Next part of course: **CLASSICAL DYNAMICS**

Gravity is important to all fields of astronomy & astrophysics.

Gravitational "celestial mechanics" is applicable over $\sim$20 orders of magnitude in size scale: from comets (1 km) to galaxy superclusters (100 Mpc)!

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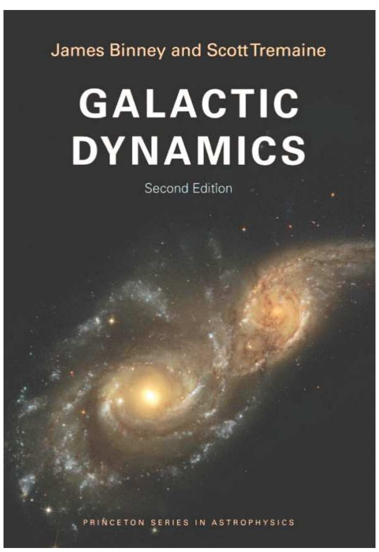
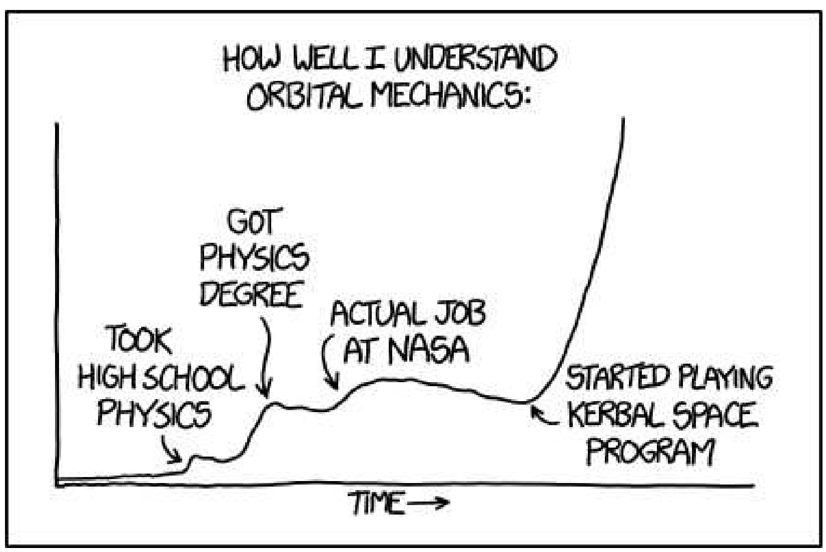
We'll start with a review of Newtonian motion of a single particle, then apply it to mutual gravitation of multiple bodies:

- Lecture 08: The two-body problem ($N = 2$) of Keplerian orbits,
- Lecture 09: The three-body problem ($N = 3$) of Roche lobes, tides, etc.,
- Lecture 10: $N \gg 1$ systems like galaxies, extending our earlier studies of statistical mechanics, collisions, and even similar applications of the Boltzmann equation...

Even without collisions, the $N \gg 1$ body problem is non-trivial; i.e., what is the motion of a "test particle" in a smooth gravitational potential $\Phi(\mathbf{r})$ caused by millions of other particles?





Newtonian Physics leads to Conservation Laws

First, we can start with Newton's laws as "axioms" and review how some basic conservation laws arise as a natural consequence.

THEN we'll derive them using even more fundamental principles (Euler-Lagrange theory).

Kepler's law's of planetary motion can be derived from either Newton *OR* from Euler-Lagrange... but the latter is more generalizable to more complex systems.

Isaac Newton (1643-1727)

1. A body will remain at rest, or moving at a constant velocity, unless it is acted on by an unbalanced force **F**.
2. **F** = d**p**/dt = m**a**.
3. When body 1 exerts a force on body 2, body 2 simultaneously exerts an equal & opposite force on body 1: $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

There is also the unofficial fourth law: Newton's universal law of gravitation. The force on particle 1, due to particle 2 is written in vector form as

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

where vectors $\mathbf{r}_i$ point to the locations of particles $i = 1, 2$, and $|\mathbf{r}_1 - \mathbf{r}_2|$ is the scalar distance between the two particles (which we often just call r).

Let's now derive conservation laws for the two-body problem...

The usual first one is **conservation of mass.** In our situation, it's trivially satisfied:

$$\frac{d}{dt}(m_1 + m_2) = 0 \qquad \text{i.e.,} \qquad m_1 + m_2 = \text{constant.}$$

Even if we extend it to allow for one body to "donate" some of its own mass to the other, this still works.

Next is **conservation of momentum.** For particle i, the classical definition of momentum is $\mathbf{p}_i = m_i \mathbf{v}_i$, and it's clear that Newton's 2nd law is more simply described in terms of it:

$$\mathbf{F}_i \; = \; m_i \, \mathbf{a}_i \; = \; m_i \frac{d\mathbf{v}_i}{dt} \; = \; \frac{d\mathbf{p}_i}{dt} \; .$$

For our two-body system, how does the total momentum ($\mathbf{p}_{\rm tot} = \mathbf{p}_1 + \mathbf{p}_2$) change in time?

$$\frac{d\mathbf{p}_{\rm tot}}{dt} \; = \; \frac{d}{dt}(\mathbf{p}_1 \, + \, \mathbf{p}_2) \; = \; \mathbf{F}_{12} \, + \, \mathbf{F}_{21} \; = \; 0$$

where Newton's 3rd law is embedded in the form of the universal law of gravitation: i.e., the force on 1 (due to 2) is **equal & opposite** to the force on 2 (due to 1).

Thus, $\mathbf{p}_{\rm tot}$ is constant over time, and total momentum is conserved.

Notes about this:

- I'll probably use $\mathbf{F}_i$ and $\mathbf{F}_{ij}$ interchangeably for the force on particle i (due to j).
- Conservation of total momentum is true for N-body systems, too. For every particle i being tugged on by j, there's also the equal and opposite force on j by i... in the big sum, all pairs of forces cancel out!

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Next is **conservation of angular momentum.** This is another vector quantity that's defined as

$$\mathbf{L}_i \; = \; \mathbf{r}_i \, \times \, \mathbf{p}_i$$

and it's important to note that because $\mathbf{L}_i$ depends on $\mathbf{r}_i$, it is always defined relative to a specific choice of origin.

For a single particle, how does it change in time?

$$\frac{d\mathbf{L}_i}{dt} \; = \; \left(\mathbf{r}_i \, \times \, \frac{d\mathbf{p}_i}{dt} \right) \, + \, \left(\frac{d\mathbf{r}_i}{dt} \, \times \, \mathbf{p}_i \right) \; = \; (\mathbf{r}_i \, \times \, \mathbf{F}_i) \, + \, (\mathbf{v}_i \, \times \, \mathbf{p}_i) \; = \; \mathbf{r}_i \, \times \, \mathbf{F}_i \; .$$

Because $\mathbf{v}$ is parallel to $\mathbf{p}$, the second cross product is zero.

The quantity $\mathbf{r}_i \, \times \, \mathbf{F}_i$ is called the **torque** on particle i.

For our 2-body system,

$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = (\mathbf{r}_1 \times \mathbf{F}_{12}) + (\mathbf{r}_2 \times \mathbf{F}_{21}) = (\mathbf{r}_1 \times \mathbf{F}_{12}) - (\mathbf{r}_2 \times \mathbf{F}_{12})$$

$$= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} \ .$$

However, the vector $\mathbf{r}_1 - \mathbf{r}_2$ points along a line joining the two particles. So does $\mathbf{F}_{12}$! Thus, this is a cross product of two parallel vectors, and

$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = 0 \qquad \text{i.e.,} \qquad \mathbf{L}_{\text{tot}} = \text{constant over time}$$

and the total angular momentum is conserved.

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Lastly, we'll examine **conservation of total energy.**

For a single particle, we begin by defining its **kinetic energy** (a scalar):

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{p_i^2}{2m_i}$$

where we use the shorthand $v_i^2 = |\mathbf{v}_i|^2 = (\mathbf{v}_i \cdot \mathbf{v}_i)$, and so on.

How does it change in time? Let's use the product rule...

$$\frac{dK_i}{dt} = \frac{m_i}{2} \frac{d}{dt} (\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{m_i}{2} \left[2 \left(\frac{d\mathbf{v}_i}{dt} \cdot \mathbf{v}_i \right) \right]$$

$$= \frac{d\mathbf{p}_i}{dt} \cdot \mathbf{v}_i = \mathbf{F}_i \cdot \mathbf{v}_i = \mathbf{F}_i \cdot \frac{d\mathbf{r}_i}{dt} \ .$$

Mathematicians hate when we do this, but let's write this relationship as

$$dK_i = \mathbf{F}_i \cdot d\mathbf{r}_i \ .$$

Over time, consider that particle i moves from position a to position b along a given path. If we integrate along the path,

$$K_i(b) - K_i(a) = \int_a^b \mathbf{F}_i \cdot d\mathbf{r}_i = W_{ab}$$

where we define the integral as W_{ab}, the total **work** done by the force on the particle over the path from a to b.

The above is called the **work-energy theorem,** and it's essentially Newton's 2nd law, but expressed in "energy language." A force exerted on a particle gives it an acceleration $\Longrightarrow$ work done on a particle gives it kinetic energy.

Work is a scalar, but it has a sign: $W > 0$ means net work was done ON the particle, so the kinetic energy at the *end* of the path is higher than at the beginning. If $W < 0$, it means work was done BY the particle ON the environment, so its kinetic energy must decrease.

(Notice that work is done only when there's a nonzero component of the force projected *parallel to* the particle's path. If $\mathbf{F}$ is perpendicular to the path, you can change a particle's direction, but no work is done... so you don't change the particle's kinetic energy!)

By itself, the work-energy theorem is NOT conservation of energy.
A system's total K_i can change over time!

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The above was true for any kind of force. However, what comes next is true for only some forces...

Gravity is a "conservative" force, which means that W_{ab} is the same, no matter what path is chosen between a and b. (One can think about *non*-conservative forces, like friction...)

In this case, we can take point a to be any arbitrary $\mathbf{r}$, and take point $b \to \infty$ (i.e., very far away from any neighboring bodies), so that the force of gravity on particle i there is zero.

Then, we define the particle's **potential energy** function such that

$$U_i(\mathbf{r}) = W_{\mathbf{r}\infty} = \int_{\mathbf{r}}^{\infty} \mathbf{F}_i \cdot d\mathbf{r}_i'$$

i.e., the potential energy at $\mathbf{r}$ = the work done by a conservative force to bring the particle from $\mathbf{r}$ to ∞.

Because conservative forces are path-independent, then

$$(W_{ab} + W_{b\infty}) \quad \text{ought to always be equal to} \quad W_{a\infty} \ .$$

Thus,

$$W_{ab} = W_{a\infty} - W_{b\infty} = U_i(a) - U_i(b)$$

and, for the above finite path ($a \to b$), the work done by a conservative force is equal to *minus* the change in potential energy associated with that force.

So, if

$$K_i(b) - K_i(a) = U_i(a) - U_i(b) \ , \quad \text{then} \quad \boxed{K_i(a) + U_i(a) = K_i(b) + U_i(b)}$$

i.e., conservation of total (kinetic + potential) energy for the particle.

What is the **total energy** of our two-body system? We can see that

$$E \ = \ \sum_{i=1}^{2} (K_i + U_i) \ = \ \text{constant over time.}$$

Each particle has its own kinetic energy K_i, so their total is just a simple sum.

When we work out the sum over the U_i terms, something interesting happens:

$$U_{\text{tot}} \ = \ \int_{\mathbf{r}}^{\infty} (\mathbf{F}_{12} \cdot d\mathbf{r}_1 \ + \ \mathbf{F}_{21} \cdot d\mathbf{r}_2)$$

$$= \ \int_{\mathbf{r}}^{\infty} \mathbf{F}_{12} \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) \ = \ \int_{\mathbf{r}}^{\infty} \mathbf{F}_{12} \cdot d(\mathbf{r}_1 - \mathbf{r}_2) \ .$$

If we use the shorthand $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, then

$$\mathbf{F}_{12} \ = \ -\frac{Gm_1m_2\mathbf{r}}{|\mathbf{r}|^3} \qquad \text{so} \qquad U_{\text{tot}} \ = \ -Gm_1m_2 \int_{\mathbf{r}}^{\infty} \frac{\mathbf{r} \cdot d\mathbf{r}}{r^3}$$

and if we're correct that the choice of path doesn't matter, then let's just assume $\mathbf{r}$ and $d\mathbf{r}$ are parallel, and we integrate outwards along the line-of-centers between the two particles:

$$U_{\text{tot}} \ = \ -Gm_1m_2 \int_{r}^{\infty} \frac{dr'}{(r')^2} \ = \ \frac{Gm_1m_2}{\infty} - \frac{Gm_1m_2}{r} \ = \ -\frac{Gm_1m_2}{r}$$

and the two-body system has only one "mutual" potential energy term.

(An N-body system will have $N(N-1)/2$ potential energy terms, corresponding to the number of unique pairings between the bodies.)

Thus,

$$E \ = \ \frac{1}{2}m_1|\mathbf{v}_1|^2 \ + \ \frac{1}{2}m_2|\mathbf{v}_2|^2 \ - \ \frac{Gm_1m_2}{r} \ = \ \text{constant over time.}$$

Calculus of Variations (all of it I hope you'll ever need!)

Now we switch gears for a bit, and **re-derive** some of the above bits of fundamental physics from different "first principles." This starts with a bit of pure/abstract math, but you'll soon see that the physics applications are profound....

Consider a 3D trajectory of a particle $\mathbf{x}(t)$, which we examine between times $t_1 \le t \le t_2$.

Later we'll define a *functional* called the Lagrangian, which may depend on position $\mathbf{x}(t)$, velocity $\dot{\mathbf{x}}(t)$, and time itself:

$$\mathcal{L}\left[\mathbf{x}(t),\, \dot{\mathbf{x}}(t),\, t\right] \qquad \text{(leave it general for now)} .$$

Let's also define the path integral:
$$I \equiv \int_{t_1}^{t_2} dt\; \mathcal{L}\left[\mathbf{x}(t),\, \dot{\mathbf{x}}(t),\, t\right] .$$

With $\mathcal{L}$ and $\mathbf{x}$ specified, I is just a scalar number that we can compute.

Let's see what happens if we find the one (unique?) trajectory $\mathbf{x}(t)$ that causes the value of I to be a local extremum (i.e., either minimum or maximum) compared to all neighboring trajectories. This turns out to be the path the particle **actually takes.**

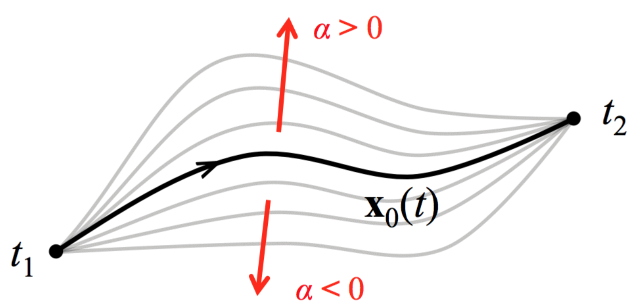
Binney & Tremaine (§B.7) derive this one way; I'll follow Marion's *Classical Dynamics* book. Start by assuming we know the "extremum path" $\mathbf{x}_0(t)$.

We parameterize a given set of "neighbor trajectories" as

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \alpha\, \mathbf{x}_1(t)$$

and we fix the endpoints at t_1 & t_2,

$$\mathbf{x}_1(t_1) = \mathbf{x}_1(t_2) = 0 .$$



(We could "fill the space" around $\mathbf{x}_0(t)$ by specifying any number of unique $\mathbf{x}_1$ perturbations, but let's just look at one at a time.)

Anyway, we'd like to know how to specify the constraint that $I(\alpha)$ must have an extremum at $\alpha = 0$.

In other words, if $\mathbf{x}_0$ truly is the extremum path, then

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0 \qquad\qquad \text{for this particular choice of } \mathbf{x}_1$$

...and then later for all possible $\mathbf{x}_1$'s.

Our goal is now to figure out what conditions need to hold, in order for the above extremum condition to be *TRUE.*

This will put constraints on how $\mathcal{L}$ evolves in space & time.

We evaluate the α derivative by noting that the integration limits are fixed, so $\partial/\partial\alpha$ affects only the integrand. Use chain rule:

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial \alpha}\right]$$

and we know

$$\frac{\partial \mathbf{x}}{\partial \alpha} = \mathbf{x}_1 \qquad\qquad \frac{\partial \dot{\mathbf{x}}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{d\mathbf{x}}{dt}\right) = \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \alpha}\right) = \dot{\mathbf{x}}_1 \qquad\qquad \frac{\partial t}{\partial \alpha} = 0$$

where the last one can be seen by realizing that the α parameter is really just a function of space, not time.

$$\text{Thus,} \qquad \frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} dt \left[\mathbf{x}_1 \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{x}} + \dot{\mathbf{x}}_1 \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right] \; .$$

The 2nd term can be integrated by parts. Look at one Cartesian component at a time:

$$\int_{t_1}^{t_2} dt\, \frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{dx_1}{dt} = \left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\, x_1\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) x_1$$

and the 1st term on RHS $= 0$ because $\mathbf{x}_1(t_1) = \mathbf{x}_1(t_2) = 0$. Thus,

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} dt \left\{\mathbf{x}_1 \cdot \left[\frac{\partial \mathcal{L}}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)\right]\right\}$$

Even though α doesn't appear explicitly, this still formally depends on it (since $\mathbf{x}$ & $\dot{\mathbf{x}}$ depend on α). However, we want to evaluate $(\partial I/\partial\alpha)_{\alpha=0} = 0$.

Also, we can realize that when we write $\mathbf{x}$, we're really referring to the "central" trajectory $\mathbf{x}_0$.

Since $\mathbf{x}_1$ is a completely arbitrary perturbation, we see the only way to make

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0 \qquad \text{is to require} \qquad \boxed{\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \nabla \mathcal{L} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)}$$

everywhere along the trajectory. This is the **Euler–Lagrange (E–L) eqn.**

To sum up, it's the condition that *must* hold true if we are on that one unique trajectory that minimizes I.

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Historically, many people realized that Nature seems to always want to minimize the "action" (i.e., time-integrated energy along a path) in a system.

Hero of Alexandria (∼50 AD) $\longrightarrow$
Fermat, Leibniz, Euler, Maupertuis, Lagrange (1700s) $\longrightarrow$
Hamilton, who unified classical dynamics (quote from *Marion*):

> In two papers published in 1834 and 1835, Hamilton‡ announced the dynamical principle upon which it is possible to base all of mechanics and, indeed, most of classical physics. Hamilton's Principle may be stated as follows:
>
> > Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.
>
> ‡ Sir William Rowan Hamilton (1805–1865), Scottish mathematician and astronomer, and later, Irish Astronomer Royal.

...and you know that Feynman took it even further, into quantum mechanics.

(FYI: In all cases we'll encounter, the "extrema" are all minima.)

The relevant functional $\mathcal{L}$ is called the **Lagrangian** of the particle:

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) \equiv K(\dot{\mathbf{x}}) - U(\mathbf{x}, \dot{\mathbf{x}})$$

where the kinetic energy K depends only on $\dot{\mathbf{x}}$ (velocity).

We'll tend to encounter **conservative** force fields, for which the potential energy U is a function only of position $\mathbf{x}$.

Consider a force derivable from a potential energy: $\mathbf{F} = -\nabla U$.
What does the E–L equation imply?

$$\mathcal{L} \;=\; K - U \;=\; \frac{1}{2}m|\dot{\mathbf{x}}|^2 \;-\; U(\mathbf{x}) \qquad \text{so} \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \;=\; -\nabla U \;=\; \mathbf{F}$$

and also,

$$K = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad \frac{\partial K}{\partial \dot{x}} \;=\; m\dot{x} \;\;, \quad \text{etc.,} \quad \text{so} \qquad \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \;=\; m\dot{\mathbf{x}} \;.$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) \qquad \Longrightarrow \qquad \begin{array}{rcl} -\nabla U & = & m\ddot{\mathbf{x}} \\ \mathbf{F} & = & m\mathbf{a} \quad ! \end{array}$$

Newton's 2nd law is *derivable* from Hamilton's principle.

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Advantages of Lagrangian dynamics:

- $\mathcal{L}$ is a scalar, whereas forces & momenta (in the traditional "equation of motion") are vectors.
- Sometimes it's difficult to specify the full list of *forces* acting on a body (including nebulous "forces of constraint"). Not needed here!
- This works even for non-Cartesian coordinates & non-inertial frames. In fact, for an N-dimensional system, if you can uniquely define some other set of **generalized coordinates,**

  $$\left.\begin{array}{rcl} q_i & = & q_i(x_1, x_2, \ldots, x_N) \\ \dot{q}_i & = & \dot{q}_i(x_1, x_2, \ldots, x_N\,;\; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N) \end{array}\right\} \quad i = 1, 2, \ldots, N$$

  then you can go through the chain rule to show that

  $$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) \qquad \text{is valid for each coordinate, too,}$$

  even if the $\mathbf{q}$'s *do NOT all have units of length!*
  Thus, the E–L equation is the same in all coordinate systems.

Aside: There's neat symmetry if we also define a generalized momentum,

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \qquad \text{(which can be verified from the full expression for } K\text{)}$$

so that the E–L equation can be written as: $$\dot{\mathbf{p}} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \ .$$

Soon, we'll look at 2 examples of generalized coordinates:

- spherical coordinates $(r,\ \theta,\ \phi)$ for relative motions between two bodies,
- rotating-frame coordinates, which let us derive the centrifugal & Coriolis forces from the E–L equation.

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First, though, there are several important general principles we can PROVE as consequences of symmetry. ($\equiv$ **Noether's theorem**)

Emmy Noether (1882-1935)

Every fundamental symmetry of nature has a corresponding conservation law:

1. A system obeying translation symmetry (i.e., laws are the same at any location) conserves **momentum.**
2. A system obeying rotational invariance conserves **angular momentum.**
3. A system obeying time invariance conserves **energy.**

(1) Energy conservation: Consider a closed system, for which

$$\frac{\partial}{\partial t}\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t) = 0 \qquad \text{i.e., } \mathcal{L} \text{ doesn't depend explicitly on absolute time } t \ .$$

The total derivative with respect to time can be written

$$\frac{d\mathcal{L}}{dt} = \dot{\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{x}} + \ddot{\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = \underbrace{\dot{\mathbf{x}} \cdot \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)}_{\text{from E–L}} + \ddot{\mathbf{x}} \cdot \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)$$

Thus, from the chain rule,

$$\frac{d\mathcal{L}}{dt} \;=\; \frac{d}{dt}\left(\dot{\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)$$

which can be rearranged to $$\frac{d}{dt}\left(\dot{\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} - \mathcal{L}\right) \;=\; 0$$

and the quantity in parentheses is a constant. Call it the **Hamiltonian:**

$$\mathcal{H} \;\equiv\; \dot{\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} - \mathcal{L} \;=\; \text{constant} \;.$$

However, if we think back to the definition of the Lagrangian ($\mathcal{L} = K - U$), and assume $U = U(\mathbf{x})$, we see that

$$\dot{\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \;=\; m|\dot{\mathbf{x}}|^2 \;=\; 2K \quad , \qquad \text{so} \qquad \mathcal{H} \;=\; 2K - (K - U) \;=\; K + U$$

i.e., **total energy is conserved** as a consequence of time invariance.

(2) Momentum conservation: A closed system ought to also be invariant to absolute *translations* (of the entire system) in space;

i.e., $\mathcal{L}$ should remain fixed if we replace $\mathbf{x} \longrightarrow \mathbf{x} + \delta\mathbf{x}$,
where $\delta\mathbf{x}$ is a fixed (small?) displacement.

With that, $\mathcal{L}$ should $\longrightarrow \mathcal{L} + \delta\mathcal{L}$, via Taylor expansion,

$$\delta\mathcal{L} \;=\; \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \cdot \delta\mathbf{x} \;+\; \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \cdot \delta\dot{\mathbf{x}}$$

However, if the displacement is fixed, $$\delta\dot{\mathbf{x}} \;=\; \frac{d}{dt}(\delta\mathbf{x}) \;=\; 0 \;,$$

and we want to also specify $\boxed{\delta\mathcal{L} = 0}$. Thus,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \cdot \delta\mathbf{x} \;=\; 0 \;, \qquad \text{so for arbitrary } \delta\mathbf{x}, \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \;=\; 0 \;.$$

From the E–L equation, this means

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) \;=\; 0 \;, \qquad \text{so} \qquad \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \;=\; \text{constant} \;.$$

Lastly, we know

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}} = \mathbf{p}$$

so **linear momentum is conserved** as a consequence of spatial (translational) invariance.

Very similarly, we could also show that **angular momentum is conserved** as a consequence of rotational invariance, i.e., that

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \text{constant}$$

for a closed system like this.

The upshot of these results: whereever you choose to define the *origin* (in space or in time), all physical laws should remain the same.

Central Force Motion: The N-Body Problem

Much of the rest of the dynamics part of the course will involve N bodies moving around & being gravitationally influenced by one another.

For point masses, we can write the Lagrangian $\mathcal{L} = K - U$. For the full N-body system,

$$\mathcal{L} \;=\; \sum_{i=1}^{N} \left(\frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2 \right) \;-\; \sum_{\text{pairs}} \left(-\frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \right)$$

Note that K is a sum over all N bodies, and U is a sum over all $N(N-1)/2$ unique pairs.

Now let's set $N = 2$ and focus back onto the **two-body problem.**

We've already thought about this a bit (Coulomb collisions).

Gravity is similar enough to the electrostatic force, that it simplifies things to go into barycentric, or center-of-mass (CM) coordinates:

$$\text{Recall:} \qquad \mathbf{R} \;=\; \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \qquad\qquad \mathbf{r} \;=\; \mathbf{r}_1 - \mathbf{r}_2$$

Here, let's take advantage of the fact that $\mathbf{U} = \dot{\mathbf{R}}$ *doesn't change* when two particles interact with one another.

Going fully into the CM frame (in which we set $\mathbf{R} = \text{constant} \equiv 0$),

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 \;=\; 0 \qquad \Longrightarrow \qquad \mathbf{r}_1 \;=\; + \left(\frac{m_2}{m_1 + m_2} \right) \mathbf{r} \quad , \quad \mathbf{r}_2 \;=\; - \left(\frac{m_1}{m_1 + m_2} \right) \mathbf{r} \; .$$

One can show that

$$m_1 r_1^2 + m_2 r_2^2 \;=\; m r^2 \qquad \text{where the reduced mass is} \qquad m \;=\; m_{12} \;=\; \frac{m_1 m_2}{m_1 + m_2} \; .$$

With this, kinetic energy can be simplified. For $N = 2$, the usual Lagrangian has 2 terms for K and 1 term for U, but in the CM frame,

$$\mathcal{L} \;=\; \left\{ \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 \right\} - U(\mathbf{r}) \;=\; \frac{1}{2} m |\dot{\mathbf{r}}|^2 - U(\mathbf{r})$$

and this reduces the 2-body problem to an equivalent 1-body problem.

Once we know the solution of $\mathbf{r}(t)$ for a "particle" of mass m, we can convert back to $\mathbf{r}_1(t)$ & $\mathbf{r}_2(t)$ for the real particles.

We do know more about the form of $U(\mathbf{r})$, but let's hold off writing it down. For now, note that it's *only* a function of $r = |\mathbf{r}^2|^{1/2}$.

What can we learn about this equivalent 1-body system? We can take advantage of the fact it's a "closed system" to use Noether's conservation laws defined above:

(1) Linear momentum: The system's *total* $\mathbf{p} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 \propto \dot{\mathbf{R}}$ is constant (and $= 0$ in the CM frame), so its conservation isn't very interesting.

(2) Angular momentum: We know that the system's total

$$\mathbf{L} \;=\; \sum_i \mathbf{r}_i \times \mathbf{p}_i \;=\; \text{constant.}$$

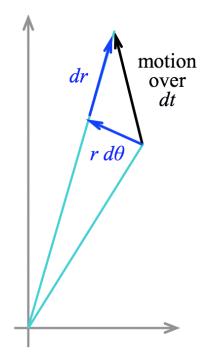
For the two-body problem, it's possible to write this in the CM frame as

$$\mathbf{L} \;=\; \mathbf{r}_1 \times m_1\mathbf{v}_1 \,+\, \mathbf{r}_2 \times m_2\mathbf{v}_2 \;=\; \rightsquigarrow\rightsquigarrow \;=\; m\mathbf{r}\times\mathbf{v}\ .$$

However, if we remember that $\mathbf{L}$ is constant in time (i.e., always pointing in the same fixed direction), then this means $\mathbf{r}$ and $\mathbf{v}$ must always stay in the **same 2D plane** perpendicular to $\mathbf{L}$.

(I think we knew this already, intuitively, from Coulomb collisions...)

Thus, we can write everything in 2D polar coordinates (r,θ):



$$\boxed{\mathcal{L} \;=\; \frac{1}{2}mv^2 - U(r) \;=\; \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)}$$

where r & θ are our generalized coordinates (q_1 & q_2).

What does the E–L equation tell us for the θ coordinate?

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = 0$$

The LHS $= 0$ because $\mathcal{L}$ doesn't depend explicitly on θ.

This means θ is an "ignorable coordinate," and

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \equiv \ell$$

which is essentially the magnitude of the system's total angular momentum.

(3) Energy: This is assured for a closed system like this, so

$$E = K + U = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + U(r) = \text{constant} .$$

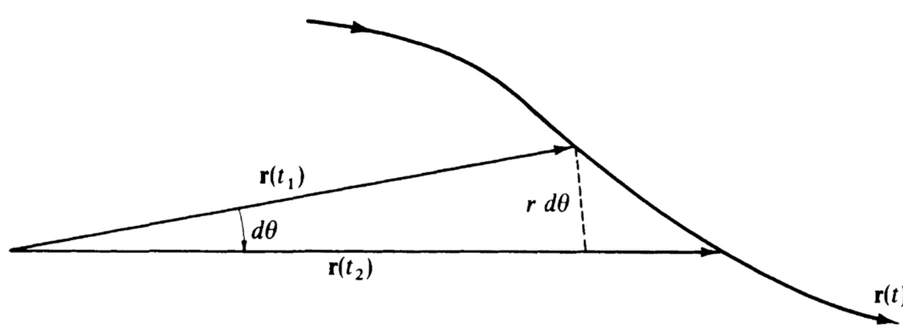
Thus, we have two **constants of motion:** E and ℓ.

....

We haven't integrated any equations of motion yet, but we now know enough to derive **Kepler's 2nd law** (i.e., equal areas in equal times).

Strangely, it's more fundamental than the 1st, in that it *doesn't* depend on the precise functional form of $U(r)$.

Define the area A swept out by the position vector $\mathbf{r}$ of the particle's path (in the CM frame) between time t and $t + dt$:



For very short times, $\mathbf{r}(t) \approx \mathbf{r}(t + dt)$, so the triangle has area

$$dA = \frac{1}{2}r(r\,d\theta) = \frac{1}{2}r^2 d\theta . \qquad \text{(assuming } d\theta \ll 1\text{)}$$

Thus,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2m} = \text{constant} .$$

and this is Kepler's 2nd law: planets sweep out "equal areas over equal times."

Interestingly, it doesn't depend on the orbits being of any particular shape, or even on $U(r)$ having any particular form.

....

Actual Equations of Motion for the Two-Body Problem

Our goal is a complete solution: $\mathbf{r}(t)$, i.e., $r(t)$ and $\theta(t)$. In general, that is not trivial, but we can bite off some pieces...

There are several ways to proceed. Right now, let's just look at the consequences of energy conservation.

Solve $E =$ constant for $\dot{r}$ and we get a differential equation:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\Big[E - U(r) - \frac{\ell^2}{2mr^2}\Big]} = \sqrt{\frac{2}{m}\Big[E - V(r)\Big]}$$

where one often sees the **effective potential**

$$V(r) = U(r) + \frac{\ell^2}{2mr^2}$$

as the sum of $U(r)$ and a centrifugal potential (corresponding to what some call a "fictitious force"). Once we settle on a form for $U(r)$, we'll plot it.

Assuming we know the constants E & ℓ and the form of $U(r)$, we could:

- Solve for dt & integrate to get $t(r)$

$$\text{i.e.,} \quad \frac{dr}{dt} = f(r) \quad \rightsquigarrow \quad \int \frac{dr}{f(r)} = \int dt = t \ .$$

- Invert the solution to get $r(t)$.
- Integrate the definition of ℓ to get $\theta(t)$.

In general, this needs to be done numerically, so we'll put a pin in this approach for now.

HERE's where we now start thinking about specific forms for $U(r)$.

Note that only for *some* forms of $U(r)$ do there exist "closed" orbits – i.e., paths for which $r(t)$ returns to its original value exactly when one loops around a full 2π radians in θ.

Bertrand (1873) proved that there exist only two forms for which ALL orbits are closed:

$$U(r) = -\frac{\gamma}{r} \quad \text{(gravity)} \qquad \text{or} \qquad U(r) = \tfrac{1}{2}kr^2 \quad \text{(simple harmonic oscillator)}$$

How can we learn more? A useful alternate approach – which will help us learn about the range of possible *geometric shapes* for orbital paths – is to use the E–L equation for the r coordinate:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)$$

$$mr\dot{\theta}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

Or, after some rearranging,

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U}{\partial r} \equiv F(r) \qquad \text{(RHS: the “force law”)} .$$

We could simplify this by using $\ell = mr^2\dot{\theta}$ and $g = F/m$, to get

$$\ddot{r} = g(r) + \frac{\ell^2}{m^2 r^3} \qquad \text{1D equation of motion; integrate twice to get } t(r).$$

However, there is a popular change of variables ($u = 1/r$) that lets us write this as a simpler 2nd order ODE for the orbit **shape** $u(\theta)$.

$$\text{Using:} \qquad \frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{\dot{r}}{r^2\,\dot{\theta}} = -\frac{m\dot{r}}{\ell} \qquad (\text{with } \ell = mr^2\dot{\theta})$$

and so on for $d^2u/d\theta^2$, we eventually eliminate time, to get

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2 u^2}F(u)} \qquad \textbf{Binet's equation .}$$

Binet's equation can be used in two ways:

- If we know F, solve the differential equation for the orbit, $u(\theta)$.
- If we know the orbit, *easily* solve for F (i.e., for any unknown forces acting on the particles).

Doing the former is straightforward if we (finally!) specify a classical gravitational potential:

$$U(r) \;=\; -\frac{Gm_1m_2}{r} \;=\; -\frac{\gamma}{r} \qquad \Longrightarrow \qquad F(r) \;=\; -\frac{\gamma}{r^2} \;=\; -\gamma u^2$$

and thus the RHS of Binet's equation is a constant.

If we perform yet another change of variables,

$$y \;=\; u \;-\; \frac{m\gamma}{\ell^2} \qquad \Longrightarrow \qquad \text{and Binet's equation is} \qquad \frac{d^2y}{d\theta^2} + y \;=\; 0$$

whose solution is a sinusoid. In general we can write it as a sum of sines and cosines, or as

$$y(\theta) \;=\; y_0\,\cos(\theta - \theta_0) \qquad \text{which has 2 constants of integration.}$$

Converting back to real units, we see: $$\boxed{r(\theta) \;=\; \frac{\lambda}{1 \,+\, e\,\cos(\theta - \theta_0)}}$$

which are **conic sections,** with one focus at the origin (i.e., the center of mass).

The two new constants are

$$\lambda = \frac{\ell^2}{m\gamma} \;\; \text{(radius of curvatuure)} \qquad\qquad e = \frac{\ell^2\,y_0}{m\gamma} \;\; \text{(eccentricity)}$$

Note: λ tells us the overall spatial scale of the orbit, while e tells us more about its shape. θ_0 sets the overall orientation of the orbit.

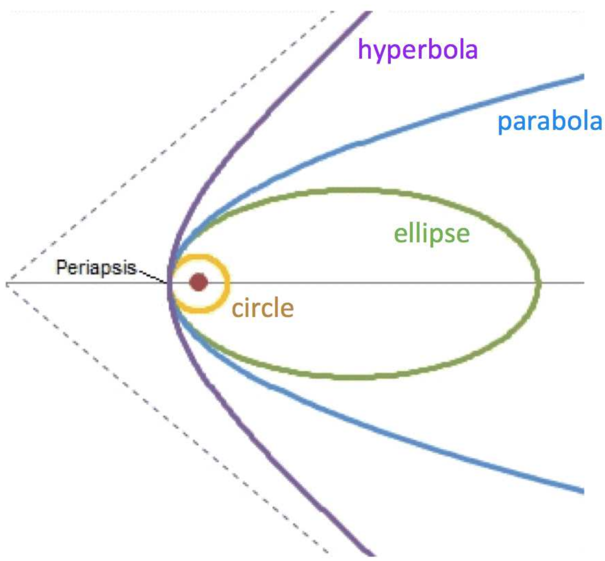
If we took this solution, substituted into energy conservation,

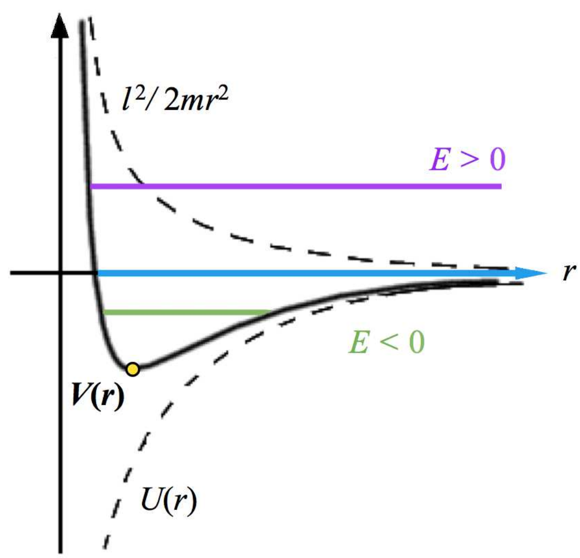
$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{\gamma}{r} \qquad \left(\text{using} \quad \dot{r} = \frac{dr}{d\theta}\,\dot{\theta}\right)$$

we'd be able to solve for e as a function of total energy:

$$e = \sqrt{1 + \frac{2E\ell^2}{m\gamma^2}} = \sqrt{1 + \frac{2\lambda E}{\gamma}} \qquad (\text{a nicer way to write } e)$$

What do the orbits look like... and how does E compare to $V(r)$?





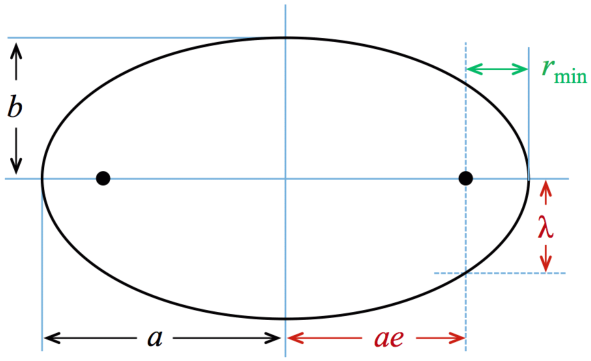
$e > 1$	$E > 0$	hyperbola	($v > 0$ as $r \to \infty$)
$e = 1$	$E = 0$	parabola	($v = 0$ as $r \to \infty$)
$0 < e < 1$	$V_{\text{min}} < E < 0$	ellipse	
$e = 0$	$E = V_{\text{min}}$	circle	($r = \lambda$)
$e < 0$	$E < V_{\text{min}}$	not allowed	($\dot{r}^2 < 0$, imaginary velocity!)

where some additional algebra can be used to show that

$$r_{\text{min}} = \frac{\lambda}{1+e} \qquad r_{\text{max}} = \frac{\lambda}{1-e} \qquad V_{\text{min}} = -\frac{m\gamma^2}{2\ell^2} = -\frac{\gamma}{2\lambda}\ .$$

Note that the plot for $V(r)$ is only for a single value of ℓ. There's really a whole family of $V(r,\ell)$ for all possible orbits between 2 bodies of known masses.

Kepler himself thought a lot about the **elliptical** case **(1st law).**



$$a \;=\; \frac{\gamma}{2|E|} \;=\; \text{semi-major axis}$$

$$b \;=\; \frac{\ell}{\sqrt{2m|E|}} \;=\; \text{semi-minor axis}$$

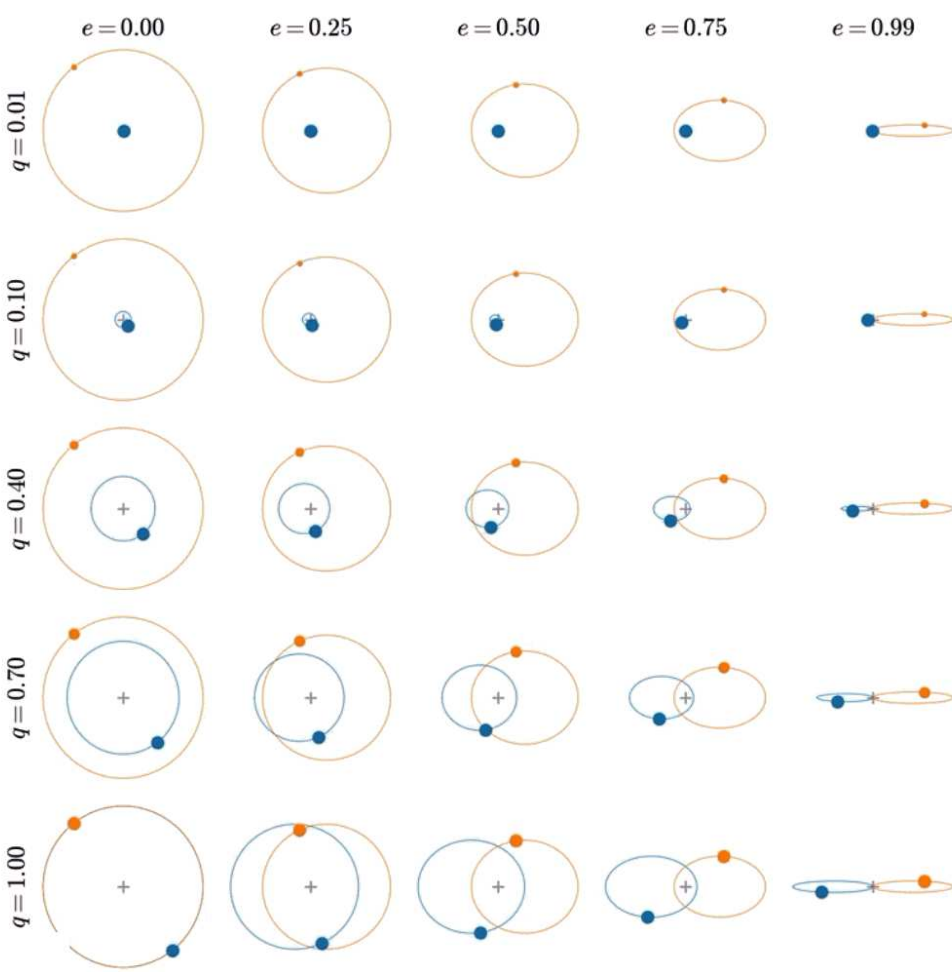
$$\text{Area} = \pi ab\,, \quad \frac{b}{a} = \sqrt{1-e^2}\,, \quad \lambda = \frac{b^2}{a}$$

The orbit around one focus ranges between the **apsides:**

$$\left\{ \begin{array}{c} \text{periapsis / pericenter / perigee} \\ \text{apapsis / apocenter / apogee} \end{array} \right\} \qquad \begin{array}{l} r_{\rm min} \;=\; a(1-e) \;=\; \lambda/(1+e) \\ r_{\rm max} \;=\; a(1+e) \;=\; \lambda/(1-e) \end{array}$$

. . . .

A gallery of elliptical orbits by Adrian Price-Whelan (adrianprw), varying both e & mass ratio $q = m_2/m_1$, converted back to the inertial frame:



We can also derive **Kepler's 3rd law** by recalling the 2nd law:

$$dt = \frac{2m}{\ell}\, dA \qquad \text{(for an ellipse).}$$

Both sides can be integrated over an exact period: $\begin{cases} t = 0 \to \mathcal{P} \\ A = 0 \to \pi ab \end{cases}$

$$\text{Thus,} \qquad \mathcal{P} = \frac{2m}{\ell}\pi ab = \frac{2m}{\ell}\pi a^{3/2}\sqrt{\lambda} = \frac{2m}{\ell}\pi a^{3/2}\sqrt{\frac{\ell^2}{m\gamma}}\ .$$

Kepler squared both sides. The ℓ's cancel, and we see that

$$\mathcal{P}^2 = \left(\frac{4\pi^2 m}{\gamma}\right) a^3$$

which isn't *quite* Kepler's 3rd law (i.e., the square of a planet's period is proportional to the cube of its semimajor axis), because the term in parentheses isn't constant. It depends on planet mass. Writing it in full:

$$\mathcal{P}^2 = \left(\frac{4\pi^2 \cancel{m_1 m_2}}{(m_1+m_2)\, G\, \cancel{m_1 m_2}}\right) a^3$$

and for the solar system, $M_{\text{tot}} = (m_1 + m_2) = M_\odot + m_{\text{planet}} \approx M_\odot$.

The 3rd law is only *approximately* true, but it's pretty close.

For circular orbits, $r = a$, and astronomers tend to write it as

$$\Omega = \frac{2\pi}{\mathcal{P}} = \sqrt{\frac{GM_{\text{tot}}}{r^3}} \propto r^{-3/2}\ .$$

. . . .

We haven't quite solved the full Kepler problem completely, since we still don't know the detailed time dependence $r(t)$ & $\theta(t)$ in closed form.

We could have integrated Kepler's 2nd law to get $t(\theta)$ for times less than $\mathcal{P}$. Noting the proportionality,

$$\frac{t}{\mathcal{P}} = \frac{A(\theta)}{\pi ab} = \frac{1}{\pi ab}\int_0^\theta \tfrac{1}{2}r^2 d\theta' = \frac{\lambda^2}{2\pi ab}\int_0^\theta \frac{d\theta'}{(1+e\cos\theta')^2}$$

(taking $\theta_0 = 0$). For an ellipse, it's analytically integrable, but complicated:

$$\frac{2\pi t}{\mathcal{P}} = 2\,\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\,\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\,\sin\theta}{1+e\cos\theta}\ .$$

We'll consider that knowing $t(\theta)$ is equivalent to knowing $\theta(t)$, since it's a single-valued function that can be tabulated numerically and "inverted" via lookup-table interpolation.

However, from the 1700s to the 1900s, a *HUGE* amount of effort was spent to invert it analytically. I won't go into **Kepler's equation** (which depends on θ-like quantitites called the "mean anomaly" & "eccentric anomaly") and is still a key component of celestial mechanics classes.

Most recently: Philcox et al. (arXiv:2103.15829) did find an exact solution for elliptical orbits... if you're comfortable with complex contour integrals.

. . . .

There's a family of interesting physics problems involving making **changes** to an elliptical (or circular) orbit.

$$\text{Two ways to do it:}\quad \left\{\begin{array}{ll} \textbf{(a)} & \text{impulsive “}\Delta v\text{”} \\ \textbf{(b)} & \text{gradual gas drag} \end{array}\right\}\quad (E \text{ can go} \uparrow \text{or} \downarrow).$$

To make any progress working out the numbers, we need to examine some additional consequences of energy conservation:

$$E = \frac{1}{2}mv^2 - \frac{\gamma}{r} \qquad \Longrightarrow \qquad v^2 = |\dot{\mathbf{r}}|^2 = \frac{\gamma}{m}\left(\frac{2}{r} + \frac{2E}{\gamma}\right)$$

Using the expressions derived for ellipses, this reduces to

$$\boxed{v^2 = \frac{\gamma}{m}\left(\frac{2}{r} - \frac{1}{a}\right) = GM_{\text{tot}}\left(\frac{2}{r} - \frac{1}{a}\right)} \qquad \text{the “vis–viva” equation}$$

Gottfried Leibniz (1646–1716)

"The force in question is *Living Force*, which arises from an infinity of continual impressions of dead force."

Qui-Gon Jinn (80 BBY – 32 BBY)

"Be mindful of the *Living Force*, my young Padawan."

Its history is... interesting... but its usefulness for determining the speed at *any* point along an elliptical orbit is also clear.

In some ways, it's yet another way of thinking about Kepler's 2nd law. The closer in you go, the faster the orbit becomes.

Applications:

(a) If you're in one circular or elliptical orbit, and you'd like to get to a different one, there are multiple ways you can fire your rockets to do it.

The most efficient way is the **Hohmann transfer orbit,** which lets you do it with the minimum thrust.

What do we mean by "thrust?" A given rocket burns with essentially a known & constant force F. If you burn it for a time Δt, then turn it off, you've changed your velocity by a given amount:

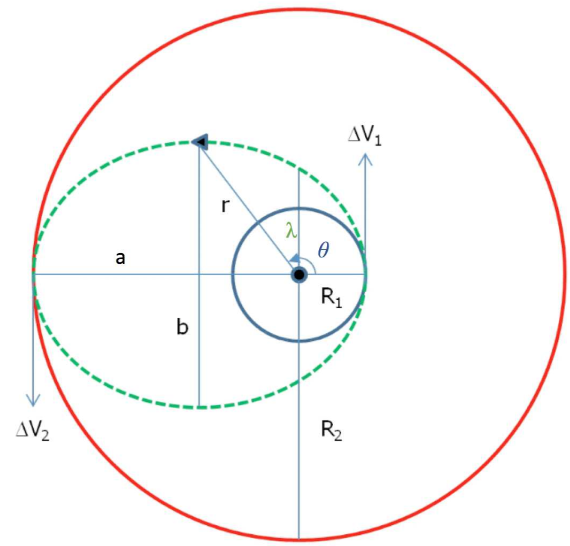
$$F = \frac{\Delta p}{\Delta t} = \frac{m_\mathrm{R}\Delta v}{\Delta t} \qquad \Longrightarrow \qquad \Delta v = \frac{F\,\Delta t}{m_\mathrm{R}}$$

where m_R is the current mass of the rocket. To accelerate, you point the rocket *behind* your current velocity vector $\mathbf{v}$. To decelerate, you point the rocket *along* $\mathbf{v}$.

So how do you choose the Δv that will get you to your new orbit? **Vis-viva!**

Let's say we're in a "low" circular orbit around the Earth with radius r_1.

We want to get to a higher circular orbit with $r_2 > r_1$.



There must be two rocket burns:

1. Boost from the low circular orbit to an elliptical orbit with the same perigee (r_1), and an apogee of r_2.

2. Once you reach the apogee of r_2, boost again to change the orbit from elliptical to circular.

To determine the required Δv for each step, just solve the vis–viva equation for the speeds...

The first burn:

$$\Delta v_1 \;=\; v(\text{ellip. at } r_1) - v(\text{circ. at } r_1)$$

and it's straightforward to use vis-viva to evaluate

$$v(\text{circ. at } r_1) \;=\; \sqrt{\frac{GM_{\text{tot}}}{r_1}} \;\; .$$

The other term requires us to recognize that $2a = r_1 + r_2$, so that

$$v(\text{ellip. at } r_1) \;=\; \sqrt{GM_{\text{tot}}\left(\frac{2}{r_1} - \frac{2}{r_1+r_2}\right)} \;=\; \sqrt{\frac{2GM_{\text{tot}}\, r_2}{r_1\,(r_1+r_2)}}$$

and thus

$$\Delta v_1 \;=\; v(\text{ellip. at } r_1) - v(\text{circ. at } r_1) \;=\; \sqrt{\frac{GM_{\text{tot}}}{r_1}}\left[\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right] > 0 \;\; .$$

Similar math could be done for the second burn (which would also have $\Delta v_2 > 0$), but we won't go through it.

It's time to point out something strange about Keplerian orbits. Notice that we had to speed up (twice!) to get from the r_1 circular orbit to the r_2 circular orbit. However, the orbital speed is **slower** at r_2. Recall the orbital frequency:

$$\dot{\theta} \;=\; \omega \;=\; \sqrt{\frac{GM}{r^3}} \qquad \Longrightarrow \qquad v_{\text{circ}} \;=\; \omega r \;=\; \sqrt{\frac{GM}{r}}$$

and as r goes up, v_{circ} goes down.

However, a circular orbit with larger r has a **larger** angular momentum

$$\ell \;=\; mr^2\dot{\theta} \;=\; mr^2\sqrt{\frac{GM}{r^3}} \;=\; m\sqrt{GMr}$$

and a **larger** (i.e., less negative) total energy

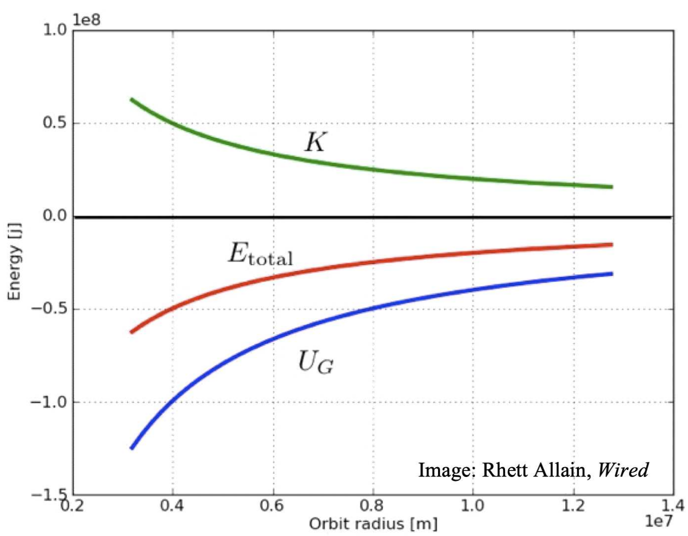
$$E \;=\; V_{\text{min}} \;=\; -\frac{Gm_1m_2}{2r} \qquad \text{(recalling that for a circular orbit, } r = \lambda\text{)}$$

so as r increases, both ℓ and E increase, too.

It's counter-intuitive, but it's how the physics works. If you're in orbit around the Earth at a given radial distance, and you wanted to "pass" a satellite in a neighboring orbit (i.e., blow by it at a higher speed), you'd have to:

- Fire your rockets in a forward direction, to slow down.
- This decreases your ℓ and E and drop you into a lower orbit,
- in which you'll have a faster orbital speed!

Maybe a plot will be helpful. For circular orbits at various r,



$$K \;=\; \frac{1}{2}\mu v^2 \;=\; +\frac{Gm_1m_2}{2r}$$

$$E_{\text{total}} \;=\; -\frac{Gm_1m_2}{2r}$$

$$U_G \;=\; -\frac{Gm_1m_2}{r}$$

By going to a lower radius, you end up speeding up (higher K), but you've lost total energy (i.e., E is lower = more negative). The only way to do that is to do "negative work."

Want to build more intuition? Play *Kerbal Space Program?*

. . . .

(b) A spacecraft orbiting in a gas atmosphere will undergo **gas drag** (i.e., **"aerobraking"** when intentional!) which causes "total" energy E to decrease.

The frictional force on an object moving with speed v through a gas with density ρ is given by Rayleigh's drag equation,

$$F_{\rm drag} = \tfrac{1}{2} C_{\rm D} \rho v^2 A$$

where $C_{\rm D}$ is an order-unity drag coefficient, and A is the cross-sectional area of the object.

(Theoretically, this should be derivable from $\Psi_{\rm visc}$ in the non-ideal fluid conservation equations... but in practice it was found via dimensional analysis & verified experimentally.)

You may have used $F_{\rm drag} = mg$ to solve for the **terminal speed** of a falling object due to "air resistance."

The corresponding loss of kinetic energy is

$$\mathbf{v} \cdot \left\{ m \frac{d\mathbf{v}}{dt} = -\mathbf{F}_{\rm drag} \right\} \qquad \Longrightarrow \qquad \frac{dE}{dt} = -C_{\rm D} \rho v^3 A$$

An initially circular orbit will slowly *decay.* The drag is exerted ~constantly around the orbit, and

$$E = V_{\rm min} = -\frac{GM_{\rm tot}}{2r} \qquad \Longrightarrow \qquad r = -\frac{GM_{\rm tot}}{2E} = \frac{GM_{\rm tot}}{2|E|}$$

and decay makes E more negative. $|E| \uparrow$, so $r \downarrow$.

An initially elliptical orbit (around a body with an atmosphere) will *circularize,* then decay. The strongest drag is at pericenter, because:

- ρ drops off exponentially with r
- v is highest at smallest r (vis–viva).

This is like an inverse Hohmann Δv (i.e., pointing rockets in the opposite direction of the orbit), and the lower E will result in the "next" orbit being a lower-e ellipse with the same r_1.

Circularization is important in close binary star systems, too.

For satellites, though, there's great practical interest in this problem, because it's a confluence of **money** (how long will my valuable satellite live?) and **risk** (when & where will it crash?).

Also, ρ in Earth's upper atmosphere depends on solar activity, so **space weather prediction** is needed to model the long-term effects.

. . . .

Another interesting application of gas drag: **planetary migration.**

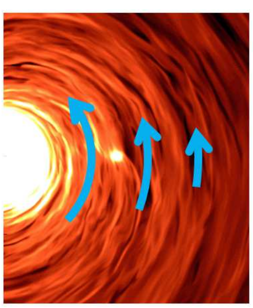
"Hot Jupiter" exoplanets were discovered in 1995, but their formation is a puzzle. It's easy to form gas giants outside the "snow line" ($\sim$3 AU for the Sun, where it's cold enough for dust grains to condense), but these planets are well inside it.

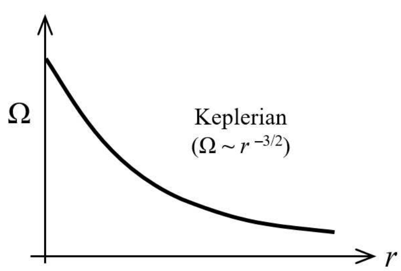
Maybe they formed at large distances, then migrated inwards. How?

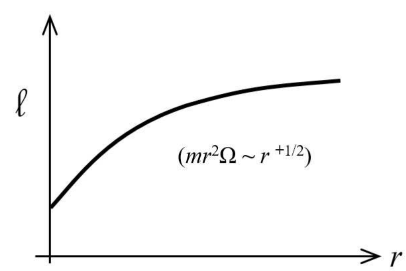
- Large-angle scattering in close encounters? (rare)
- Viscous drag with disk gas (preferred model?)

As the planet plows through the disk, it experiences drag with neighboring parcels of gas. Friction "wants" to make it all rigid (constant Ω), so

$$\text{Coupling with} \left\{ \begin{matrix} \text{inner, higher-}\Omega \\ \text{outer, lower-}\Omega \end{matrix} \right\} \text{gas} \left\{ \begin{matrix} \text{speeds up} \\ \text{slows down} \end{matrix} \right\} \text{the planet.}$$







Which wins? Depends on disk $\rho(r, \phi)$: more gas $\rightarrow$ more friction.

Low-mass planets: the latter tends to win. Planets tend to lose $\mathbf{L}$ (and/or E) & migrate inwards.

High-mass: the planet's gravity clears out a gap in the disk, so it feels $\sim$no local drag force. But it's an *accretion disk,* so everything gets brought in $\Rightarrow$ planet tends to migrate inwards.

How fast does it occur? Assume a diffusion timescale: $t_{\rm mig} \sim \dfrac{R^2}{\nu}$

where the viscosity can be given by Shakura & Sunayev's α model...

$$\nu \sim \alpha H^2 \Omega \qquad H = \text{disk vertical scale height, } \alpha \sim 0.01.$$

$$\text{Thus,} \qquad t_{\rm mig} \sim \frac{1}{\alpha}\left(\frac{R}{H}\right)^2 \sqrt{\frac{R^3}{GM_*}}$$

and for $H/R \sim 0.1$, $M_* \sim M_\odot$, and $R = 1$–5 AU, we get $t_{\rm mig} \sim 10^3$–10^4 years.

This is much shorter than disk lifetimes of $\sim 10^6$ years.

What *stops* the migration (i.e., prevents it from colliding with star)?

- Disk is truncated by star's strong magnetic field?
- Tidal interactions with star, once it comes very close?
- If planet formed "late," disk may dissipate before migration done?
- Other planets may have paved the way; ate up disk gas?

Maybe "super-earths" are failed hot Jupiters, whose migration was halted?

Beyond circular and elliptical orbits

Many other applications in astrophysics depend on the special case of **parabolic** ($E = 0$) orbits: e.g.,

- star formation (infall accretion of mass from large distances)
- "single apparation" comets coming in from the Oort cloud
- the lowest-energy way to do spacecraft orbit insertion ("capture orbit").

Consider the accretion problem inside a Giant Molecular Cloud (GMC) in the disk of our galaxy. They become turbulent and break into protostellar "cores" that will eventually collapse into stars. Typical properties:

$$\begin{aligned} R_{\rm core} &\approx 0.1 \text{ parsec} \\ M &\approx 1 \text{ solar mass} \\ \omega &\approx 10^{-15} \text{ rad/s} \end{aligned}$$

Rotation is slow; it only gets a gentle kick from galactic shear motions:

$$\mathcal{P} = 2\pi/\omega = \text{ 200 million years.} \qquad v_{\rm circ} = \omega R_{\rm core} = \text{ 0.003 km/s.}$$

Let's assume there's already a protostar forming at the center of the GMC, which dominates the total mass M, and has a radius $\approx R_\odot \ll R_{\rm core}$.

Because of the slow rotation of the cloud, it's a good approximation to assume that a small clump of gas from its outer edge falls inwards on a **parabolic orbit;** i.e., with $E \approx 0$, and thus zero kinetic energy at its outer starting point (essentially $r \to \infty$).

We want to know: **Will the clump impact the star?**

The answer depends on how much angular momentum it has. Assume $m_1 \approx M$ is the star and $m_2 \ll M$ is the clump, then the reduce mass $m \approx m_2$. We set the angular momentum at the outermost "initial condition:"

$$\ell = mr^2\dot{\theta} = mR_{\rm core}^2\omega = \text{ constant.}$$

For a parabolic orbit with $e = 1$,

$$r(\theta) = \frac{\lambda}{1 + \cos\theta}$$

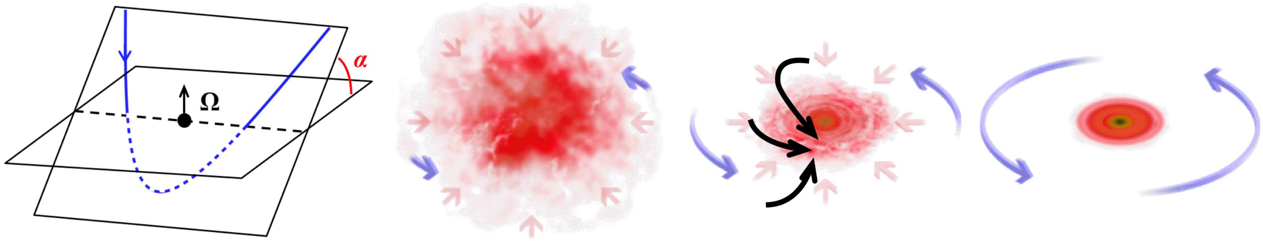
and the radial distance of closest-approach to the star will occur at $\theta = 0$, or

$$r_{\rm min} = \frac{\lambda}{2} = \frac{\ell^2}{2GMm^2} = \frac{\omega^2 R_{\rm core}^4}{2GM}$$

Plugging in the above numbers: $r_{\rm min} \approx 490\, R_\odot \approx \boxed{\text{2.3 AU.}}$

Thus, **NO,** most parcels won't impact the star, because they've got *too much angular momentum* and are unable to move so far in.

Instead, many infalling parcels end up interacting with one another, usually when they try to pass through the rotational midplane:



There are really many parcels, all coming in with random values of α between 0 and 2π.

Infall may start as random, but if there's a net overall sense of non-radial (rotational) flow, the north/south motions can cancel out, and the east/west motions remain to flow in one predominant direction.

The flow flattens into an **accretion disk,** and our derived size of "a few AU" is close to what is observed.

Interestingly, vis-viva says the parcels first cross the disk-plane (at $\theta = \pm\pi/2$) with $v^2 = 2GM/\lambda$, but they become rapidly "circularized" into a Keplerian orbit at that distance, which has $v^2 = GM/\lambda$.

Roughly ∼half of their kinetic energy went into **heating up** the gas in the disk!

This is a **frictional/viscous** effect that ultimately causes the orbiting parcels to lose energy, and thus spiral into the star very slowly. (*That's* why it's called an accretion disk, and not an orbiting-forever disk...)

. . . .

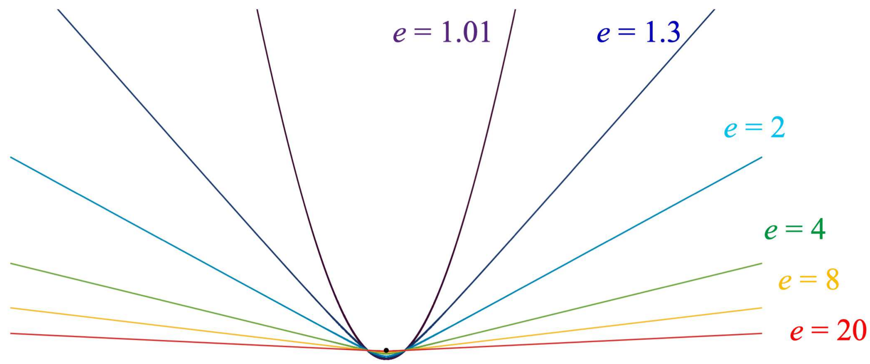
The parabolic orbit ($E \approx 0$) is always "on the edge" of either capture or escape. It's also possible to use gravity to make small nudges.

Let's look at one more problem in the case of **hyperbolic** ($E > 0$) orbits. Later we'll look in more detail at gravitational scattering-type interactions in N-body systems (similar to Coulomb collisions in plasmas), which are also essentially hyperbolic.

The Gravitational Slingshot Effect

You're in a rocket, and you want to change your speed by some Δv. However, you don't have enough fuel. What do you do...?

There's hope. Consider a **hyperbolic** flyby between a spacecraft and a planet or moon. The eccentricity determines the "opening angle" of the hyperbola:



These trajectories show the relative motions described by $\mathbf{r}$ & $\mathbf{v}$.

There's one interesting fact to learn about $\mathbf{v}$ in hyperbolic orbits. Consider the "initial" and "final" conditions (way before & way after the closest approach).

For both conditions, $r \to \infty$, so the potential energy U_{tot} is essentially zero in both places. Thus,

$$E \;=\; \frac{1}{2}m|\mathbf{v}|^2 - \frac{Gm_1m_2}{r} \;\approx\; \frac{1}{2}m|\mathbf{v}|^2 \;=\; \text{constant}$$

so we see that $\qquad |\mathbf{v}_{\text{init}}| = |\mathbf{v}_{\text{final}}| \qquad$ in the CM frame.

However, we're going to have to transform back into the inertial frame. Let's assign m_1 = spacecraft, and m_2 = planet. Thus, $m_2 \gg m_1$. The velocity of the CM frame is

$$\mathbf{V} \;=\; \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \;\approx\; \mathbf{v}_2$$

i.e., the planet dominates the total momentum of the system.

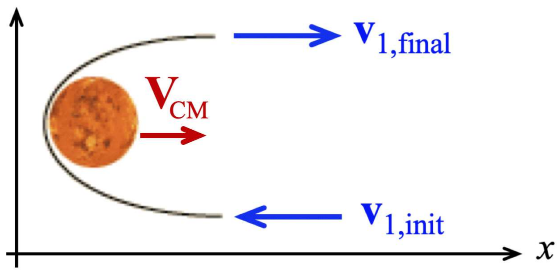
Rather than just jump into the CM frame, we should be aware that the planet is in orbit around its star, too. Thus, over short time intervals, we can consider $\mathbf{v}_2 \approx \mathbf{V}$ to be a known planetary orbital velocity.

Anyway, we see that

$$\mathbf{v} \;=\; \mathbf{v}_1 - \mathbf{v}_2 \;\approx\; \mathbf{v}_1 - \mathbf{V}$$

i.e., the relative velocity ≈ velocity of spacecraft in CM frame.

Let's assume we're in a low-eccentricity hyperbolic orbit, and let's remember that $|\mathbf{v}_{\rm init}| = |\mathbf{v}_{\rm final}|$.



$$\mathbf{v}_{1,\rm init} \;\equiv\; -W\,\hat{\mathbf{e}}_x \qquad \text{(let's set } W > 0\text{)}$$

$$\mathbf{v}_{\rm init} \;=\; \mathbf{v}_{1,\rm init} - \mathbf{V} \;=\; (-W - V)\,\hat{\mathbf{e}}_x$$

$$\mathbf{v}_{\rm final} \;=\; -\mathbf{v}_{\rm init} \;=\; (W + V)\,\hat{\mathbf{e}}_x$$

Thus, going back to the inertial frame,

$$\mathbf{v}_{1,\rm final} \;=\; \mathbf{v}_{\rm final} + \mathbf{V} \;=\; (W + 2V)\,\hat{\mathbf{e}}_x$$

i.e., the x magnitude of the spacecraft velocity changed from W to $W + 2V$.

If the spacecraft approaches the planet "head-on" in the planet's orbit (i.e., $V > 0$), the spacecraft speeds up after the slingshot.

- Similar to the terrestrial analogy of a tennis ball being thrown at an approaching wall... it bounces back faster (see also "Fermi acceleration").

If the spacecraft approaches the planet "along" the planet's orbit (i.e., $V < 0$), the slingshot slows down the spacecraft.

- This is how *Parker Solar Probe* is inching its way closer to the Sun... by using Venus to shed its angular momentum.
