rho = \frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2} \propto r^{-5} \quad \text{as } r \to \infty$$

(b) Hénon's (1959) **"isochrone potential:"**

$$\Phi = -\frac{GM}{a+\sqrt{r^2+a^2}} \qquad \longrightarrow \qquad \rho \text{ is complicated.} \quad \text{As } r \to \infty, \quad \rho \approx \frac{aM}{2\pi r^4}$$

(c) For **dark matter halos,** one often starts with a parameterized density, then derives the potential,

$$\rho = \frac{\rho_0}{(r/a)^\alpha [1+(r/a)]^{\beta-\alpha}}$$

and some popular choices of the exponents are

$\alpha=2,\ \beta=4$	(Jaffe 1983)	$\Phi = -4\pi G\rho_0\, a^2\, [\ln(1+a/r)]$
$\alpha=1,\ \beta=4$	(Hernquist 1990)	$\Phi = -2\pi G\rho_0\, a^2\, (1+r/a)^{-1}$
$\alpha=1,\ \beta=3$	(NFW: Navarro et al. 1995)	$\Phi = -4\pi G\rho_0\, a^3/r\, [\ln(1+r/a)]$
$\alpha=1,\ \beta=3.5$	(super-NFW: Lilley et al. 2018)	$\Phi =$ complicated...

Of course, most galaxies are flattened (non-spherical) systems, so one often generalizes to **axisymmetry** in cylindrical coordinates (R, φ, z), where ρ and Φ depend only on R & z.

So what? Who cares? We'd like to know how Φ and ρ behave in our own galaxy. How do we measure them? Stellar orbits!

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Stellar Orbits in an Axisymmetric Potential

We don't need to know the full form of $\Phi(R,z)$ in order to learn something useful about how stars behave. Let's work out Lagrangian dynamics for a single test particle of mass m, in cylindrical coordinates:

$$\mathcal{L} = m\left\{\frac{1}{2}\left[\dot{R}^2 + (R\dot{\varphi})^2 + \dot{z}^2\right] - \Phi(R,z)\right\} \ .$$

The resulting E–L equations of motion are similar to those of the two-body problem. Start with angular momentum conservation from the φ equation:

$$\frac{d}{dt}\left(R^2\,\dot{\varphi}\right) = 0 \qquad \longrightarrow \qquad j = R^2\,\dot{\varphi} = \text{constant}$$

and define an effective potential,

$$\Phi_{\text{eff}} = \Phi + \frac{1}{2}R^2\,\dot{\varphi}^2 = \Phi + \frac{j^2}{2R^2}$$

where the 2nd term is a positive "centrifugal barrier" like in the two-body $V(r)$ plot. Thus, the E–L equations for R and z are given by

$$\boxed{\ddot{R} + \frac{\partial \Phi_{\text{eff}}}{\partial R} = 0} \qquad \text{and} \qquad \boxed{\ddot{z} + \frac{\partial \Phi_{\text{eff}}}{\partial z} = 0} \ .$$

We've "effectively" transformed these equations into the rotating frame. The equations of motion govern motion in a (corotating) meridional plane (R,z).

Can circular orbits exist?

Like in the Keplerian two-body problem, they would occur at local minima in Φ_{eff}. This occurs when

$$\frac{\partial \Phi_{\text{eff}}}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial \Phi_{\text{eff}}}{\partial R} = 0 = \left(\frac{\partial \Phi}{\partial R} - \frac{j^2}{R^3}\right) = \left(\frac{\partial \Phi}{\partial R} - R\dot{\varphi}^2\right) \ .$$

The 1st condition occurs anywhere in the equatorial plane $(z=0)$ as long as $\Phi(R,z)$ is symmetric about $z=0$.

Aside: Stars that *remain* in the equatorial plane at all times have no way of knowing that Φ isn't spherically symmetric. Thus, its orbit will be the same as that for a spherical potential.

The 2nd condition above occurs at the so-called "guiding-center radius" R_{gc}. Thus, circular orbits occur when

$$R_{gc}\dot{\varphi}^2 = \left(\frac{\partial \Phi}{\partial R}\right)_{R=R_{gc},\, z=0} \qquad \text{and} \qquad \Omega = \dot{\varphi} = \sqrt{\frac{1}{R_{gc}}\left(\frac{\partial \Phi}{\partial R}\right)_{R=R_{gc},\, z=0}}$$

which usually isn't solvable analytically for R_{gc}. However, if you know R_{gc}, you can solve for Ω.

. . . .

In disk galaxies, many stars *are* indeed on (nearly) circular orbits. Thus, it's useful to derive approximate solutions for orbits with $R \approx R_{gc}$.

For a given Ω, put a star at $R \neq R_{gc}$ and $z \neq 0$. What will it do?

$$\text{Define:} \qquad x = R - R_{gc} \qquad \text{and let's assume} \quad |x| \ll R_{\rm gc}$$

so that the local minimum of $\Phi_{\rm eff}$ occurs exactly at $x = 0$ and $z = 0$. Expand $\Phi_{\rm eff}$ in a Taylor series about this point...

$$\Phi_{\rm eff}(x,z) = \Phi_{\rm eff}(0,0) + \cancel{x\left(\frac{\partial \Phi_{\rm eff}}{\partial R}\right)_0} + \cancel{z\left(\frac{\partial \Phi_{\rm eff}}{\partial z}\right)_0} +$$

$$+\frac{1}{2}\left[x^2\left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_0 + \cancel{2xz\left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R\,\partial z}\right)_0} + z^2\left(\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right)_0\right] + \cdots$$

where subscript 0 means to evaluate it at $x = 0$ and $z = 0$.

The 1st order terms vanish because we're expanding around the point at which they're *defined* to be zero. The 2nd order cross-term ($\propto 2xz$) vanishes because of symmetry around the $z = 0$ plane.

Truncating the expansion at 2nd order is called the **epicycle approximation,** and we define

$$\kappa^2 \equiv \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_0 \qquad \text{(the radial/epicycle frequency)}$$

$$\nu^2 \equiv \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right)_0 \qquad \text{(the vertical frequency)}$$

Here, these are "constants." If we know the full form of $\Phi(R,z)$, we can compute how κ, ν, and Ω all vary with R_{gc}.

Thus,

$$\Phi_{\rm eff}(x,z) \;\approx\; \Phi_{\rm eff}(0,0) + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2$$

and we can get the equations of motion by taking $\partial\Phi_{\rm eff}/\partial R$ and $\partial\Phi_{\rm eff}/\partial z$.

We see that x and z evolve sinusoidally, like displacements of harmonic oscillators, because the equations of motion reduce to

$$\ddot{x} + \kappa^2 x = 0 \qquad \text{and} \qquad \ddot{z} + \nu^2 z = 0$$

with frequencies in the two directions that don't have to be equal. In general, $\kappa \neq \nu \neq \Omega$. Particles bob in/out and up/down relative to the undisturbed circular orbit.

Note: since $\dot{\varphi} = j/R^2$, and R oscillates in/out, that φ must also oscillate "forwards/backwards" along the orbit. To 0th order, we can assume $\dot{\varphi} \approx \Omega$.

Recall the condition for circular orbits:

$$\Omega^2 = \frac{1}{R_{gc}}\left(\frac{\partial\Phi}{\partial R}\right)_0 , \qquad \text{so we can evaluate} \;\rightsquigarrow\; \kappa^2 = \left(R\frac{\partial\Omega^2}{\partial R} + 4\Omega^2\right)_0 .$$

Note that ideal Keplerian rotation (in the outer regions of galaxies where $\Phi \sim -GM/r$) implies

$$\Omega^2 \propto \frac{1}{R^3} \qquad \Longrightarrow \qquad \kappa = \sqrt{-3\Omega^2 + 4\Omega^2} = \Omega$$

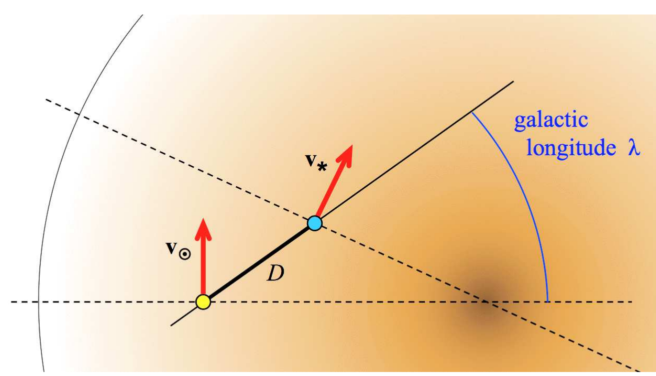
and perturbed orbits are ellipses, like we derived in homework.

On the other hand, for either rigid rotation <u>or</u> the constant-ρ model ($\partial\Phi/\partial r \propto r$) in the inner regions of galaxies,

$$\Omega = \text{constant} \qquad \Longrightarrow \qquad \kappa = 2\Omega$$

and in most galaxies, we see that $\boxed{\Omega \lesssim \kappa \lesssim 2\Omega}$.

In the 1920s, Jan Oort measured these properties with local stars around us at $R \sim 8$ kpc. He was first to prove that the galaxy rotates *differentially.*



Observations give us the relative velocity: $\mathbf{V}_{\rm obs} = \mathbf{v}_\odot - \mathbf{v}_*$, via

- Doppler shifts of spectral lines $\Longrightarrow$ $V_{{\rm obs},r}$ (line-of-sight component).
- Proper motions $\Longrightarrow$ $V_{{\rm obs},t}$ (transverse "plane-of-sky" component).

If both the Sun & the other star are in ~circular orbits around the galactic center, then the data ought to be fit well by

$$\begin{aligned} V_{{\rm obs},r}/D &= A \sin 2\lambda \\ V_{{\rm obs},t}/D &= A \cos 2\lambda + B \end{aligned}$$

Frequency-like fitting parameters A & B are the **Oort constants:**

- A describes relative *shear* between us & the other star.
- B describes mean galactic rotation (*vorticity*) in our neighborhood.

Given that these can also be described by $\Omega(R)$, the formal definitions of the Oort constants are

$$\begin{aligned} A(R) &= -\frac{R}{2}\frac{d\Omega}{dR} &&= \frac{1}{2}\left(\frac{V_c}{R} - \frac{dV_c}{dR}\right) \\ B(R) &= -\frac{R}{2}\frac{d\Omega}{dR} - \Omega &&= -\frac{1}{2}\left(\frac{V_c}{R} + \frac{dV_c}{dR}\right) \end{aligned}$$

where the circular speed $V_c = R\Omega = \sqrt{R|\nabla\Phi|}$.

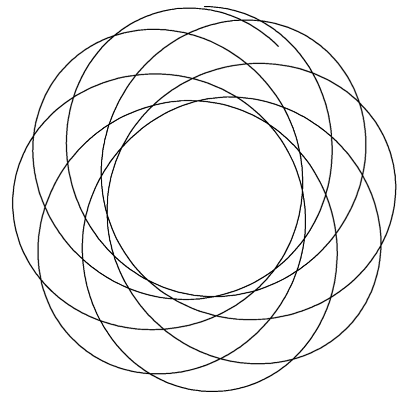
Other useful identities can be derived:

$$\Omega = A - B \quad , \qquad \kappa^2 = 4B(B-A) \quad , \qquad \frac{\kappa}{\Omega} = \sqrt{\frac{4B}{B-A}} \ .$$

Limiting cases (& observations):

Rigid rotation (Ω = constant)	$A = 0$	$B = -\Omega$	$\kappa/\Omega = 2$
Flat rotation curve ($\Omega \propto R^{-1}$)	$A = 0.5\Omega$	$B = -0.5\Omega$	$\kappa/\Omega = \sqrt{2}$
Keplerian rotation ($\Omega \propto R^{-3/2}$)	$A = 0.75\Omega$	$B = -0.25\Omega$	$\kappa/\Omega = 1$
Hipparcos (1997) observations	$A \approx 0.545\Omega$	$B \approx -0.455\Omega$	$\kappa/\Omega \approx 1.35$
Gaia DR1 (2016) observations	$A \approx 0.563\Omega$	$B \approx -0.438\Omega$	$\kappa/\Omega \approx 1.323$
Gaia DR2 (2020) observations	$A \approx 0.554\Omega$	$B \approx -0.446\Omega$	$\kappa/\Omega \approx 1.336$

Because κ is not an integer multiple of Ω, our local rotation curve is close to *flat,* and the Sun's orbit is a non-closed "rosette!"



FYI, in our local neighborhood:

$R_{gc} \approx 8$ kpc
$\Omega \approx 27.2$ km/s/kpc
$V_c \approx 218$ km/s
Rotation period $(2\pi/\Omega) \approx 226$ Myr

. . . .

What is the vertical frequency ν? Measuring it is harder than for κ.

Jan Oort realized that in our highly **flattened** disk galaxy,

$$\nu^2 \;=\; \frac{\partial^2 \Phi_{\rm eff}}{\partial z^2} \;\approx\; \left|\frac{\partial^2 \Phi}{\partial z^2}\right| \;\gg\; \left\{ \left|\frac{\partial^2 \Phi}{\partial R^2}\right|,\; \left|\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|,\; \kappa^2,\; A^2,\; B^2,\; \Omega^2 \right\}$$

(especially if the rotation curve is close to "flat")

Poisson's equation connects these 2nd derivatives to the local mass density

$$4\pi G\rho \;=\; \nabla^2\Phi \;=\; \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} \;\approx\; \frac{\partial^2 \Phi}{\partial z^2} = \nu^2 \;.$$

In our local solar neighborhood, $\left\{ \begin{array}{l} \nu/\Omega \;\approx\; 2.5 \rightarrow 3.2 \\ \rho \;\approx\; 0.1 \rightarrow 0.15\; M_\odot/\mathrm{pc}^3 \end{array} \right\}$

and locally, ν contributes $\sim$80% of the total $\nabla^2\Phi$.

Counting stars is one way to estimate ρ (to then compute ν), but that may miss gas, dark matter, and possibly dim white dwarfs. We'd need the total ρ that exerts gravitational force.

One can use **stellar dynamics** to do a better job, via ν. Recall the collisionless Vlasov equation for stars (with $\mathbf{a} = -\nabla\Phi$),

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} \;=\; 0 \;.$$

We can take the standard fluid-moment equations by integrating over velocity space. (Galaxy people call these the **Jeans equations.**)

Recalling $\quad n = \int d^3\mathbf{p}\ f\ ,\quad$ & assuming $\quad n\langle\mathbf{v}\rangle = n\mathbf{u} = \int d^3\mathbf{p}\ \mathbf{v}\, f \approx 0\ ,$

the 1st moment (momentum conservation equation) is, in general,

$$\frac{\partial}{\partial t}\left(n\langle\mathbf{v}\rangle\right) + \nabla\cdot\left(n\langle\mathbf{v}\mathbf{v}\rangle\right) + n\nabla\Phi = 0\ .$$

If we use cylindrical coordinates and assume a time-steady ($\partial/\partial t = 0$) and axisymmetric ($\partial/\partial\phi = 0$) system, we can simplify the 3 components of this equation. It's kind of just hydrostatic equilibrium...

When deriving $\rho \approx \nu^2/(4\pi G)$, it's customary to just worry about the z component of the equation...

$$\frac{\partial}{\partial R}\left(n\langle v_R v_z\rangle\right) + \frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right) + \frac{n\langle v_R v_z\rangle}{R} + n\frac{\partial\Phi}{\partial z} = 0\ .$$

To obtain ν^2, one just needs to solve the above for $\partial\Phi/\partial z$, then take another z derivative.

Caveats:

- As noted above, in "flattened" galaxies, the $\partial/\partial z$ term tends to dominate, such that one often sees
$$\rho \approx \frac{1}{4\pi G}\frac{\partial^2\Phi}{\partial z^2} \approx -\frac{1}{4\pi G}\frac{\partial}{\partial z}\left[\frac{1}{n}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\right]\ .$$
- These statistical **variances** of nearby stellar velocities $\langle v_z^2\rangle$ & $\langle v_R v_z\rangle$ can be measured via Doppler shifts & proper motions.
- One needs only to measure the *relative* spatial variation of $n(z)$.
Its absolute normalization divides out of every term.
- Taking two z-derivatives of noisy data makes for an uncertain result!
Bahcall (1984, *ApJ,* 276, 169) found some clever tricks to reduce, but not eliminate, the uncertainties.

. . . .

It's time to be more exact about what's going on here in the **Milky Way Galaxy.**

There have been several decades of gradual improvement in empirical models of $\Phi(R, \varphi, z)$. Putting aside non-axisymmetry (bars, triaxial halos), let's look at a recent one that was optimized for accuracy over a large dynamic range: $r = 5$ pc to 100 kpc!

Scott Kenyon et al. (2008, *ApJ,* 680, 312) specified 4 components:

$$\Phi(R, z) \; = \; \Phi_{\rm BH}(r) \, + \, \Phi_{\rm bulge}(r) \, + \, \Phi_{\rm disk}(R, z) \, + \, \Phi_{\rm halo}(r)$$

(a) **Supermassive Black Hole:** For scales larger than a parsec or so, Sgr A* is more or less just a Newtonian point-mass: $\Phi_{\rm BH} \approx -GM_{\rm BH}/r$.

(b) **Central Bulge:** Mostly gas-free and composed of old Pop II stars, a spherically symmetric Hernquist (1990) profile is a good fit, with $a \approx 0.1$ kpc.

(c) **Galactic Disk:** An axisymmetric spheroidal can account for both the "thin" Pop I disk of recent star formation, and for the slightly older spheroidal halo that extends out to the GCs. Miyamoto & Nagai (1975) postulated:

$$\Phi_{\rm disk}(R, z) \; = \; -\frac{GM_{\rm disk}}{\sqrt{R^2 + [a + (z^2 + b^2)^{1/2}]^2}}$$

which reduces to the Plummer (1911) model in the midplane $z = 0$.

The Kenyon et al. model uses $a = 2$ kpc, $b = 0.3$ kpc.

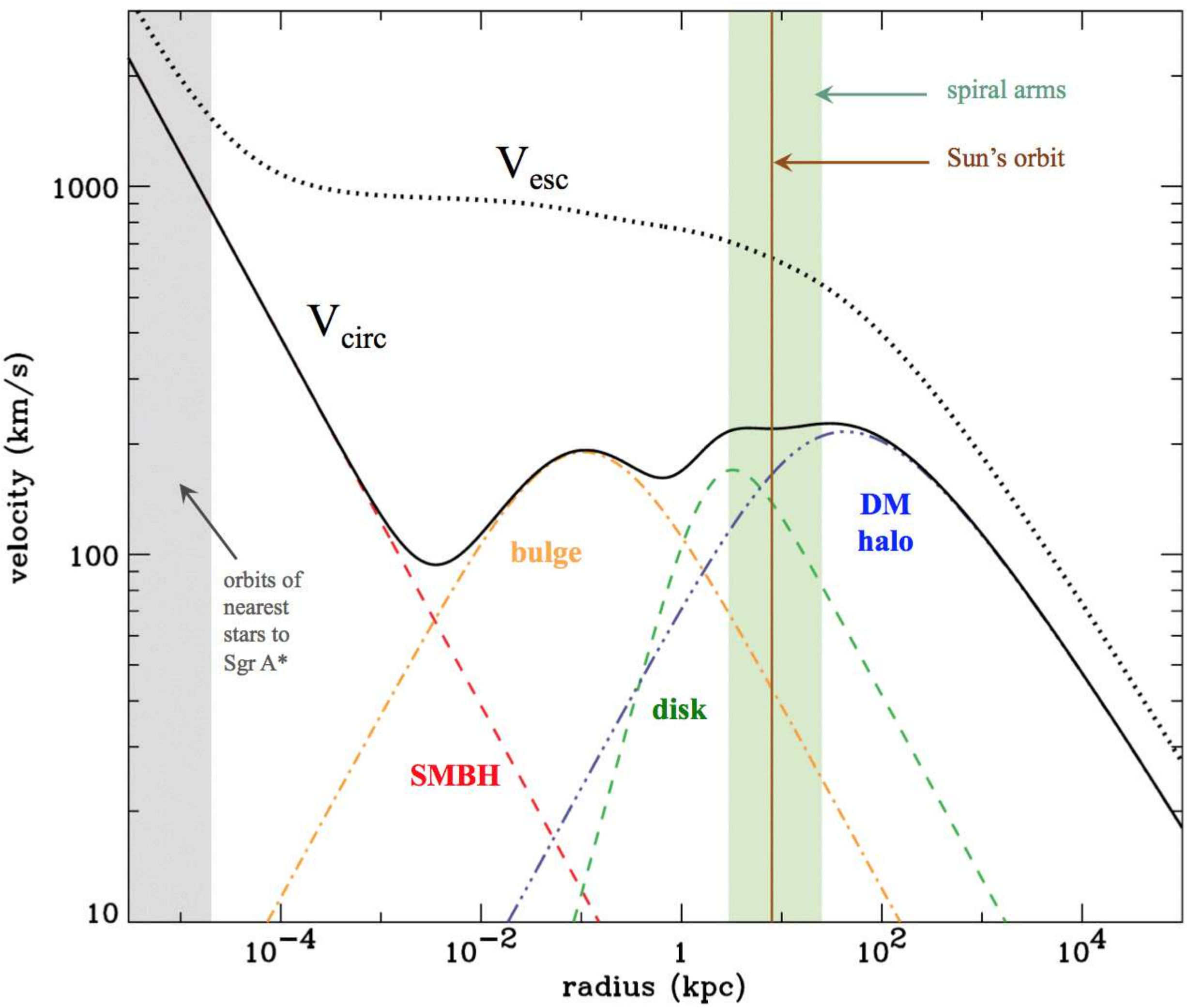
(d) **Dark Matter Halo:** Even though it's probably more of a triaxial ellipsoid, a spherical NFW model (with $a = 20$ kpc) does a good job of modeling its overall gravitational effects.

More recently than 2008, it's been realized that there's also a very hot (10^6–10^7 K) **extended halo** of plasma going out a few hundred kpc, which doesn't contain much mass, but may carry *half* of the MW's total angular momentum (e.g., Hodges-Kluck et al. 2016, *ApJ,* 822, 21).

For the midplane ($z = 0$), we'll plot:

$$V_c(R) \;=\; R\Omega \;=\; \sqrt{R(\partial\Phi/\partial R)} \qquad \text{(matching grav. \& centripetal accel.)}$$

$$V_{\rm esc}(R) \;=\; \sqrt{2|\Phi|} \qquad \text{(matching kinetic \& potential energy: } \tfrac{1}{2}mV_{\rm esc}^2 = m|\Phi|)$$



Rotation is indeed pretty "flat" in the vicinity of the spiral arms. The local differential-rotation *shear* (which spins up GMCs) is difficult to see in this plot.

At the Sun's orbit, $V_c = 219$ km/s, $V_{\rm esc} = 638$ km/s.

In the inner parts of the galaxy, $V_{\rm esc} \sim 1000$ km/s!

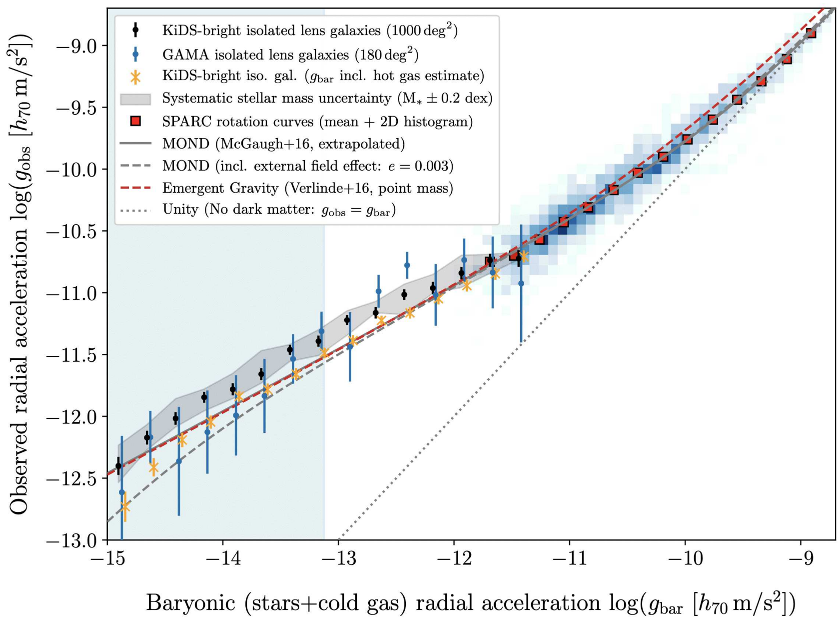
. . . .

We'll talk a bit more about the motivations for **dark matter (DM)** later, but it's interesting to note a recent (and still hotly debated!) piece of observational information from a big-data study of ∼150 galaxies.

McGaugh et al. (2016, *PRL,* 117, 201101) & Brouwer et al. (2021, *A&A,* 650, A113) used rotation curves to estimate $g_{\rm obs} = V_c^2/R$. Dominated by DM?

They also used observed intensity profiles to estimate the $\rho(R, z)$ of luminous matter, used Poisson's equation to get Φ, and took $g_{\rm baryons} = |\partial\Phi/\partial R|$.

Each galaxy produces its own independent curve in $(g_{\rm obs}, g_{\rm bar})$ space. Amazingly, the curves all *overlap* with one another:



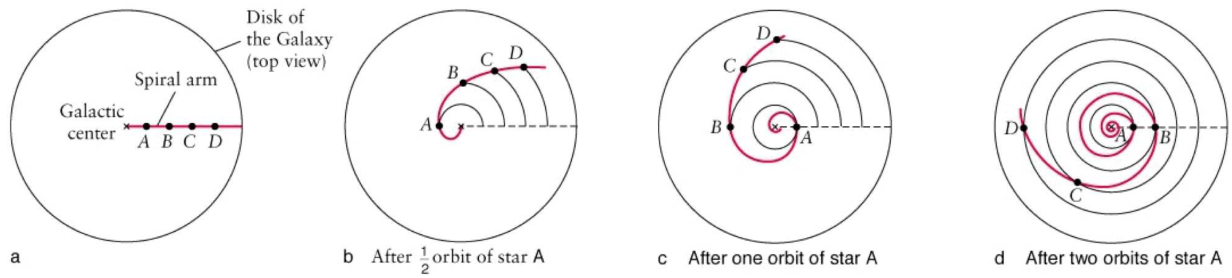
Of course, $g_{\rm obs} > g_{\rm bar}$, and it's well-known that the tiniest galaxies have the *most* relative contribution from dark matter (i.e., $g_{\rm obs}/g_{\rm bar} \gg 1$) because their low gravity allows gas/plasma to be easily lost via jets/winds.

But the tight correlation is surprising. One conclusion is that the dark matter is "fully specified" by the baryons. Maybe that's MOND...?

There are other ideas... Many think the DM halo determines the galaxy's overall potential well, and baryons "fill up" the wells proportionally in different galaxies. Or maybe young galaxies accrete both DM & baryons together (Marasco et al., arXiv:2105.10508)?

The baryonic matter in galaxy disks often exhibits **Spiral Arms.**

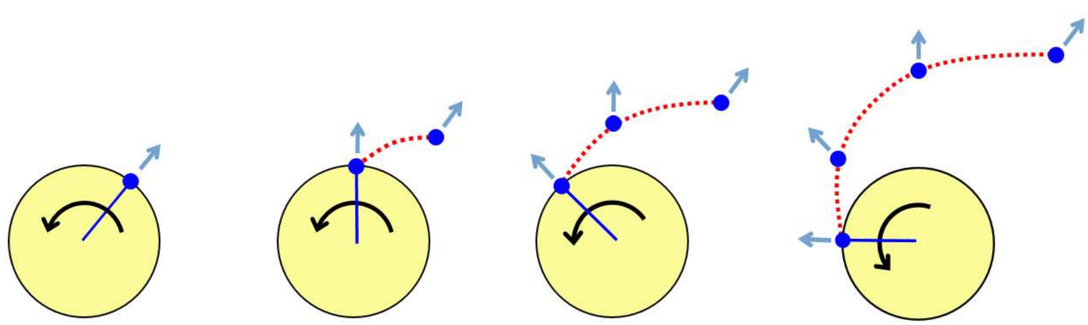
If a galaxy rotates differentially [$\Omega(R) \neq$ constant], it's not hard to see that any longitudinal "spoke" of higher ρ would get "wound up" into a spiral. For a flat rotation curve,



Winding problem: Over the lifetimes of most galaxies, there would be way too much winding – which we don't see.

Several possible solutions continue to be discussed, but the most accepted one came in the 1960s. Lin & Shu realized that spiral arms could be **density waves;** i.e., patterns of higher ρ that propagate through the disk and maintain their "pitch-angle coherency" over time.

Spirals in rotating fluids don't have to be a single rigid structure. Think about the "garden sprinkler" analogy, in which one longitude of a central source that rotates and spits out parcels that flow out radially:



Even though **v** is radial (in the nonrotating frame), the *streakline* pattern is a spiral that rotates with Ω, and is fixed in a frame rotating with Ω.

Galactic spiral density-wave patterns are a close cousin of this effect. They consist of stars on closed orbits in *some* fixed rotating frame of reference (not necessarily at Ω).

In fact, closed orbits are a necessary (but not sufficient) condition for orbiting patterns to stay **coherent** over long times.

We won't delve into full-on density-wave theory, but it is straightforward to show that there *can* be a rotating frame in which star orbits remain closed over a range of distances corresponding to observed spiral arms.

Recall the epicycle perturbations from a circular orbit.

The orbit bobs in & out in R with frequency κ (period $t_R = 2\pi/\kappa$).

To zero order, the star also orbits around in φ with angular frequency Ω (period $t_\varphi = 2\pi/\Omega$).

Consider $\Delta\psi \,=\, \{\text{the azimuthal angle traversed over time } t_R\} \,\approx\, \Omega t_R$.

Orbits are closed only if

$$\Delta\psi \,=\, 2\pi\,\frac{n}{m} \qquad \text{for } n, m = \text{integers.}$$

However, we *could* choose to view a parcel in a frame that rotates with a **pattern speed** Ω_p defined such that the orbit *is* closed. In this frame,

$$\Delta\psi_p \,=\, \Delta\psi \,-\, \Omega_p t_R \,=\, 2\pi\,\frac{n}{m}\ .$$

Using the definitions above, we can solve for

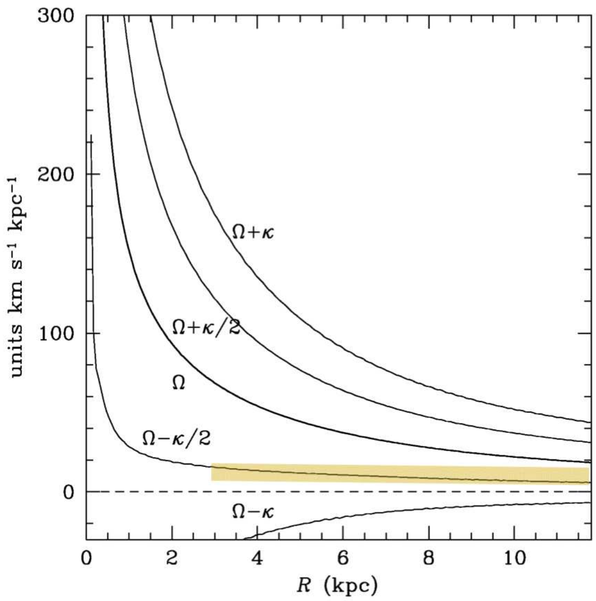
$$\Omega_p \,=\, \Omega \,-\, \frac{n\,\kappa}{m}\ .$$

In our galaxy, we can compute $\Omega_p(R)$ for a number of choices for n & m. It turns out that if we choose $n = 1, m = 2$, then

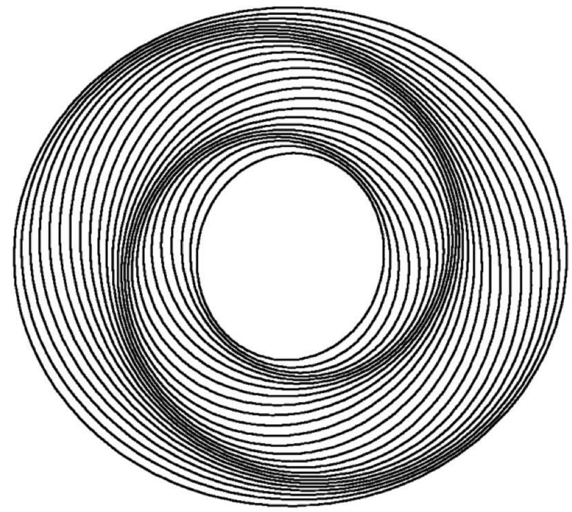
$$\Omega_p \,\approx\, \text{constant}, \quad \text{over the range} \quad R \,\approx\, 3 \rightarrow 15 \text{ kpc.}$$

Closed orbits in this frame collect into a nested pattern that maintains a ∼constant pattern speed consistent with the observed spiral arms.

(See figures on next page.)



Figures from Binney & Tremaine, chapter 6



Note that each concentric "ring" is a closed ellipse, but their longitudes of periapsis are varying gradually with increasing R.

Because $\Omega \neq \Omega_p$ at any radius, stars pass through the arms, like cars through a "gaper-delay" traffic jam.

Binney & Tremaine § 6.2 discuss more about the resonances (Lindblad) and stability criteria (Toomre) that help maintain spiral density waves over long times, even when Ω_p isn't exactly constant.

It is possible to measure Ω_p observationally, too... see Peterken et al. (arXiv:1809.08048).

. . . .

In the final part of this section, we'll look at one more way to *squeeze* some useful insights out of statistics...

(3) Collisionless Equilibrium Statistics $\longrightarrow$ The Virial Theorem

How do the *spatially integrated* properties of gravitationally-bound objects relate to one another? Consider conservation laws:

$$\begin{bmatrix} \text{Boltzmann/} \\ \text{Vlasov eqns} \end{bmatrix} \rightarrow \int d^3\mathbf{p} \rightarrow \begin{bmatrix} \text{Fluid/Jeans} \\ \text{moment eqns} \end{bmatrix} \rightarrow \int d^3\mathbf{r} \rightarrow \begin{bmatrix} \text{virial theorem} \end{bmatrix}$$

Formally, one can start with the fluid momentum conservation equation from earlier. Multiplying each term by the mass of an object converts n to ρ,

$$\frac{\partial}{\partial t}(\rho\langle\mathbf{v}\rangle) + \nabla\cdot(\rho\langle\mathbf{v}\mathbf{v}\rangle) + \rho\nabla\Phi = 0 \ .$$

Each term is a vector. There are 2 things we could do.

- Take tensor/outer product of each term with $\mathbf{r}$, then integrate over the system volume $d^3\mathbf{r}$. This gives the **tensor virial theorem.**
- Take scalar/dot product of each term with $\mathbf{r}$, then integrate over the system volume $d^3\mathbf{r}$. This gives the **scalar virial theorem.**

Either way, we'd have to take the 1st moment in $\mathbf{p}$, then the 1st moment in $\mathbf{r}$. **We won't actually go through with it.**

Binney & Tremaine explore both methods, and eventually note that the trace of the tensor equation gives the scalar equation. The math is complicated, and this is a physics class! Thus, we'll derive the **scalar** virial theorem using an alternate approach from Goldstein's *Classical Mechanics* § 3.4:

Consider a system of N point-masses with masses m_i, position vectors $\mathbf{r}_i$, and momenta $\mathbf{p}_i$. The force on particle i is $\mathbf{F}_i$, for which Newton's 2nd law says

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} \ .$$

$$\text{Define the summed quantity} \qquad \Gamma \equiv \sum_{i=1}^{N} \mathbf{r}_i\cdot\mathbf{p}_i \ .$$

What is it? For non-relativistic particles,

$$\Gamma = \sum_i m_i\left(\mathbf{r}_i\cdot\frac{d\mathbf{r}_i}{dt}\right) = \sum_i m_i\frac{1}{2}\frac{d}{dt}\Big(\mathbf{r}_i\cdot\mathbf{r}_i\Big) = \frac{1}{2}\frac{d}{dt}\underbrace{\Big(\sum_i m_i|\mathbf{r}_i|^2\Big)}_{I} = \frac{1}{2}\frac{dI}{dt}$$

where I = the total moment of inertia of the system, with respect to the adopted origin.

{ For an extended system, its total angular momentum $\mathbf{L} = I\boldsymbol{\Omega}$, and an analogue of Newton's 2nd law is that the applied torque $= d(I\boldsymbol{\Omega})/dt$. }

Alternately, we can examine the time derivative of Γ, from its definition,

$$\frac{d\Gamma}{dt} = \sum_i \left(\frac{d\mathbf{r}_i}{dt} \cdot \mathbf{p}_i \right) + \sum_i \left(\mathbf{r}_i \cdot \frac{d\mathbf{p}_i}{dt} \right)$$

$$= \sum_i m_i \left(\frac{d\mathbf{r}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt} \right) + \sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)$$

This lets us write the scalar virial theorem in its usual form

$$\frac{d\Gamma}{dt} = \boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2\mathcal{E}_{\mathrm{K}} + \mathcal{E}_{\mathrm{G}}}$$

where $\mathcal{E}_{\mathrm{K}}$ is the **total kinetic energy** of the system,

$$\mathcal{E}_{\mathrm{K}} = \sum_i \left[\frac{1}{2} m_i \left(\frac{d\mathbf{r}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt} \right) \right] = \sum_i \left(\tfrac{1}{2} m_i v_i^2 \right)$$

and

$$\mathcal{E}_{\mathrm{G}} = \sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)$$

can be shown to be the **total gravitational potential energy** of the system. (Recall the work-energy theorem.) Note that for our system of N particles, we write

$$\mathbf{F}_i = \sum_{j \neq i} G m_i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

which we should note is an attractive force. Particle i feels a force that points *toward* each of the other j particles. Thus,

$$\mathcal{E}_{\mathrm{G}} = \sum_i \sum_{j \neq i} G m_i m_j \left(\mathbf{r}_i \cdot \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3} \right) .$$

This is a sum over all possible $N(N-1)$ pairings of particles.

The sum includes reciprocal pairs, for which we know $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$

$$\mathcal{E}_{\rm G} \;=\; \cdots + Gm_a m_b \left(\mathbf{r}_a \cdot \frac{\mathbf{r}_b - \mathbf{r}_a}{|\mathbf{r}_b - \mathbf{r}_a|^3} \right) + \cdots + Gm_b m_a \left(\mathbf{r}_b \cdot \frac{\mathbf{r}_a - \mathbf{r}_b}{|\mathbf{r}_a - \mathbf{r}_b|^3} \right) + \cdots$$

and since $N(N-1)$ is always an even number, it's always possible to group each reciprocal pair together,

$$\mathcal{E}_{\rm G} \;=\; \cdots + \frac{Gm_a m_b}{|\mathbf{r}_b - \mathbf{r}_a|^3} \Big[-(\mathbf{r}_b - \mathbf{r}_a) \cdot (\mathbf{r}_b - \mathbf{r}_a) \Big] + \cdots$$

and now there are only $N(N-1)/2$ terms in the sum, which correspond only to the full list of *unique* pairs, and

$$\mathcal{E}_{\rm G} \;=\; -\sum_i \sum_{j<i} \frac{Gm_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|}$$

which is exactly the potential energy U we used for the $N = 2, 3$-body problem.

FYI, sometimes one sees this written as

$$\mathcal{E}_{\rm G} \;=\; -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{Gm_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|} \qquad \text{for the sum over } all \text{ pairs,}$$

which essentially "double counts," then corrects for it afterwards.

....

For those of you interested in STARS, I'll include an alternate derivation of the time-steady virial theorem (i.e., assuming $d^2I/dt^2 = 0$) for self-gravitating spheres of gas/plasma, instead of the N-body particle system being studied here:

Assume a static, spherical distribution of gas obeying Maxwell-Boltzmann statistics as derived earlier. No rotation, no big mass motions, no magnetic fields, etc.

$$\text{Total energy:} \quad \mathcal{E}_{\rm tot} \;=\; \mathcal{E}_{\rm K} + \mathcal{E}_{\rm G}$$

$\mathcal{E}_{\rm K}$ is just the total *thermal* energy due to random motions, i.e.,

$$\mathcal{E}_{\rm K} = \int dV \; U = \int dV \; \frac{P}{\gamma - 1} \quad \text{for ideal gas, or any system with const } \gamma.$$

Also,

$$V \;=\; \frac{4}{3}\pi r^3 \;\; , \qquad dV \;=\; 4\pi r^2 \, dr \; .$$

What about $\mathcal{E}_{\rm G}$? For test particles of mass m in the field of a gravitating body of mass M_*, we know that

$$\mathcal{E}_{\rm G} = -\frac{GM_* m}{r}$$

where r is the distance between the two bodies. So, for a star in *its own* potential well, it's analogous to write

$$\mathcal{E}_{\rm G} = -\int dM_r \, \frac{GM_r}{r}$$

where the equation describing mass conservation in concentric shells is given by

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho \ .$$

We also already know about one main "link" between gravity and thermal pressure: hydrostatic equilibrium. This will serve as our starting point for building the virial theorem:

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \ .$$

Multiply both sides by the volume V,

$$V\frac{dP}{dr} = -\frac{4}{3}\pi r^3 \rho \, \frac{GM_r}{r^2} = -\frac{1}{3}\left(4\pi r^2 \rho\right) \frac{GM_r}{r} = -\frac{1}{3}\left(\frac{dM_r}{dr}\right) \frac{GM_r}{r}$$

where the last equality takes advantage of mass conservation. Juggle a bit, to get

$$V \, dP = -\frac{1}{3} dM_r \, \frac{GM_r}{r} \ .$$

Now we can integrate over the stellar volume. The RHS is simply $\mathcal{E}_{\rm G}/3$. The LHS requires integration by parts:

$$\int_{\rm core}^{\rm surface} V \, dP = [PV]_{\rm core}^{\rm surface} - \int_{\rm core}^{\rm surface} dV \, P$$

The first term has two parts, each of which is **zero:** (1) At $r = 0$, we know that $V = 0$. (2) At $r = R_*$, we often assume that $P \to 0$, when the star is surrounded by the vacuum of space.

The second term is closely related to $\mathcal{E}_{\rm K}$, so the static form of the virial theorem is

$$\boxed{\frac{\mathcal{E}_{\rm G}}{3} = (1-\gamma)\mathcal{E}_{\rm K}} \qquad \text{i.e.,} \quad 2\mathcal{E}_{\rm K} + \mathcal{E}_{\rm G} = 0 \qquad \text{for } \gamma = 5/3.$$

To simplify things a bit, consider constant-density, constant-temperature spheres of gas with arbitrary values of M, R, and T. We can write

$$\mathcal{E}_{\rm K} \approx \frac{M k_{\rm B} T}{\mu m_{\rm H}} \qquad \text{and} \qquad \mathcal{E}_{\rm G} \approx -\frac{GM^2}{R} \ .$$

If these values are completely arbitrary, there are three possibilities:

$$\begin{array}{lll} \text{If} & 2\mathcal{E}_{\rm K} > |\mathcal{E}_{\rm G}| \ , & \text{the sphere is unstable to expansion} \\ & 2\mathcal{E}_{\rm K} = |\mathcal{E}_{\rm G}| \ , & \text{virial equilibrium holds; it's bound \& stable} \\ & 2\mathcal{E}_{\rm K} < |\mathcal{E}_{\rm G}| \ , & \text{the sphere is unstable to collapse} \end{array}$$

This leads to the **Jeans criterion** for a molecular cloud to begin collapsing into a protostar. If the third option above is true, then

$$\frac{GM}{R} \gtrsim \frac{k_{\rm B} T}{\mu m_{\rm H}}$$

which is kind of equivalent to $V_{\rm esc}^2 \gtrsim v_{\rm th}^2$.

In other words, for collapse to be possible, *most* particles in the distribution must be trapped in the sphere's potential well.

$$\text{Thus, using} \qquad \rho = \frac{M}{4\pi R^3/3} \qquad \Longrightarrow \qquad R = \left(\frac{3M}{4\pi\rho} \right)^{1/3}$$

one can define a "Jeans mass"

$$M_{\rm J} \approx \left(\frac{k_{\rm B} T}{\mu m_{\rm H} G} \right)^{3/2} \frac{1}{\rho^{1/2}}$$

and if a cloud has $M > M_{\rm J}$, it will collapse. If you collect together enough mass in a small enough space, gravity will win.

On the other hand, when one is **in virial equilibrium,** it's interesting that the internal temperature is given (approximately) by

$$T \approx \frac{GM\mu m_{\rm H}}{k_{\rm B} R} \ .$$

A star's core temperature is *not* set by esoteric physics of nuclear burning. It's just gravity!

. . . .

Now back to galaxies...

Applications of the Virial Theorem

In practice, the time-steady virial theorem ($2\mathcal{E}_{\rm K} + \mathcal{E}_{\rm G} = 0$) is used to study statistical equilibrium properties of large systems like galaxy clusters.

Big assumption: **Ergodicity** $\longrightarrow$ [time averages $\approx$ ensemble averages]

i.e., we often assume that if the system is left to its own devices, it will live ~forever, and (eventually) pass arbitrarily close to ~every point in phase space.

In practice, an *instantaneous snapshot/sample* involving sums over all $N \gg 1$ particles "ought" to trend to the time/ensemble average, too.

Fritz Zwicky (1933, 1937) first applied the virial theorem as a **cluster mass estimator.** What quantities can actually be measured?

Kinetic energy:

$$\mathcal{E}_{\rm K} = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2} M_{\rm tot} \langle v^2 \rangle \quad \text{where} \quad \langle v^2 \rangle = \frac{\sum m_i v_i^2}{\sum m_i} \sim \frac{1}{N}\sum_i v_i^2 \ .$$

Our goal is to solve for $M_{\rm tot}$.

Distant galaxies don't show proper motions on the sky, so all we really have is the Doppler radial velocity v_r. Assuming isotropy,

$$\langle v_r^2 \rangle \approx \frac{\langle v^2 \rangle}{3} \ .$$

Potential energy:

$$\mathcal{E}_{\rm G} = -\sum_i \sum_{j<i} \left(\frac{G m_i m_j}{r_{ij}} \right) \equiv -\frac{G M_{\rm tot}^2}{r_g}$$

which essentially defines the *gravitational radius* r_g. More specifically,

$$\frac{1}{r_g} = \frac{1}{M_{\rm tot}^2} \sum_i \sum_{j<i} \frac{m_i m_j}{r_{ij}} \ .$$

Although the masses & separations are coupled together inside the sum, one often sees assumptions that they're each sampled from independent distributions.

In other words, we can note that

$$\sum_i \sum_{j<i} m_i m_j \;\approx\; \frac{N(N-1)}{2}\langle m^2\rangle \;\approx\; \frac{N^2\langle m^2\rangle}{2} \;\approx\; \frac{M_{\rm tot}^2}{2} \qquad (\text{for } N \gg 1),$$

and we can estimate

$$\mathcal{E}_{\rm G} \;\approx\; -G\left[\sum_i \sum_{j<i} m_i m_j\right]\left\langle \frac{1}{r}\right\rangle \;\approx\; -\frac{1}{2}GM_{\rm tot}^2\left\langle \frac{1}{r}\right\rangle \;, \qquad \text{where}$$

$$\left\langle \frac{1}{r}\right\rangle \;=\; \frac{\sum_i \sum_{j<i}(1/r_{ij})}{\sum_i \sum_{j<i}(1)} \;\approx\; \frac{2}{N^2}\sum_i \sum_{j<i}\left(\frac{1}{r_{ij}}\right) \;.$$

Thus, the time-steady virial theorem is

$$2\mathcal{E}_{\rm K} \;=\; -\mathcal{E}_{\rm G} \qquad \Longrightarrow \qquad M_{\rm tot}\langle v^2\rangle \;=\; \frac{GM_{\rm tot}^2}{2}\left\langle \frac{1}{r}\right\rangle$$

i.e., the virial mass estimator

$$M_{\rm tot} \;=\; \frac{2\langle v^2\rangle}{G}\left\langle \frac{1}{r}\right\rangle^{-1} \;.$$

Practially, though, we observe $\langle v_r^2\rangle \approx \langle v^2\rangle/3$, and we only see *projected* separations on the sky. Thus, we can only really measure

$$\left\langle \frac{1}{r_\perp}\right\rangle \approx \frac{2}{N^2}\sum_i \sum_{j<i}\left(\frac{1}{r_{\perp ij}}\right) \;.$$

Of course, $r_{\perp ij} < r_{ij}$ in general, so a statistical correction factor is applied.

Limber & Mathews (1960, *ApJ,* 132, 286) showed that, if the separation vectors $\mathbf{r}_{ij}$ are distributed randomly in space, then

$$\left\langle \frac{1}{r_\perp}\right\rangle \approx \frac{\pi}{2}\left\langle \frac{1}{r}\right\rangle \;.$$

Thus, the observable virial mass estimator is

$$\boxed{M_{\rm tot} \;=\; \frac{3\pi\langle v_r^2\rangle}{G}\left\langle \frac{1}{r_\perp}\right\rangle^{-1}} \;.$$

There have also been several decades of improvements to this technique; see, e.g., Bahcall & Tremaine (1981, *ApJ,* 244, 805), and Watkins et al. (2010, *MNRAS,* 406, 264). Sometimes it just takes careful tweaks in how one uses the data to properly extract quantities like $\langle v^2 \rangle$.

Note also that many galaxy clusters are "unrelaxed" (i.e., still evolving, so not in time-steady virial equilibrium)! Their actual mass tends to be lower than the virial estimate.

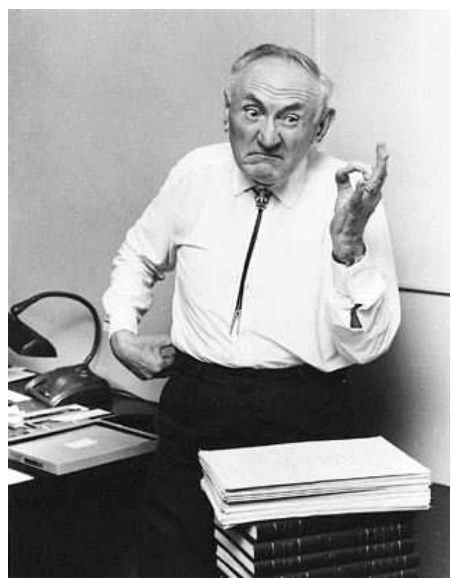
There are other ways to estimate virial masses.

For galaxies & clusters too dense to count individual components, many models show that if you can measure the **half-mass radius** r_h (where the enclosed mass is half the total), then for most realistic $\Phi(R, z)$ models,

$$\frac{r_h}{r_g} \approx 0.4 \rightarrow 0.52 \qquad \text{(a relatively narrow range!)}$$

and thus,

$$M_{\text{tot}} \approx \frac{2.2 \, \langle v^2 \rangle \, r_h}{G} \ .$$



r_h often has been taken to be $\approx$ the **half-light radius** (i.e., the point at which integrated intensity drops to half its peak value), but that ignores dark matter.

Of course, Zwicky (1933, 1937) is most famous for concluding that M_{tot} must $\gg$ mass from luminous matter in clusters, by factors of >100, thus requiring copious amounts of *"dunkle Materie"* (dark matter).

For more about how Zwicky may have also ruled out MOND-type theories, see arXiv:1610.01543

However, there's a good case to be made that Knut Lundmark assembled all necessary pieces of the *"dunkle Materie"* story in 1930, prior to Zwicky. But this contribution was forgotten until 2015!?