Central Force Motion: The Three-Body Problem

2 mutually gravitating particles: solvable as equivalent 1-body system.
3 mutually gravitating particles: **unsolvable** (in general)!

There are some simple geometries for which there are "just-so" analytic solutions (e.g., equilateral triangle, figure-8, all 3 particles remain collinear).

Most configurations are complex; some are highly *chaotic* (sensitive to tiny changes in initial conditions).

In astronomy, a simplified problem of importance is the **restricted 3-body problem,** in which two massive particles (M_1, M_2) exert gravitational pull on a low-mass test particle (m_3) that doesn't influence the others.

M_1 and M_2 are essentially a 2-body problem, and to start we often assume they're in a **circular** binary orbit.

Applications of the *circular* restricted 3-body problem (CR3BP):

- Stability of satellites in a star/planet or planet/moon system.
- Distribution of shapes of distorted stars in a close binary (i.e., the m_3 test particles are 'parcels' of gas).
- Motions of stars in a galaxy with a neighboring satellite galaxy.

Thus, studying the motion of particle 3 is best done in the rotating frame of the binary, centered on its CM (barycenter). Consider angular rotation rate $\mathbf{\Omega}$ pointing normal to the orbital plane, and

$$|\mathbf{\Omega}| = \Omega = \sqrt{\frac{G(M_1 + M_2)}{|\mathbf{r}_1 - \mathbf{r_2}|^3}} = \text{constant.}$$

The full Lagrangian is given by

$$\mathcal{L} = \sum_{i=1}^{3} \tfrac{1}{2} m_i \left(\dot{r}_i^2 + r_i^2 \dot{\theta}_i^2 \right) - \left[-\frac{GM_1 M_2}{r_{12}} - \frac{GM_1 m_3}{r_{13}} - \frac{GM_2 m_3}{r_{23}} \right]$$

(where r_{ij} is just the mutual distance between i & j.)
This can be simplified greatly, as long as we stick with CR3BP...

- Kinetic energy terms for particles 1 & 2 are **constants,** because if orbits are circular, $\dot{r} = 0$ and $\dot{\theta} = \Omega =$ constant in the CM frame.
- The U_{12} term is also constant, because r_{12} doesn't change.

A very complete derivation: Hadjidemetriou (1975, *Celest. Mech.*, 12, 155).

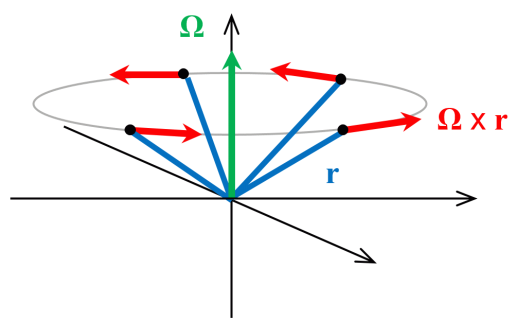
Because the E–L equations of motion depend on *derivatives* of $\mathcal{L}$, we can ignore constant terms, and treat the Lagrangian as

$$\mathcal{L} \;=\; \tfrac{1}{2} m_3 |\mathbf{v}|^2 \;-\; m_3 \Phi \qquad\qquad \Phi \;=\; -\frac{GM_1}{|\mathbf{r}-\mathbf{r}_1|} \;-\; \frac{GM_2}{|\mathbf{r}-\mathbf{r}_2|}$$

$\mathbf{v}$ is the velocity of particle 3 in the inertial frame; $\mathbf{r}$ points to particle 3.

Finally, we can transform into the rotating CM frame. U terms don't change, since the r_{ij} relative distances don't change. However,

$$\begin{aligned} \mathbf{v} \;&=\; \mathbf{v}' \;+\; \mathbf{\Omega} \times \mathbf{r}' \\ &=\; \dot{\mathbf{r}}' \;+\; \mathbf{\Omega} \times \mathbf{r}' \end{aligned}$$



Primes: in the rotating frame.

2nd term: velocity of rotating frame with respect to inertial frame.

For sanity's sake, I'll now remove the primes and assume we're in the rotating frame. All coordinates (e.g., $\mathbf{r}$, $\dot{\mathbf{r}}$) are in the rotating frame:

The E–L equation (all vector components in one): $$\frac{\partial \mathcal{L}}{\partial \mathbf{r}} \;=\; \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}}\right)$$

Performing the derivatives requires the use of some vector identities. For example, one can expand out:

$$|\mathbf{v}|^2 \;=\; |\dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r}|^2 \;=\; |\dot{\mathbf{r}}|^2 \;+\; 2\mathbf{\Omega} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) \;+\; [\Omega^2 r^2 - (\mathbf{\Omega} \cdot \mathbf{r})^2]$$

$$\text{Thus,} \quad \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} \;=\; m_3(\dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r}) \qquad \Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}}\right) \;=\; m_3(\ddot{\mathbf{r}} + \mathbf{\Omega} \times \dot{\mathbf{r}})$$

$$\text{and} \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{r}} \;=\; -m_3 \nabla \Phi \;-\; m_3 \left[\mathbf{\Omega} \times \dot{\mathbf{r}} \;+\; \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\right] \;\; .$$

Each term in the E–L equation of motion contains m_3, which we can divide out to obtain:

$$\ddot{\mathbf{r}} = -2\boldsymbol{\Omega} \times \dot{\mathbf{r}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - \nabla\Phi$$

1st term on RHS: Coriolis force. 2nd term: centrifugal force.

$$\left\{ \begin{array}{lcl} \ddot{x} &=& -(\partial\Phi/\partial x) + 2\Omega\dot{y} + \Omega^2 x \\ \ddot{y} &=& -(\partial\Phi/\partial y) - 2\Omega\dot{x} + \Omega^2 y \\ \ddot{z} &=& -(\partial\Phi/\partial z) \end{array} \right\} \quad \text{with } \Phi(x,y,z) \text{ known.}$$

(Remember: x and y are coordinates in the rotating frame.)

There's a lot happening on the RHS. Édouard Roche simplified the problem by defining an effective potential containing a centrifugal term:

$$\Phi_{\text{eff}} = \Phi(x,y,z) - \frac{1}{2}\Omega^2 (x^2 + y^2) \qquad \textbf{(the Roche potential).}$$

This simplifies the equations of motion:

$$\begin{array}{rcl} \ddot{x} - 2\Omega\dot{y} &=& -\partial\Phi_{\text{eff}}/\partial x \\ \ddot{y} + 2\Omega\dot{x} &=& -\partial\Phi_{\text{eff}}/\partial y \\ \ddot{z} &=& -\partial\Phi_{\text{eff}}/\partial z \end{array}$$

and if we're looking for the **equilibrium** properties of the system, that means we're looking for properties for which particle 3 is at rest in the corotating system: $\boxed{\dot{x} = 0,\ \dot{y} = 0}$. (Stay tuned for a more complete treatment.)

Without the Coriolis forces, the equation of motion is just $\ddot{\mathbf{r}} = -\nabla\Phi_{\text{eff}}$

i.e., forces push particles along the direction of "steepest descent" down the gradient of $\Phi_{\text{eff}}(x,y,z)$. Thus, the net force is **zero** along equipotential surfaces (i.e., surfaces of constant Φ_{eff}).

Let's study these equipotentials: **"Roche surfaces."** For simplicity,

- Define separation $D = |\mathbf{r}_1 - \mathbf{r}_2|$. Thus, $\Omega = \sqrt{G(M_1+M_2)/D^3}$.
- Assume $M_1 > M_2$, and define binary mass ratio $q = M_2/M_1 < 1$.
- Place M_1 & M_2 along the x axis, with $x = 0$ being the CM. Thus,

$$x_1 = \left(\frac{M_2}{M_1+M_2}\right) D = \left(\frac{q}{1+q}\right) D \ , \qquad y_1 = z_1 = 0$$

$$x_2 = -\left(\frac{M_1}{M_1+M_2}\right) D = -\left(\frac{1}{1+q}\right) D \ , \qquad y_2 = z_2 = 0$$

Using Kepler's 3rd law for Ω, we obtain

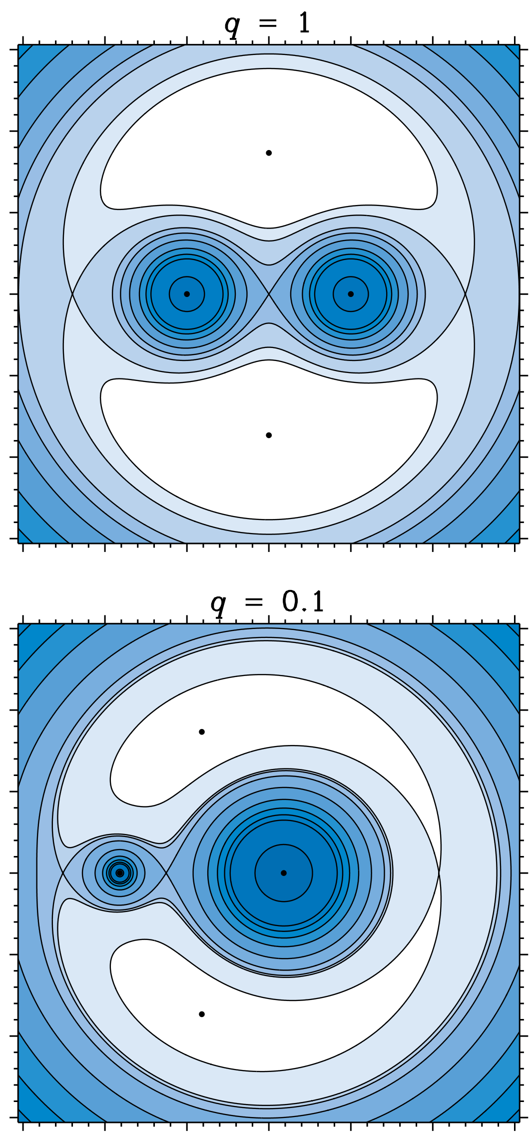
$$\Phi_{\rm eff} = -\frac{GM_1}{\sqrt{(x-x_1)^2+y^2+z^2}} - \frac{GM_2}{\sqrt{(x-x_2)^2+y^2+z^2}} - \frac{G(M_1+M_2)}{2D^3}(x^2+y^2)$$

Equipotentials are 3D surfaces, but we can still learn a lot by restricting ourselves to the orbital plane ($z=0$).

Nondimensionalize by defining $\tilde{x} = x/D$, $\tilde{y} = y/D$, and multiplying $\Phi_{\rm eff}$ by $D/[G(M_1+M_2)]$,

$$\widetilde{\Phi}_{\rm eff} = -\frac{1}{1+q}\left\{\left[\tilde{x} - \left(\frac{q}{1+q}\right)\right]^2 + \tilde{y}^2\right\}^{-1/2} - \frac{q}{1+q}\left\{\left[\tilde{x} + \left(\frac{1}{1+q}\right)\right]^2 + \tilde{y}^2\right\}^{-1/2} - \frac{1}{2}\left(\tilde{x}^2+\tilde{y}^2\right)$$

and the shapes of the surfaces depend only on one parameter: q

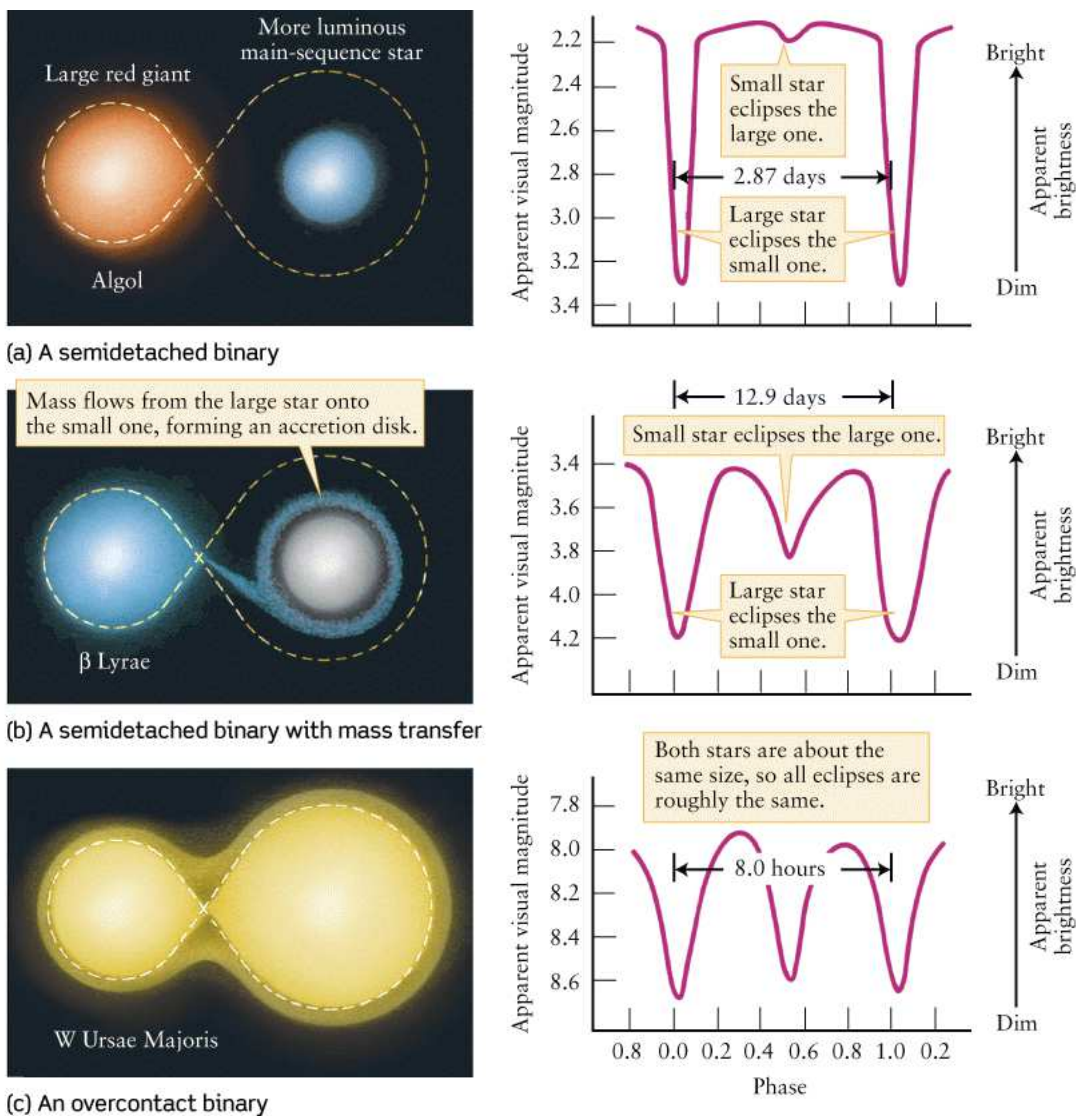


Darker colors = deeper in the potential wells. Note that the centrifugal force "wins" as $|x|$ & $|y| \to \infty$ [i.e., the system acts like 1 point-mass (M_1+M_2) at the CM, in the rotating frame].

Close to sources 1 or 2 ($r \ll D$), the local gravity wins.

Close binary stars contain fluid/plasma that often evolves to *synchronized* rotation; i.e., time-steady hydrodynamics in the rotating CM frame.
$\Longrightarrow$ (It's the freely-flowing gas "parcels" that act as m_3 in this case.)

Thus, the net gravity felt by the stars is $\mathbf{g}_{\text{eff}} = -\nabla\Phi_{\text{eff}}$, which is anti-parallel to ∇P in hydrostatic equilibrium. $\longrightarrow$ Shapes of stars follow Roche potentials:

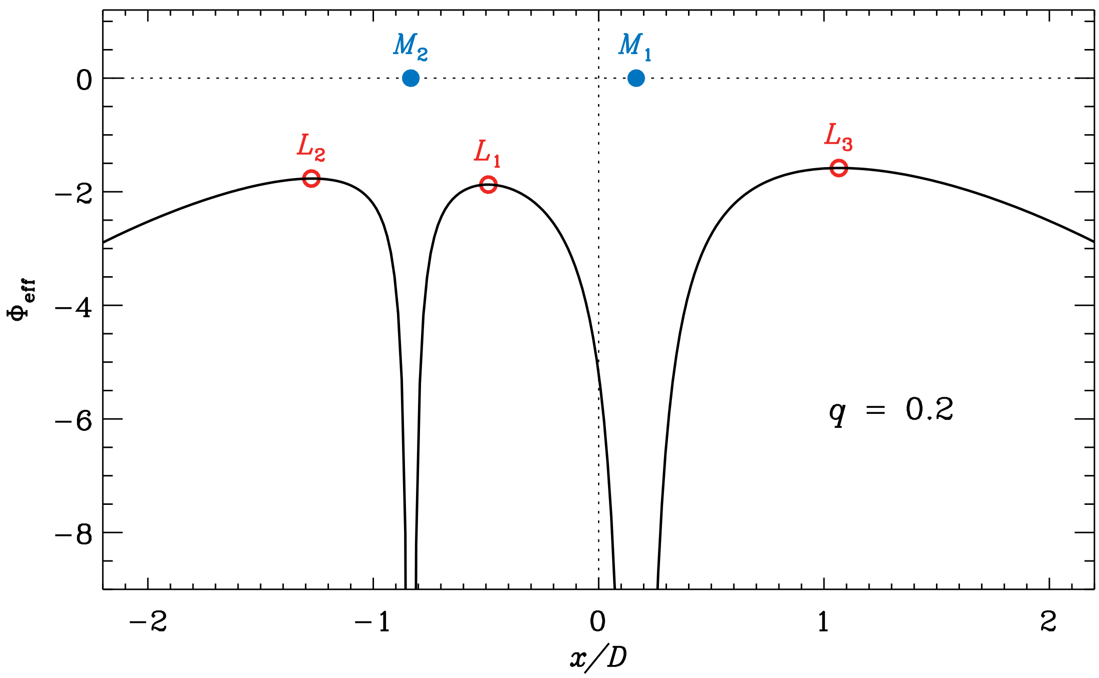


(Of course, this isn't *exactly* true, since actual stars are not point masses.)

Shapes of surfaces *inside* the Roche lobes are a combination of 2 effects: (1) oblateness (from rotation), and (2) prolateness (around the line of centers, from tidal forces). Thus, in general, a star's $R_x > R_y > R_z$.

Looking back at the equipotential surface plots, note there are 5 points at which $\nabla\Phi_{\rm eff} = 0$. Here, there are no forces on $m_3 \longrightarrow$ possible equilibria. These are the **Lagrange points** $L_1 \rightarrow L_5$.   [all in the $z = 0$ plane]

The 3 points along the x axis (L_1, L_2, L_3) are saddle points: local maxima in x, but local minima in y.



You can also think of them as "keyholes" through which particles go to change between M_1-centric orbits, M_2-centric orbits, and circumbinary orbits.

Right at L_1, L_2, and L_3, motions are **unstable** to small perturbations (in the x,y plane). Still, putting spacecraft there is possible: they can undergo quasi-stable "halo orbits" *around* the points (lots of work done at NASA in 1970s... Bob Farquhar, Kathleen Howell). See, e.g.,

- Sun–Earth L_1: space weather monitors *SOHO, ACE, DSCOVR*
- Sun–Earth L_2: astro telescopes *WMAP, Gaia, Herschel, Planck, JWST*

In general, finding their locations analytically involves solving a 5th order polynomial.   Set $\partial\Phi_{\rm eff}/\partial x = 0$.

As q becomes small (i.e., for M_1 being a star and M_2 being a planet), the L_1 & L_2 points shrink to ∼equal distances on either side of the planet, and their distances from the planet become

$$\text{For } q \ll 1, \qquad r_{\rm L1} \approx r_{\rm L2} \equiv r_{\rm H} \approx (q/3)^{1/3} D$$

This denotes the **Hill (or Jacobi) radius,** which we can derive.

(The "Hill sphere" around a planet = the region where the planet's gravity dominates; i.e., where moons can exist!)

Let's derive the distance to L_1 in the $q \ll 1$ limit. For points along the line of centers ($\tilde{y} = 0$, $x_1 > x > x_2$), and we can evaluate quantities like $\sqrt{(x - x_1)^2}$ as either $(x - x_1)$ or $(x_1 - x)$. Choosing the positive ones,

$$\widetilde{\Phi}_{\rm eff} = -\frac{1}{(1+q)(\tilde{x}_1 - \tilde{x})} - \frac{q}{(1+q)(\tilde{x} - \tilde{x}_2)} - \frac{\tilde{x}^2}{2}$$

and thus

$$\frac{\partial \widetilde{\Phi}_{\rm eff}}{\partial \tilde{x}} = -\frac{1}{(1+q)(\tilde{x}_1 - \tilde{x})^2} + \frac{q}{(1+q)(\tilde{x} - \tilde{x}_2)^2} - \tilde{x} = 0 \ .$$

We want to solve for the value of x at which the local maximum occurs. Write it as dimensionless distance $\tilde{r}$ (from M_2), with

$$\tilde{r} = \tilde{x} - \tilde{x}_2 \qquad \text{and} \qquad 1 - \tilde{r} = \tilde{x}_1 - \tilde{x} \qquad\qquad (\text{with } \tilde{r} \ll 1)$$

so that, after multiplying by $(1+q)$, the polynomial becomes

$$-\frac{1}{(1-\tilde{r})^2} + \frac{q}{\tilde{r}^2} + 1 - (1+q)\tilde{r} = 0 \ .$$

We know from the numerical solutions that for small mass ratios, we have $r \ll D$ (i.e., $\tilde{r} \ll 1$), so we can:

- expand $(1 - r)^{-2} \approx 1 + 2r + \cdots$
- ignore the second-order (tiny) $q\tilde{r}$ term (since $q \ll 1$ and $\tilde{r} \ll 1$).

Thus, the only surviving terms are

$$-3\tilde{r} + \frac{q}{\tilde{r}^2} = 0 \qquad \Longrightarrow \qquad \boxed{\tilde{r} = \left(\frac{q}{3}\right)^{1/3} = \frac{r_{\rm H}}{D}} \qquad \checkmark$$

. . . .

The other two L points occur in equilateral triangle points with M_1 & M_2...
i.e., take a cut along the $\tilde{y}$ direction along the line of centers: $\tilde{x}_c = (\tilde{x}_1 + \tilde{x}_2)/2$.

For M_1 = Sun and M_2 = planet,

$$\left\{ \begin{array}{ll} L_4 \text{ leads planet by } 60^\circ & (\text{“Greek” asteroids with Jupiter}) \\ L_5 \text{ trails planet by } 60^\circ & (\text{“Trojan” asteroids with Jupiter})^* \end{array} \right\}$$

*First one (“Achilles”) discovered in 1906, now >6000 known!

L_4 and L_5 points are true maxima in Φ_{eff}, but they may be **stable.**

But how can this be, if the points are at the *tops* of the Φ_{eff} mountains? A particle that starts at L_4 or L_5 starts to move ($\dot{\mathbf{r}} \neq 0$). Because it's off-axis, **Coriolis forces** kick in and steer the particle into “orbit” around the point.

General stability criterion: $\quad 27(M_1 M_2 + M_1 m_3 + M_2 m_3) \; < \; (M_1 + M_2 + m_3)^2$.

If $M_1 \gg \{M_2, m_3\}$, it's usually stable.

If $m_3 \to 0$, the stability condition is $\quad q \; < \; \dfrac{\sqrt{27} - \sqrt{23}}{\sqrt{27} + \sqrt{23}} \; \approx \; 0.04$,

usually okay for M_2 being a planet orbiting around M_1 star.

Earth–Moon L_5: proposed in 1970s as good location for space colonization!?

Earth–Sun L_5: definitely a good location for heliophysics **(space weather)** monitoring: sees solar features 4–5 days prior to them rotating around to point “down the barrel” at the Earth.

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Lastly, we should discuss what happens once particle m_3 is in motion. We cannot ignore Coriolis forces, but we can look for new constants of motion. Examine the equations of motion again:

$$\left\{ \begin{array}{llcl} \ddot{x} - 2\Omega\dot{y} & = -\partial\Phi_{\text{eff}}/\partial x & \Longrightarrow & \text{multiply each term by } \dot{x} \\ \ddot{y} + 2\Omega\dot{x} & = -\partial\Phi_{\text{eff}}/\partial y & \Longrightarrow & \text{multiply each term by } \dot{y} \\ \ddot{z} & = -\partial\Phi_{\text{eff}}/\partial z & \Longrightarrow & \text{multiply each term by } \dot{z} \end{array} \right\}$$

and we sum them up to obtain

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = -\dot{x}\frac{\partial\Phi_{\text{eff}}}{\partial x} - \dot{y}\frac{\partial\Phi_{\text{eff}}}{\partial y} - \dot{z}\frac{\partial\Phi_{\text{eff}}}{\partial z}$$

i.e., $$\frac{d}{dt}\left[\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)\right] = \frac{d}{dt}\left(\frac{v^2}{2}\right) = -\frac{d\Phi_{\rm eff}}{dt}$$

and we can integrate both sides, with an arbitrary constant:

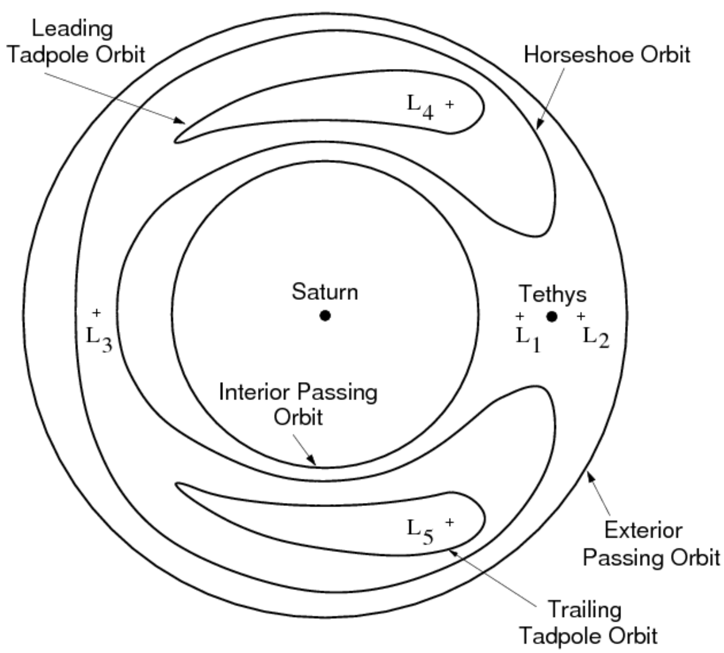
$$\frac{v^2}{2} + \Phi_{\rm eff} = E_{\rm J} = \quad \textbf{(the Jacobi integral, or Jacobi constant).}$$

The integration constant $E_{\rm J}$ is a conserved quantity (in the CR3BP). Particles with constant $E_{\rm J}$ are free to go "down" to lower $\Phi_{\rm eff}$ regions... they just have to speed up when they do.

Since $|\dot{\mathbf{r}}|^2 = v^2 \geq 0$, a particle that starts at some specific value of $E_{\rm J}$ (determined by $\mathbf{r}$ and $\dot{\mathbf{r}}$ at some initial time) must always obey

$$E_{\rm J} - \Phi_{\rm eff} \geq 0 \ .$$

- Equipotential surfaces with $E_{\rm J} = \Phi_{\rm eff}$ are called **zero-velocity surfaces.**
- The particle cannot ever access regions with $\Phi_{\rm eff} > E_{\rm J}$.
  - Some particles are trapped inside closed surfaces (Hill spheres) surrounding M_2
  - Some particles are free to wander around in "horseshoe orbits."
  - Some particles are quasi-trapped near L_4 or L_5 in "tadpole orbits" (which are *not* identical to the zero-velocity surfaces).



Central Force Motion: Orbital Resonances (true 3-body effects)

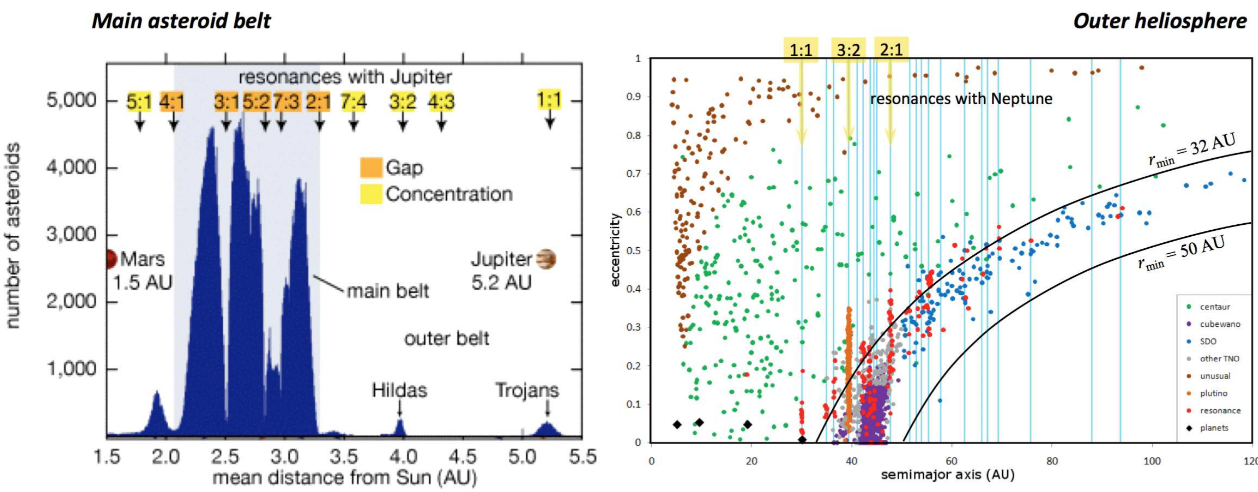
Consider 2 small-mass bodies in orbit around a massive central source.

Usually, 2 nearby orbits produce occasional close encounters, but on average (in most cases) their gravitational effects average out over long times.

However, if the 2 orbits have **regular & periodic** close encounters, the gravitational forces can add up!

One commonly seen type of interactions are **mean motion resonances (MMRs),** in which orbital periods of the 2 bodies are close to a ratio of small integers. For more info, see chapters 6–9 of Murray & Dermott.

In the solar system, we see asteroids & comets sometimes avoid MMRs, and sometimes collect around them:



(1) Unstable MMR: In some cases, repeated close encounters keep shifting the orbit of one (or both) bodies until they get **ejected** out of resonance.

Examples:

- Major planets that "clear their own orbit" by ejecting (or accreting?) objects in nearly 1:1 resonances.
- There are **Kirkwood gaps** in the asteroid belt, due to 3:1, 5:2, 7:3, and 2:1 MMRs with Jupiter.
- **Cassini's division** in Saturn's rings appears to be caused by a 2:1 resonance with Mimas.

(2) Stable MMR: In other cases, patterns arise in which the 2 bodies tend to avoid one another on repeated orbits. Sometimes the avoidance becomes self-correcting, so that objects become **trapped.**

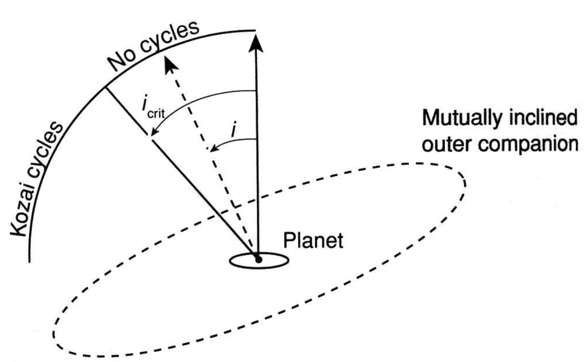
Examples:

- Jupiter's Greek & Trojan asteroids (at L_4, L_5) are in stable 1:1 MMRs.
- Pluto and Neptune are in a 2:3 MMR with one another. Some other resonances (involving elliptical perihelion precession) keep it locked in.
- Ganymede/Europa/Io are in a mutual 1:2:4 MMR, in which "triple conjunctions" are impossible.

R. Malhotra proposed that some of these resonances (e.g., Pluto & other Plutinos at 2:3) can "fill up" with a large number of bodies because the larger planet underwent substantial early **migration.** This perturbed a large swath of the protoplanetary disk, and "snowplowed up" the Plutinos.

The math is developed nicely in Chapters 6–9 of Murray & Dermott.

In addition to MMRs, there are many other types of resonances (e.g., secular & spin–orbit) that often involve interactions between the elliptical orbital elements (eccentricity, inclination, longitude of perihelion, oblateness).



One particularly dramatic secular resonance is the **Lidov-Kozai mechanism.**

Perturbations from an outer planet cause an inner planet to undergo huge swings in e & i.

May be responsible for highly inclined exoplanets (spin–orbit misalignment).

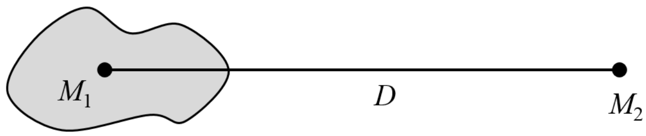
Some combination between MMRs & Lidov-Kozai may be responsible for the herding of Sedna-like objects into similar orbits by *"Planet Nine?"*

Lidov & Kozai's work in 1960s was actually preceded by Hugo von Zeipel (1910), who probably ought to get credit, too (see arXiv:1911.03984).

Central Force Motion: TIDES

We're going back to the two-body problem, but replacing one of the point-masses with an *extended body.*

Distant point-mass M_2 exerts a gravitational force on the resolved body M_1, and there are differences in $\mathbf{g}$ from one side of M_1 to the other.



Usually we need to consider only M_2's mass, *not* its shape.

M_1 is deformed, and the deformation **moves** as M_2 orbits or passes by.

Changes in shape vs. time cause **tidal torque** on M_1. To zero order, it produces a net dissipation of kinetic energy. What can that do?

- It slows M_1's rotation rate (synchronization).
  - angular momentum transfer can occur: M_1's rotation slows down, while orbital ℓ increases (e.g., the increasing Earth–Moon distance).
- If the rotation rate slows down to the point of **tidal locking** (i.e., synchronous rotation), then the torque can cause...
  - eccentricity $e \to 0$ (circularization)
  - additional angular momentum losses from the system?
- Prior to full-on circularization, a planet in a highly eccentric orbit can be nudged into *pseudo-synchronous* rotation. It's driven to be close to tidal-locking when near periastron. Hut (1981, A&A, 99, 126) predicted how M_1 would be driven toward a unique rotation period that depends on D, M_2, & the eccentrity e.
- Also, tidal torques can cause *internal* energy losses: **tidal heating** (e.g., volcanism on Io).

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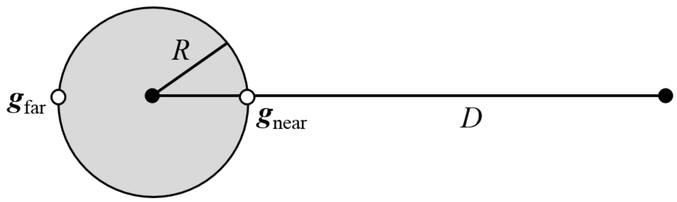
Basic idea: Look at the differential acceleration between 2 extreme points on M_1:



The mean, zero-order accel. on M_1 due to M_2 is

$$g \approx -\frac{GM_2}{D^2}$$

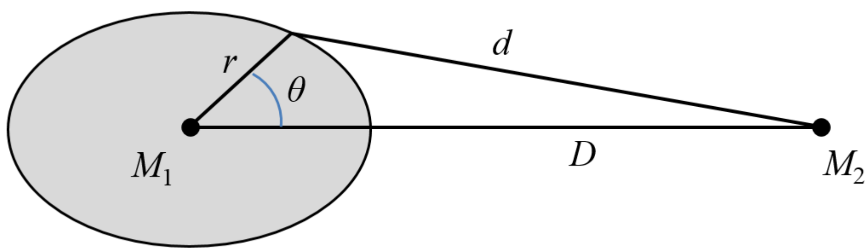
But consider $\Delta\mathbf{g} = \mathbf{g}_{\text{near}} - \mathbf{g}_{\text{far}}$.

If $R \ll D$, then differences are differentials:

$$\frac{\Delta g}{\Delta r} \approx \frac{dg}{dr} \;, \quad g = -\frac{GM_2}{r^2} \;, \quad \frac{dg}{dr} = \frac{2\,GM_2}{r^3} \;, \quad \text{so} \quad \Delta g \approx \frac{2\,GM_2}{D^3}\,\Delta r$$

and $\Delta r \approx 2R$, or maybe just R if the differential is taken between the center of M_1 and either extreme point.

However, to really figure out how a star or planet is distorted by a companion, let's look at the force in more detail.



In this case, let's simplify the gravity of M_1 as if it's a point-mass, and put aside the orbital (Ω-dependent) terms. Thus, the gravitational potential felt by a test particle at (r, θ) is

$$\boxed{\Phi \;=\; \Phi_1 \,+\, \Phi_2 \;=\; -\frac{GM_1}{r} - \frac{GM_2}{d}}$$

This is the Roche point-mass approximation, and will lead to the classical Roche equipotential surfaces if computed *exactly* (in the rotating frame for a circular orbit).

Here, however, let's expand d in terms of other known quantities, in the limit of $r \ll D$. Use the "law of cosines" for triangles:

$$d^2 \;=\; D^2 + r^2 - 2rD\cos\theta \quad .$$

What we really want to evaluate is

$$\frac{1}{d} = \frac{1}{D}\left[1+\left(\frac{r}{D}\right)^2 - 2\left(\frac{r}{D}\right)\cos\theta\right]^{-1/2}$$

and for $r \ll D$, let's expand using the binomial formula,

$$(1+\epsilon)^{-1/2} \approx 1 - \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} - \cdots$$

Note that ϵ contains terms of order (r/D) and $(r/D)^2$, and ϵ^2 contains terms of order $(r/D)^2$ and $(r/D)^4$.

Keeping all terms up to $(r/D)^2$ consistently, we get

$$\frac{1}{d} = \frac{1}{D}\left[1 - \frac{1}{2}\left(\frac{r}{D}\right)^2 + \left(\frac{r}{D}\right)\cos\theta + \frac{3}{2}\left(\frac{r}{D}\right)^2\cos^2\theta + \cdots\right]$$

$$= \frac{1}{D}\left[1 + \left(\frac{r}{D}\right)\underbrace{\cos\theta}_{P_1} + \left(\frac{r}{D}\right)^2 \underbrace{\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}_{P_2} + \cdots\right] \propto \Phi_2 \ .$$

Collins (chapter 7) shows how this expansion keeps going in terms of higher-order Legendre polynomials $P_n(\cos\theta)$.

Let's look at each term in the expansion.

Eventually we want to know about the **acceleration** due to the potential term from M_2...

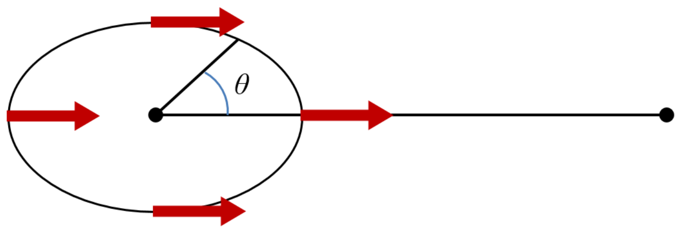
$$\mathbf{a} = -\nabla\Phi_2 = \nabla\left(\frac{GM_2}{d}\right) \ .$$

Zero-order term: The Φ_2 term is proportional to $1/D$. This is just a constant, so $\mathbf{a} = 0$.

First-order term:

$$a_r = -\frac{\partial \Phi_2}{\partial r} = \frac{GM_2}{D^2}\cos\theta$$

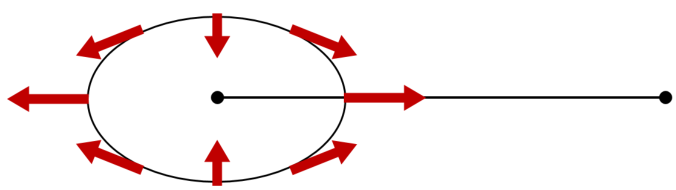
$$a_\theta = -\frac{1}{r}\frac{\partial \Phi_2}{\partial \theta} = -\frac{GM_2}{D^2}\sin\theta$$



This is straightforward gravitational **attraction** of the whole body.

If M_1 is in a circular orbit around M_2, then this is cancelled out if we go into the co-orbiting reference frame.

Second-order term: Here's the dominant tidal distortion:



$$\Phi_2 = -\frac{GM_2 r^2}{2D^3}(3\cos^2\theta - 1)$$

Earlier we speculated that time-steady *stellar surfaces* coincide with equipotentials (i.e., the net force on a particle vanishes along an equipotential surface).

Thus, if all points along the distored surface $R_*(\theta)$ have identical values of Φ, we can equate Φ at two different values of θ to derive the shape of M_1.

$$\Phi(r,\theta) = \Phi(r,0) \qquad \text{(right-hand side: “pole” along line-of-centers)}$$

$$-\frac{GM_1}{R_*} - \frac{GM_2 R_*^2}{2D^3}(3\cos^2\theta - 1) = -\frac{GM_1}{R_\mathrm{p}} - \frac{GM_2 R_\mathrm{p}^2}{D^3}$$

Simplify by specifying the *equator* on the left side ($\theta = \pi/2$), so that we're eventually solving for $x_\mathrm{e} = R_\mathrm{eq}/R_\mathrm{pol}$.

Multiply by constants to make each term non-dimensional, and we obtain a cubic equation for $x_{\rm e}$

$$\boxed{\left(\frac{Q}{2}\right) x_{\rm e}^3 \;+\; (1+Q)x_{\rm e} \;-\; 1 \;=\; 0} \qquad \text{where} \quad Q \equiv \frac{M_2}{M_1}\left(\frac{R_{\rm p}}{D}\right)^3$$

and Q is a "tidal deformation parameter."

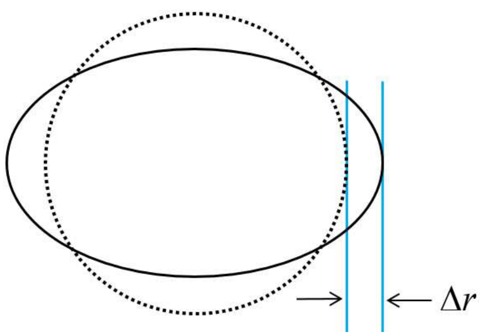
There are analytic solutions, but in a lot of cases we care about *weak* tidal effects, in which $Q \ll 1$. In that case, $x_{\rm e}$ is close to 1.

$$\text{Assuming } x_{\rm e} \approx 1+\epsilon, \text{ then to 1st order,} \qquad x_{\rm e} \;\approx\; 1 - \frac{3Q}{2} \;\; .$$

Note that $x_{\rm e} < 1$. The object is **prolate** (stretched out along its $\theta = 0$ axis).

. . . .

It's useful to estimate the so-called "bulge height" due to tidal deformation.



Compare it to a **sphere** of equivalent volume.

Also assume it's a prolate spheroid, with

$$\begin{aligned} V = (4\pi/3)abc &= (4\pi/3)R_{\rm p}R_{\rm eq}^2 \\ &= (4\pi/3)R_{\rm p}^3 x_{\rm e}^2 \end{aligned}$$

Equate it to the sphere's equivalent volume $V_0 = (4\pi/3)R_0^3$

$$\text{Thus,} \quad R_0^3 = R_{\rm p}^3\, x_{\rm e}^2$$

One way to define the bulge height is

$$\Delta r \;\equiv\; R_{\rm p} - R_0 \;=\; \left[x_{\rm e}^{-2/3} - 1\right] R_0 \;\approx\; \left[\left(1 - \frac{3Q}{2}\right)^{-2/3} - 1\right] R_0$$

and, continuing to assume $Q \ll 1$,

$$\boxed{\frac{\Delta r}{R_0} \approx Q \approx \frac{M_2}{M_1}\left(\frac{R_0}{D}\right)^3} \qquad \text{(using } R_0 \text{ in the definition of } Q\text{).}$$

Others define the tidal bulge as the difference between the max & min radii,

$$\text{flattening } f \equiv \frac{R_\mathrm{p} - R_\mathrm{eq}}{R_\mathrm{p}} = 1 - x_\mathrm{e} \approx \frac{3Q}{2} \ .$$

We've made a lot of assumptions. Not only $Q \ll 1$, but we also assumed the infinite series of Legendre polynomials can be cut off at P_2.

That assumption is essentially that $r \ll D$, and so it's similar to $Q \ll 1$ as limiting us to "weak tides."

. . . .

What happens when the tides are strong?

Well, M_1 certainly won't hold together if the tidal force along the line of centers (i.e., at $r = R_\mathrm{p}$ and $\theta = 0$) is **stronger** than its own self-gravity!

$$\text{That occurs for } \quad \frac{GM_1}{R_\mathrm{p}^2} \approx \left| \frac{\partial \Phi_2}{\partial r}(\theta = 0) \right| = \frac{2GM_2 R_\mathrm{p}}{D^3}$$

$$\text{i.e., a critical value of } \quad \frac{M_2}{M_1}\left(\frac{R_\mathrm{p}}{D}\right)^3 = \frac{1}{2} \equiv Q_\mathrm{crit} \ .$$

If Q exceeds that value, it's unlikely that M_1 will remain a single, centrally condensed body.

The force-balance above was essentially an estimate of the distance to the L_1 Lagrange point. For a synchronously rotating binary system, the Roche equipotentials include a centrifugal term, which changes the above force-balance a bit.

As seen earlier, the actual **Hill radius** result (using our current notation) in the limit of $r \ll D$ is

$$\frac{r}{D} = \left(\frac{1}{3}\frac{M_1}{M_2}\right)^{1/3} \ , \quad \text{or exactly } \ Q_\mathrm{crit} = \frac{1}{3} \ .$$

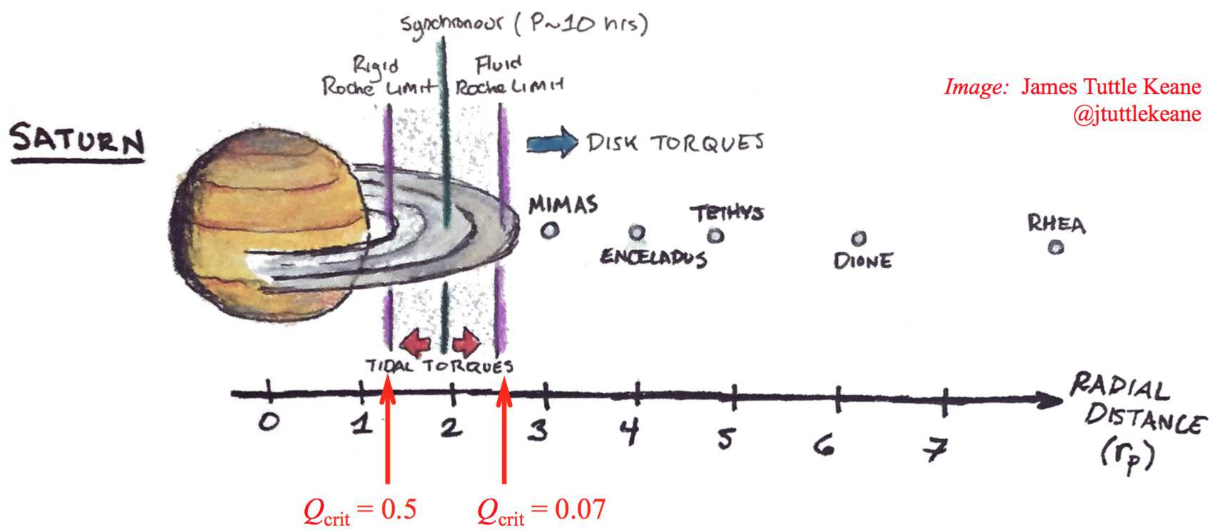
This critical point is sometimes written in terms of the *minimum distance D* that M_2 can have before its tidal forces break up M_1.

$$\text{i.e.,} \qquad D_\mathrm{crit} = R_1 \, Q_\mathrm{crit}^{-1/3} \left(\frac{M_2}{M_1}\right)^{1/3} \ .$$

Numerical models for real extended fluid bodies give values of $Q_{\rm crit}$ between about 0.07 and 0.3, depending on the internal structure of M_1.

The traditional **Roche limit** for breakup (i.e., planetary ring formation) uses the simple estimate of $Q_{\rm crit} = 1/2$ above, but more realistic (smaller) values of $Q_{\rm crit}$ give larger values for $D_{\rm crit}$.

Example: For M_2 = Saturn & M_1 = Mimas, Saturn's rings seem to encompass the full range of fluid/solid $Q_{\rm crit}$ values:



However, a more complete description of what actually confines Saturn's rings is given by Tajeddine et al. (2017, *ApJ Suppl.,* 232, 28).

All of the above is true if the equipotentials are allowed to "float freely," i.e., if the fluid inside the star/planet is **perfectly elastic.**

For rocky planets & moons, it's not so elastic.

Solid substances have a characteristic **shear strength** (or shear rigidity) $S_\perp$, given in units of pressure, and we'll estimate its value below.

Planetary scientists define a so-called "tidal Love number,"

$$k_{\rm T} \equiv \frac{3/2}{1 + [S_\perp/(\rho g R)]}$$

where the quantity $\rho g R$ is a back-of-envelope estimate for the *central pressure* of a self-gravitating body.

For elastic fluids $S_\perp$ is negligible (i.e., there's no resistance to deformation), so $k_\mathrm{T} \approx 3/2$.

For rigid materials, $S_\perp$ is large, so $k_\mathrm{T} \ll 3/2$.

The general way to write the bulge "flattening" is

$$f \;\approx\; k_\mathrm{T} Q \;\approx\; k_\mathrm{T} \frac{M_2}{M_1} \left(\frac{R_0}{D} \right)^3 \quad .$$

Deriving the shear strength will tell us more about **tidal heating.**

When there's resistance to deformation, the tidal energy still has to go somewhere... friction will dissipate it as heat.

The shear strength is formally defined as

$$S_\perp \;=\; \frac{F_\perp / A}{r/R_0} \;=\; \frac{\text{applied shear stress}}{\text{relative sheared displacement}} \; .$$

In other words, in order to shear a rigid body over a distance r, one needs to apply a transverse force

$$F_\perp \;=\; A\, S_\perp \frac{r}{R_0} \;\; .$$

The **work** done by applying this force gives the amount of energy expended:

$$\Delta E \;=\; \int_0^r dr' \; F_\perp \;=\; \rightsquigarrow \rightsquigarrow \;=\; \frac{A}{2} S_\perp \frac{r^2}{R_0} \; .$$

What is the area A? Very roughly, if the force is acting over the *whole planet,* then $A \approx \pi R_0^2$.

$$\text{Neglecting order-unity constants,} \quad \Delta E \;\approx\; S_\perp R_0 \, r^2 \;\approx\; S_\perp R_0 \, (\Delta r)^2$$

In a binary system, we can estimate the average **power** released over one orbit, with $\Delta t =$ the period.

(Over one orbit, the tidal bulge gets swept through the whole planet.)

$$\text{Thus, let's try} \quad L_\mathrm{tide} \;=\; \frac{\Delta E}{\Delta t} \;\approx\; \frac{S_\perp (\Delta r)^2 R_0}{\Delta t}$$

$$\text{and if } S_\perp \text{ scales with } \; \rho g R_0 \sim \frac{M_1^2}{R_0^4} \quad \text{(from global hydrostatic balance)}$$

and if we use the bulge height approximation above (scaling out constants like $k_{\rm T}$), we get

$$L_{\rm tide} \propto \frac{M_2^2\, R_0^5}{D^6\, \Delta t} \qquad (M_1 \text{ drops out!}).$$

If we work all this out for Io's molten core, we first would look up $S_\perp \sim 10^{11}$ dynes/cm^2. This is several orders of magnitude bigger than the compressive strength of a rock or iron core.

For Io, we would get

$$\frac{\Delta r}{R_0} \approx 10^{-3} \quad \text{and} \quad L_{\rm tide} \approx 10^{25} \text{ erg/s} \ .$$

This is about $10^{-9} L_\odot$. Tiny, but since the mass of Io is about $10^{-8} M_\odot$, that's not too shabby.

Problem: Observationally, Io emits only $L_{\rm tide} \approx 10^{21}$ erg/s.

In reality, all that ΔE work isn't all going into heating.

In many systems, most of the tidal work goes into changing the angular momentum of the planet, or of the orbit.

However, if the orbit is **elliptical,** there's a net change in the magnitude of the tidal work done over the orbit. *That* component is more likely to go directly into heating.

For an orbit with eccentricity e, roughly speaking the *relative* sheared displacement

$$\text{isn't } \frac{\Delta r}{R_0} \ , \quad \text{instead it's} \quad \sim e\,\frac{\Delta r}{R_0} \ .$$

Thus, one must multiply our $L_{\rm tide}$ above by a factor of e^2.

For Io, $e \approx 0.0043$, so $L_{\rm tide}$ is reduced to about 3×10^{20} erg/s. Much closer to the observed value of $\sim 10^{21}$.
