Non-Ideal Effects: Resistive MHD & "Beyond MHD"

In ideal MHD, we neglected terms in the conservation equations that have to do with Coulomb collisions.

Earlier, we saw that the Boltzmann collision term is $\neq 0$ only in certain circumstances; e.g.,

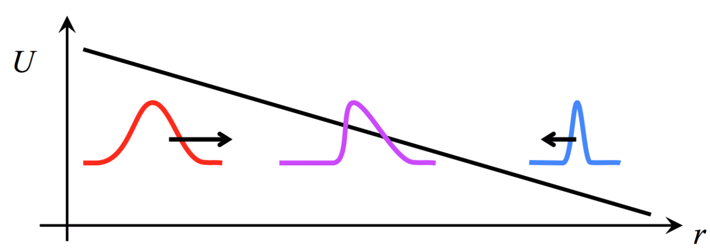
- For multiple species, $(T_i \neq T_j)$ or $(\mathbf{u}_i \neq \mathbf{u}_j)$ gives rise to equilibration terms on the RHS.
- For "self-collisions" (single MHD fluid interacting with itself), the RHS $\neq 0$ when $f(\mathbf{v})$ is non-Maxwellian.

For now, we'll investigate just the 2nd item in the list. But what *generates* non-Maxwellian distributions?

Note: A Maxwellian means we're in locally homogeneous equilibrium; i.e., there's *NO* net transport of "stuff" from point A to point B.

But if large-scale **gradients** exist in the background plasma, then collisions *MAY* start acting as catalysts to transport stuff from point A to point B.

Consider something like a star, with high thermal energy U in the interior, and low values higher up. At any one point, $f(\mathbf{p})$ is *mostly* isotropic. If collisions let different regions "talk to each other," the fact that $|\nabla U| \neq 0$ produces non-local skewness:



We'll see that skewness occurs in tandem with an energy flux $\mathbf{F} = -D\nabla U$.

Thus, if spatial gradients exist, collisions can:

1. transport momentum	• viscosity
2. transport thermal energy	• heat conduction
3. transport magnetic energy (when $\mathbf{u}_{\rm ion} \neq \mathbf{u}_e$)	• electrical resistivity

Our goal is to derive how these **transport coefficients** (i.e., diffusion coefficients D) depend on the Coulomb collision rate $\nu_{\rm coll} = 1/\tau_{\rm coll}$.

First, let's review the **macro**scopic definitions of the transport coefficients... then we can derive how they are enabled by **micro**scopic departures from a Maxwellian $f(\mathbf{p})$.

(1) Viscosity measures how collisions transport momentum... i.e., how the system responds to shear motions.

Recall the full MHD equation of momentum conservation:

$$\rho\frac{D\mathbf{u}}{Dt} + \nabla\cdot\mathbb{P} - \rho\mathbf{g} - \frac{1}{c}\mathbf{J}\times\mathbf{B} = 0$$

and it was only for a Maxwellian that the 3×3 stress tensor was given by

$$\mathbb{P} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} \qquad \text{where} \quad P = nk_{\rm B}T \quad \text{and} \quad \nabla\cdot\mathbb{P} = \nabla P \quad .$$

Recall that each component of the stress tensor looks like $\mathbb{P}_{ij} = \rho\langle v_i v_j\rangle$.

Thus, $\mathbb{P}$ is the rate at which the i component of "momentum density" (ρu_i) is carried in the j direction with speed (u_j).

Thus, the off-diagonal terms $(i \neq j)$ represent **shear,** and they are most general & physically realistic when written as

$$\mathbb{P}_{ij} = \mu\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\} \qquad (\text{for } i \neq j)$$

μ = coefficient of shear/dynamic viscosity $= \rho\nu$.
$\nu = \mu/\rho$ = coefficient of **kinematic viscosity...** units of length^2/time, same as a **diffusion coefficient.** We ought to call it $D_{\rm visc}$?

Verify that units work out: $\nabla\cdot\mathbb{P} = \dfrac{\rho\nu}{\ell}\dfrac{u}{\ell} = \dfrac{\rho(u\ell)u}{\ell^2} = \dfrac{\rho u^2}{\ell} = \dfrac{\rho u}{t}$ ✓

The traditional hydrodynamic form is

$$\nabla\cdot\mathbb{P} = \nabla P - \rho\nu\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right] \qquad (\text{for } \nu \text{ constant in space}).$$

in which we neglect the so-called "second viscosity" ζ that is related to compression/expansion in the $i = j$ components (not shear).

Maxwell showed that an ideal monatomic gas has $\zeta = 0$, and in many applications this isn't a bad assumption (*exception: shock thickness*).

Also, for incompressible flows ($\nabla \cdot \mathbf{u} = 0$), all terms related to ζ are zero, and the normal shear viscosity simplifies, too.

In that case, with no gravity or $\mathbf{B}$ (also assuming pressure equilibrium & incompressible flow), all that's left in the momentum equation is

$$\frac{D\mathbf{u}}{Dt} = \nu \nabla^2 \mathbf{u}$$

i.e., viscosity provides diffusive momentum transport for a fluid parcel when there are relative motions, leading ultimately to $\mathbf{u}$ const. in space.

How important is viscous diffusion, compared to standard fluid advection? The standard gauge is to take the ratio of back-of-envelope magnitudes for the two terms:

Reynolds number: $$\text{Re} = \frac{\text{momentum advection}}{\text{momentum diffusion}} = \frac{|(\mathbf{u}\cdot\nabla)\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \sim \frac{u^2/\ell}{\nu\, u/\ell^2} \sim \frac{u\,\ell}{\nu}$$

In astrophysical systems, usually $\text{Re} \gg 1$, but sometimes viscosity does appear to be important despite that! (There may exist "anomalous" sources of ν in addition to Coulomb collisions.)

Strong turbulence is possible only when $\text{Re} \gg 1$. The opposite case is peanut butter or molasses, with $\text{Re} \ll 1$.

. . . .

(2) Heat Conduction measures how collisions transport thermal energy through the system (again, only when it's initally inhomogeneous).

It's worth going back to the moments of $f(\mathbf{p})$,

$$\int d^3\mathbf{p}\ f(\mathbf{p}) \begin{cases} 1 & \text{(0th)} \quad \text{number density } n \quad \text{(scalar)} \\ \mathbf{v} & \text{(1st)} \quad \text{bulk flow speed } \mathbf{u} \quad \text{(vector)} \\ \mathbf{vv} & \text{(2nd)} \quad \text{pressure/stress } \mathbb{P} \quad (3\times3 \text{ dyadic tensor}) \\ \mathbf{vvv} & \text{(3rd)} \quad \text{heat conduction } \mathcal{Q} \quad (3\times3\times3 \text{ triadic tensor}) \end{cases}$$

Just as we opted to often reduce $\langle \mathbf{vv} \rangle$ to the mean thermal energy $\langle v^2 \rangle$, we usually *never* need all 27 components of $\mathcal{Q}_{ijk}$.

It's useful to think about a vector version of the 3rd moment **heat conduction flux:**

$$\mathbf{q} \;=\; n \left\langle \frac{1}{2} m v^2 \; \mathbf{v} \right\rangle$$

which follows the transport of kinetic energy per particle ($\frac{1}{2}mv^2$) in a given direction, at a given speed ($\mathbf{v}$), through the system.

Classically (going back again to Fourier in Napoleon's army), we've seen

$$\mathbf{q} = -\kappa \nabla T$$

which makes similar sense as the pressure tensor being $\propto$ velocity shear. Heat wants to flow down the gradient. Here, $\mathbf{q} \neq 0$ only when there are local variations in *thermal energy.*

Spitzer (1962) and Braginskii (1965) found that Coulomb collisions in an ionized plasma give $\boxed{\kappa \propto T^{5/2}.}$ We'll derive that in a bit, but there are many exceptions, too.

With these transport terms, the thermal energy equation is

$$\begin{aligned} \frac{DU}{Dt} + (U+P)\nabla \cdot \mathbf{u} \;&=\; \{\text{sum of heating – cooling terms}\} \\ &=\; -\nabla \cdot \mathbf{q} + \Psi_{\text{visc}} \;=\; \kappa \nabla^2 T \,+\, \Psi_{\text{visc}} \end{aligned}$$

where we shouldn't forget that viscosity results in irreversible conversion of kinetic to thermal energy:

$$\Psi_{\text{visc}} \;=\; \mu \left[\Delta_{ij}\Delta^{ij} - \frac{2}{3}(\nabla \cdot \mathbf{u})^2 \right] \quad , \qquad \Delta_{ij} \;=\; \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

For the simplest orthogonal shear flow case (i.e., $\mathbf{u}$ pointing in the i direction, but varying only in j direction):

$$\Psi_{\text{visc}} \;\sim\; \mu \left(\frac{\partial u_i}{\partial x_j} \right)^2 \qquad \text{(I think this holds only for } i \neq j\text{)} \; .$$

This cancels out a viscous loss term in the *total* fluid energy conservation equation. Remeber the terms like: $\mathbf{u} \cdot \{\text{all terms in momentum eqn}\}$?

The thermal conductivity κ is used often in astrophysics, but its units aren't very natural. Some prefer the **thermal diffusivity** $D_{\rm T}$, which is (roughly) the diffusion coefficient that occurs when $\mathbf{u} = 0$ in the energy equation:

$$\frac{\partial T}{\partial t} = \frac{5}{3} D_{\rm T} \nabla^2 T \qquad D_{\rm T} = \frac{\kappa}{\rho \widetilde{c}_P} = \frac{2\kappa}{5n\, k_{\rm B}} \quad \text{(for ideal gas)}$$

where $\widetilde{c}_P = \frac{5}{2} k_{\rm B}/\langle m \rangle$ is (one version of) the specific heat at constant pressure.

However, when $\mathbf{u} \neq 0$, it's not clear which of the 2 RHS terms in the energy equation is more important. Define another ratio:

$$\textbf{Brinkman number:} \quad \mathrm{Br} = \frac{|\Psi_{\rm visc}|}{|\nabla \cdot \mathbf{q}|} \sim \frac{\mu\, u^2/\ell^2}{\kappa T/\ell^2} \sim \frac{\mu\, u^2}{n D_{\rm T}\, k_{\rm B} T} \sim \frac{\nu}{D_{\rm T}} \frac{u^2}{c_s^2}$$

The ratio $\nu/D_{\rm T}$ arises frequently in hydrodynamics:

$$\textbf{Prandtl number:} \quad \mathrm{Pr} = \frac{\text{momentum diffusion}}{\text{thermal diffusion}} = \frac{\nu}{D_{\rm T}}$$

Typically, $\mathrm{Pr} \sim 1$, but it can be as small as 10^{-4} in stars. Thus, $\mathrm{Br} \sim \mathcal{M}^2$, where $\mathcal{M} = u/c_s$ is the **Mach number** of a flow.

Static plasmas or subsonic flows have $\mathrm{Br} \ll 1$, so conduction is much more important that viscous heat loss.

Not to overwhelm you with dimensionless numbers, but one can also compare to the macroscopic motions again, to get the

$$\textbf{Péclet number:} \quad \mathrm{Pe} = \frac{\text{thermal advection (enthalpy flux)}}{\text{thermal diffusion (heat conduction)}} = \frac{u\,\ell}{D_{\rm T}} = \mathrm{Re}\ \mathrm{Pr}$$

i.e., sort of a "thermal Reynolds number," with $D_{\rm T}$ replacing ν.
In astrophysics, $\mathrm{Pe} \gg 1$ often, too.

Of course, the higher we go in moments of $f(\mathbf{p})$, the more that subtle departures from Maxwellians can affect the transport. Recall...

$$\frac{\partial n}{\partial t} = \cdots \qquad \text{RHS contains } \mathbf{u}$$
$$\frac{\partial \mathbf{u}}{\partial t} = \cdots \qquad \text{RHS contains } P \quad \text{(\& "external" gravity, Lorentz forces)}$$
$$\frac{\partial P}{\partial t} = \cdots \qquad \text{RHS contains } \mathbf{q} \quad \text{(\& "external" heating/cooling rates)}$$
$$\frac{\partial \mathbf{q}}{\partial t} = \cdots \qquad \textit{(very seldom used, but results have been insightful!)}$$

That last equation (and some even higher moment equations) becomes important in **collisionless** plasmas.

For example, we'll see later that when there's a strong $\mathbf{B}$, we have $|\mathbf{q}_\parallel| \gg |\mathbf{q}_\perp|$.

. . . .

(3) Resistivity measures how collisions transport electromagnetic energy through the system. The term "conductivity" is often used ($\sigma = 1/\eta$), and it's important to distinguish *electrical* conductivity from *thermal* conductivity.

We've already justified its presence in the induction equation, and discussed its intrinsically diffusive nature:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + D_{\mathrm{B}} \nabla^2 \mathbf{B} \quad .$$

It's relative importance is measured by the **magnetic Reynolds number:**

$$\mathrm{Rm} = \frac{\text{magnetic advection}}{\text{magnetic diffusion}} = \frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|D_{\mathrm{B}} \nabla^2 \mathbf{B}|} \sim \frac{u\,\ell}{D_{\mathrm{B}}}$$

Because the "advection" of structure along $\mathbf{B}$ often takes place via Alfvén-like fluctuations (and sometimes we're dealing with magnetostatic systems with $\mathbf{u} = 0$), MHD people often define an analogous quantity, the

$$\textbf{Lundquist number:} \qquad S = \frac{V_{\mathrm{A}}\,\ell}{D_{\mathrm{B}}}$$

In astrophysics, both Rm and S are often $\gg 1$ (often $\gtrsim 10^{10}$ to 10^{15}).

The relative strengths of viscosity and resistivity are gauged via a

$$\textbf{magnetic Prandtl number:} \qquad \mathrm{Pm} = \frac{\text{momentum diffusion}}{\text{magnetic diffusion}} = \frac{\nu}{D_\mathrm{B}} = \frac{\mathrm{Rm}}{\mathrm{Re}}$$

In turbulence, as the cascade creates structure at ever-smaller scales, eventually dissipation takes hold. We care about what kind of dissipation occurs "first:"

viscous damping of U_K		resistive damping of U_B
(if $\mathrm{Pm} > 1$)	or	(if $\mathrm{Pm} < 1$)
e.g., neutron star disks		e.g., protoplanetary disks

. . . .

FYI, in real astrophysical systems, there usually are lots of other effects that behave similarly enough to collisions (and are *faster*), that we use **anomalous transport coefficients.**

Too often the details of these other processes are swept under the rug by calling them "turbulent..."

$$D_\mathrm{turb} \equiv u_\mathrm{turb}\ell_\mathrm{turb} \quad \gg \quad \{\,\nu\,,\,D_\mathrm{T}\,,\,D_\mathrm{B}\,\}_\mathrm{coll} \quad .$$

Next, how do we calculate ν, κ, and η (i.e., $D_{\rm visc}$, $D_{\rm T}$, $D_{\rm B}$) in terms of the micro-physics of Coulomb collisions?

Chapman-Enskog theory

Entire courses can be devoted to this (see also Braginskii 1965, *Rev. Plasma Phys.,* 1, 205), but for now we'll just look at the following procedure:

- Set up Boltzmann equation with simple BGK form of the collision term.
- Impose one type of **large-scale transport** (i.e., a gradient in a 0th-order quantity).
- Linearize $f(\mathbf{v}) = f_0 + f_1$, and solve for the small/perturbed (1st-order) part of f that responds to the gradient.
- Plug in this modified $f(\mathbf{v})$ into a "higher moment" definition of a fluid quantity that contains the desired transport coefficient.
- Solve for the transport coefficient!

In class we'll just go through this procedure for the heat conductivity κ.

. . . .

Goal: verify that $\mathbf{q} \;=\; \frac{1}{2}\rho\langle v^2\mathbf{v}\rangle \;=\; -\kappa\nabla T$ and solve for κ.

If we were to plug in a Maxwellian $f(\mathbf{v})$ into the above definition, we would get $\mathbf{q} = 0$. They're integrals of odd fuctions. There's no transport in equilibrium.

Thus, to find the non-Maxwellian f consistent with thermal energy transport, write the BGK Boltzmann equation (with no external forces):

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f \;=\; \frac{f_0 - f}{\tau}$$

where τ is a constant collision timescale.

Simplify with 4 assumptions:

- Linearize $f = f_0 + f_1$ (with $|f_1| \ll |f_0|$).
- Both f_0 and f_1 are time-steady.
- The only spatial variation is $T(z)$ in the Maxwellian f_0.
- Work in the frame of the bulk flow (i.e., $\mathbf{u} = 0$).

The Boltzmann equation becomes $$v_z \frac{\partial}{\partial z}(f_0 + f_1) \approx v_z \frac{\partial f_0}{\partial z} \approx -\frac{f_1}{\tau} \ .$$

Recall the Maxwellian f_0 is a function of n, $\mathbf{u}_0$, and T. If T is the only quantity that varies as a function of z, the chain rule gives

$$\frac{\partial f_0}{\partial z} = \frac{\partial f_0}{\partial T}\frac{\partial T}{\partial z} = \left[\frac{f_0}{T}\left(\frac{v^2}{v_{\rm th}^2} - \frac{3}{2}\right)\right]\frac{\partial T}{\partial z} \qquad \left(\text{where } \ v_{\rm th}^2 = \frac{2k_{\rm B}T}{m}\right)$$

Thus, Boltzmann's equation tells us

$$f_1(v) = -\frac{\tau\, v_z\, f_0}{T}\left(\frac{v^2}{v_{\rm th}^2} - \frac{3}{2}\right)\frac{\partial T}{\partial z}$$

and note that f_1 is a cubic polynomial (in v) times f_0. The total $f = f_0 + f_1$ is **skewed** in the v_z direction... **and not necessarily positive-definite!**

Thus, we can plug f into the definition of $\mathbf{q}$ (or at least its 1 relevant vector component):

$$q_z = \frac{1}{2}mn\langle v^2\, v_z\rangle = \frac{m}{2}\int d^3\mathbf{p}\ v^2\, v_z\ (f_0 + f_1)$$

and we can ignore the f_0 term because nice isotropic Maxwellians don't transport heat.

Algebra redacted... $$q_z = -\frac{m\tau}{2}\frac{1}{T}\frac{dT}{dz}\int d^3\mathbf{v}\ v_z^2\left(\frac{v^4}{v_{\rm th}^2} - \frac{3v^2}{2}\right)f_0(\mathbf{v})$$
$$= \ \leadsto \ -\frac{m\tau}{2}\frac{1}{T}\frac{dT}{dz}\left(\frac{5}{2}nv_{\rm th}^4\right)$$

If we replace $v_{\rm th}^2$ by $2k_{\rm B}T/m$, we indeed get

$$q_z = -\kappa\frac{dT}{dz} \qquad \text{where} \qquad \kappa = \frac{5\tau n k_{\rm B}^2 T}{m} \ .$$

Combine this with Drude's model for the resistivity η (or electrical conductivity σ) and we get

$$\frac{\kappa}{\sigma} = 5\,\frac{k_{\rm B}^2 T}{e^2}$$

which is also known as the Wiedemann–Franz law. For metals, the factor of 5 is replaced by $\pi^2/3 \approx 3.29$.

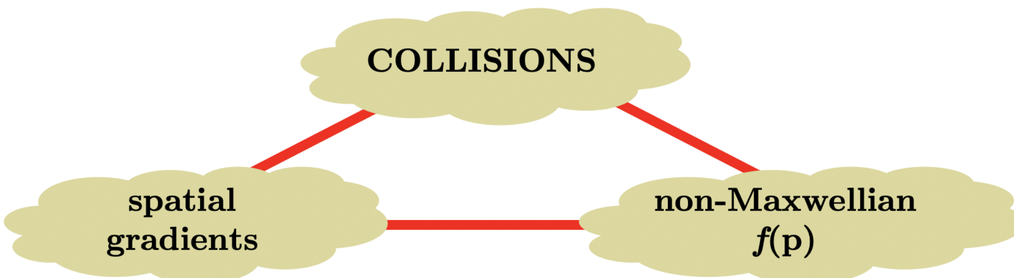
Even simpler, though, we can show that $\boxed{D_{\rm T} \;=\; \tau v_{\rm th}^2 \;=\; \ell_{\rm mfp}\, v_{\rm th}}$.

Of course, we then need to go back to Coulomb collision theory to specify $\tau = \tau_{\rm coll}$ in terms of other plasma parameters:

$$\tau_{\rm coll} \,\propto\, n^{-1} T^{3/2} \qquad \Longrightarrow \qquad \kappa \,\propto nT\,/\nu_{\rm coll} \,\propto\, T^{5/2} \;\; .$$

. . . .

In summary, we've "closed the loop" that describes how transport processes occur as a confluence of three factors:



Magnetic Reconnection

We noted that astrophysical plasmas often have $\boxed{\mathrm{Rm} \gg 1}$, which means collisional resistivity effects shouldn't be important.

However, sometimes magnetized plasmas get themselves twisted & braided into **complex topologies...**

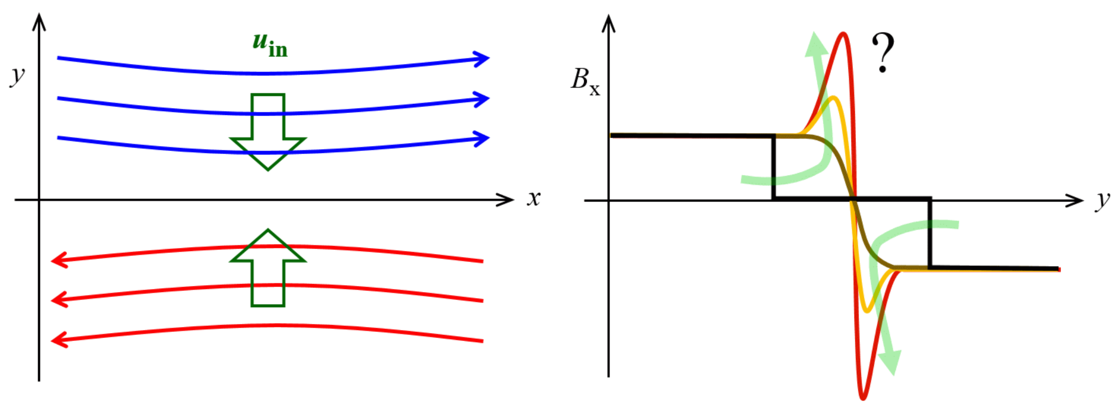
- coronal loops/flares/prominences above a convectively churning star
- magnetized plasma above & below an MRI-unstable accretion disk
- initially helical **B** in galactic jets can become tangled & chaotic
- planetary magnetospheres (2 disparate regions pressed together)

In such regions, there arise small-scale locations where $u\ell/D_{\mathrm{B}}$ is no longer $\gg 1$.

Thus, if we want to study what happens when oppositely-directed regions of **B** are pushed together, we're forced to take resistivity seriously.

. . . .

If magnetic flux was **perfectly frozen-in** to the flow, the field lines would build up in a "log-jam."



However, we know there's another term in the induction equation: **diffusion.**

Note that in the above problem, we're looking at "steep" variations in the y direction. Define the thickness of the "reconnection region" (in y) as δ, and assume spatial derivatives are strongest in y. As magnetic flux piles up, gradients get sharper, so $\boxed{\delta \text{ gets smaller.}}$

Replacing u by $u_{\rm in}$ (fixed speed at which fields are pushed together) and ℓ by δ (which is shrinking), we find that eventually,

$$\text{Rm} = \frac{u_{\rm in}\delta}{D_{\rm B}} \sim 1$$

i.e., diffusion starts to "beat" flux freezing.

i.e., the thickness of a reconnection region is $\quad \delta \sim \dfrac{D_{\rm B}}{u_{\rm in}}$

and once the region gets this thin, diffusion starts to **annihilate** magnetic energy and convert it to heat.

<u>Problem:</u> We don't know what sets the scale for *either* δ or $u_{\rm in}$.

In order to figure out what's really going on when $\mathbf{B}$ starts to get destroyed, we need to bring in more information. This step is still unsolved & controversial.

Aside: The sharp, flattened region where opposing fields meet is often called a **current sheet.** Why? In MHD,

$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B} \qquad \text{and in this geometry,} \qquad J_z \sim \frac{c\,B_x}{4\pi\,\delta}$$

because the $\partial B_x/\partial y$ term dominates. Outside the current sheet, $J \to 0$.

Also, in the reference frame moving with the flow, $\mathbf{E} = \eta \mathbf{J}$, so

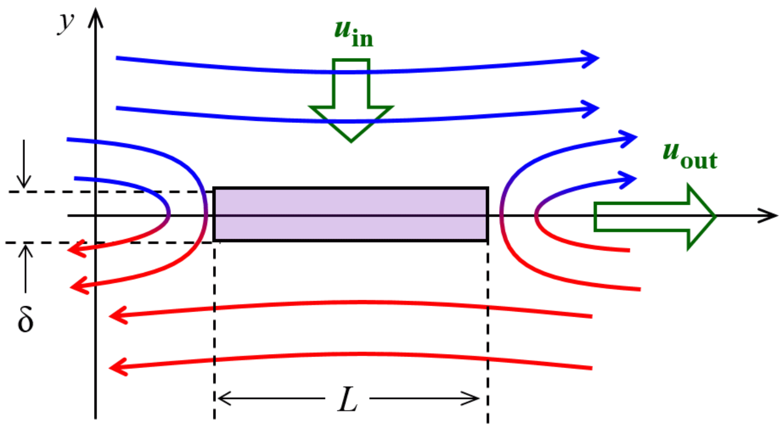
$$E_z \approx \frac{\eta\, c\, B}{4\pi\,\delta} \approx \frac{D_{\rm B}\, B}{c\,\delta} \qquad \text{i.e.,} \qquad \frac{u_{\rm in}}{c} \approx \frac{E}{B}$$

so the faster the reconnection inflow, the more of a **DC** electric field is built up in the current sheet.

Thus, once we know all the parameters, the volumetric heating rate inside the current sheet can be computed; $Q_{\rm heat} = \mathbf{J}\cdot\mathbf{E} = J_z E_z$.

I'll go over one of the earliest models of what happens when incoming field lines are "broken" and reconnected to field lines from the opposite side.

In the 2D **Sweet–Parker** model (1957), the reconnection region can be thought of as flattened & ~rectangular...



The large-scale length L of the system is something we already know, like ρ (which we can assume is uniform everywhere).

If it's steady-state, then the total mass coming in must balance the mass going out, in proportion to the dimensions,

$$u_{\text{in}} L \approx u_{\text{out}} \delta$$

i.e., mass flux depends on $\rho u A$, but the full A depends on extent in/out of the board. That's the same for both in & out motions.

$$\text{Toothpaste tube analogy...} \qquad \frac{u_{\text{out}}}{u_{\text{in}}} \approx \frac{L}{\delta} \gg 1 \ .$$

We can also make use of **energy conservation,** and assume that the **magnetic energy** going in ≈ **kinetic energy** coming out the sides.

(Inside the diffusion region, it's dominated by thermal energy, but we're staying "outside" for now.)

$\mathcal{E}$ = volume × energy density, so

$$\mathcal{E}_{\text{in}} = \Delta x \, \Delta y \, \Delta z \; U_{\text{B}} \qquad\qquad \mathcal{E}_{\text{out}} = \Delta x \, \Delta y \, \Delta z \; U_{\text{K}}$$

$$= L \, (u_{\text{in}} \Delta t) \Delta z \left(\frac{B^2}{8\pi} \right) \qquad\qquad = (u_{\text{out}} \Delta t) \, \delta \, \Delta z \left(\frac{1}{2} \rho u_{\text{out}}^2 \right)$$

$$\text{If } \ \mathcal{E}_{\text{in}} = \mathcal{E}_{\text{out}} \ , \quad \text{then} \quad u_{\text{out}} = \frac{B}{\sqrt{4\pi\rho}} = V_{\text{A}} \qquad \text{(the Alfvén speed)} \ .$$

In magnetically dominated regions (e.g., corona), $\beta \ll 1$, and thus $V_{\text{A}} \gg c_s$. Reconnection outflow is strongly *supersonic.*

Some context:

- Note that $V_{\rm A}$ is showing up as a "characteristic" macroscopic/large-scale speed of the system. It's not just a wave phase speed.
- Really, $\mathcal{E}_{\rm in} \neq \mathcal{E}_{\rm out}$, since a part of $\mathcal{E}_{\rm in}$ must go into *heating up* the diffusion region! Note that a larger $u_{\rm in}$ means larger $\mathcal{E}_{\rm in}$, so the total **reconnection heating rate** must also scale with $u_{\rm in}$.

Finally, we can put together everything we know to write

$$\begin{aligned} u_{\rm in} &= \frac{u_{\rm out}\,\delta}{L} && \text{(from mass conservation)} \\ &= \frac{V_{\rm A}\,(D_{\rm B}/u_{\rm in})}{L} && \text{(from energy conservation \& Rm} \approx 1 \text{ in box)} \end{aligned}$$

Multiplying the right side by $V_{\rm A}/V_{\rm A}$, we can write

$$u_{\rm in}^2 = \frac{V_{\rm A}^2}{S} \qquad \text{where recall that} \qquad S = \frac{V_{\rm A}\,L}{D_{\rm B}}$$

and S is specifically the Lundquist number for the macroscopic/large-scale system. Thus,

$$\boxed{\frac{u_{\rm in}}{V_{\rm A}} \approx \frac{\delta}{L} \approx \frac{1}{\sqrt{S}}}$$

Unfortunately, this is extremely **slow.** If $S \sim 10^{10}$, then $u_{\rm in}$ is $\sim 10^{-5}$ times the local Alfvén speed. In the solar corona, it would take **months to years** to fully "process" a flare's worth of $\mathbf{B}$ via reconnection this way. However, we see in flares that it can all happen in **5–10 minutes!**

Observations (and more detailed computer simulations) show that real reconnecting systems often find their way to the narrow range of

$$\boxed{\frac{u_{\rm in}}{V_{\rm A}} \approx 0.01 \text{ to } 0.1}$$

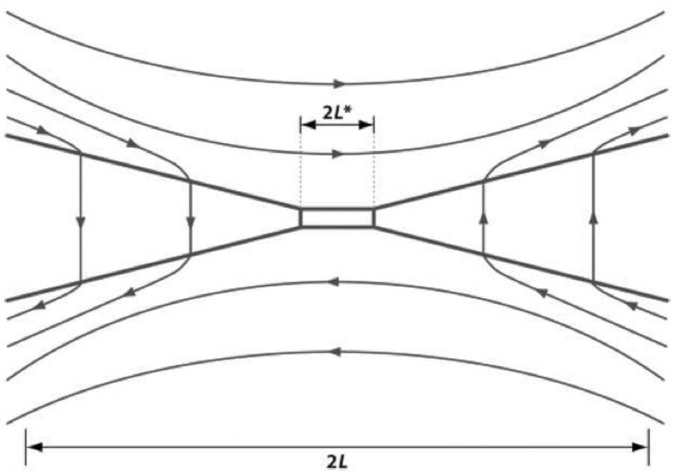
which is sufficiently fast and efficient to account for what we see. The details (how the universe gets around Sweet-Parker "constraints") are still unclear.

Ideas include:

- The thin "current sheet" (diffusion region) can be **turbulent,** in which small magnetic "islands" can form & grow along the thin interface region. Chaotic eddies produce extra anomalous diffusion:

$$\text{larger } D_{\rm B} \longrightarrow \text{ smaller effective } S \longrightarrow \text{ faster } u_{\rm in} \text{ !}$$

- Maybe the reconnecting fields come together in a series of "X-points" rather than all along a parallel line. **Petschek** proposed a model where *not all* reconnecting plasma has to go through the diffusion region.



  Some plasma short-circuits the diffusion region and forms SHOCKS along the inflow/outflow interface.

$$\text{Petschek's} \quad \frac{u_{\rm in}}{V_{\rm A}} \approx \frac{1}{\ln S} \quad \text{which isn't as tiny as Sweet-Parker's.}$$

- If the diffusion region "wants" to get smaller than the particle **Larmor radii,** then non-MHD collisionless effects can take over. (Electrons and ions decouple from a common fluid motion.) This is related to the **Hall effect** from Ohm's law. Vasyliunas (1975, *Rev. Geophys. & Space Phys.*, 13, 303) derived some straightforward modifications to the Sweet-Parker theory for finite "inertial lengths." More on these in a bit...

BEYOND MHD ...

Thus far, we've assumed that astrophysical plasma motions occur on **large** spatial & time scales, compared to scales that individual particles care about:

$$\text{MHD} \implies \quad L \gg \begin{cases} \lambda_{\mathrm{D},s} & \text{Debye length} \quad (\text{species } s) \\ \ell_{\mathrm{mfp},s} & \text{collisional mean free path} \\ r_{\perp,s} & \text{Larmor gyroradius} \end{cases}$$

Equivalently, using the "most-probable" (thermal) speed $v_{\mathrm{th},s}$ we see that MHD implies **slow** variability:

$$\text{MHD} \implies \quad \frac{1}{t} \ll \begin{cases} \omega_{ps} = v_{\mathrm{th},s}/\lambda_{\mathrm{D},s} & \text{plasma frequency} \\ \nu_{\mathrm{coll},s} = v_{\mathrm{th},s}/\ell_{\mathrm{mfp},s} & \text{collision frequency} \\ \Omega_s = v_{\mathrm{th},s}/r_{\perp,s} & \text{cyclotron frequency} \end{cases}$$

Most astrophysical plasmas have $\lambda_{\mathrm{D},s} \ll L$, and they tend to be dilute enough that $\boxed{\ell_{\mathrm{mfp}} \gg r_\perp}$, but the other orderings can run the gamut...

$$\begin{array}{ll} L > \ell_{\mathrm{mfp}} > r_\perp & \text{collisional fluid} \\ \ell_{\mathrm{mfp}} > L > r_\perp & \text{collisionless fluid} \\ \ell_{\mathrm{mfp}} > r_\perp > L & \text{collisionless kinetic} \end{array}$$

Knudsen number ($\mathrm{Kn} = \ell_{\mathrm{mfp}}/L$) is used for atmospheres $\rightarrow$ exospheres.

When single particles do what they want to do (without frequent collisions), $f(\mathbf{p})$ may no longer be even *close* to Maxwellian.

We'll go over three general effects:

1. **Non-Maxwellian anisotropy** in a strong magnetic field.
2. **Kinetic-scale waves** that can damp or grow ("micro-instabilities").
3. **Collisionless particle drifts** such as ambipolar diffusion.

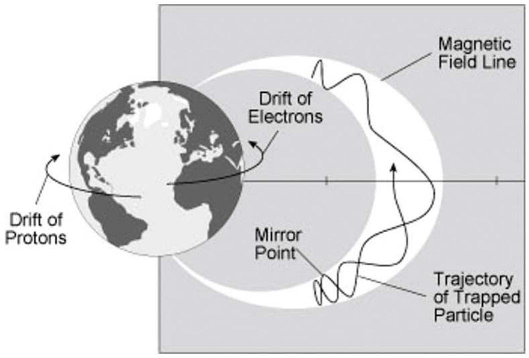
. . . .

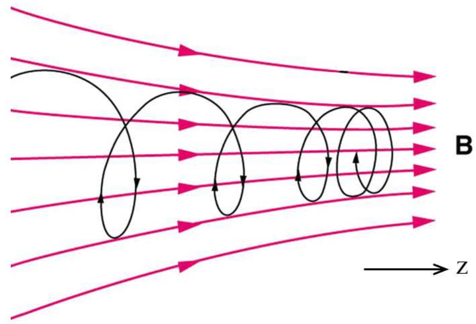
(1) Anisotropy: In collisionless & kinetic regimes with **strong B,** the system can exhibit, e.g., $T_\perp \neq T_\parallel$, $\kappa_\perp \neq \kappa_\parallel$, and so on.

This is clear to see for individual particles flowing along a magnetic field that *varies spatially.* We can describe how particle motion is affected (in a decidedly non-Maxwellian way!) by looking at:

Magnetic Moment Conservation

Consider particles gyrating around $\mathbf{B}$, but gradually moving (via $v_{\parallel}$) into a region of *increasing* field strength:





(i.e., from equator to pole along Earth's dipole)

Let's think of this like the magnetic "flux tube" from the homework:
$\mathbf{B}$ is dominated by its "axial" field B_z, and it's got

$$\begin{array}{ll} B_r > 0 & \text{when tube expands as } z \uparrow \\ B_r < 0 & \text{when tube constricts as } z \uparrow \end{array} \qquad \text{and} \qquad B_r = -\frac{r}{2}\,\frac{\partial B_z}{\partial z}$$

$$\text{and let's assume constriction; \quad i.e.,} \quad \frac{\partial B_z}{\partial z} > 0 \ ,$$

where the latter came from $\nabla \cdot \mathbf{B} = 0$ in cylindrical coordinates.

For $\mathbf{B}$ being dominated by its z component, the gyroradius $r_{\perp} = v_{\perp}/\Omega$ is $\propto 1/B_z$, i.e., $r_{\perp}$ *decreases* as B gets stronger. This makes sense; the whole thing is converging, and thus $r_{\perp}$ behaves essentially like the "tube radius."

When $\mathbf{B}$ was constant, the Lorentz force on a charged particle gave gyromotion in $v_{\perp}$ (which we'll also call v_{ϕ}) and a constant value of $v_{\parallel} = v_z$.

Now let's re-evaluate the parallel component of the Lorentz force for this case of varying B_z. In cylindrical coordinates, the gyromotion is v_{ϕ}, and there's virtually no v_r.

$$m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\,(\mathbf{v} \times \mathbf{B}) \qquad\qquad m\frac{dv_z}{dt} = -\frac{q}{c}\,v_{\phi}B_r \qquad (\text{before, } B_r \text{ was zero})\,.$$

Recall that positively charged particles ($q > 0$) are left-hand polarized.
If B_z points along the $+z$ direction, then $v_{\phi} < 0$.

Negative particles ($q < 0$) are right-hand polarized, so they'd have $v_\phi > 0$.

No matter what, the product $qv_\phi < 0$. Thus,

$$m\frac{dv_z}{dt} = +\left|\frac{qv_\perp}{c}\right| B_r \;=\; -\left|\frac{qv_\perp r}{2c}\right| \frac{\partial B_z}{\partial z} \;=\; -\mu\,\frac{\partial B_z}{\partial z}$$

where we define μ as the **magnetic moment** of a charged particle.

Note that the radius r of the tube behaves just like the gyroradius, so let's use $r_\perp$ for r...

$$\mu \;=\; \left|\frac{qv_\perp r_\perp}{2c}\right| \;=\; \left|\frac{qv_\perp}{2c}\,\frac{v_\perp}{\Omega}\right| \;=\; \left|\frac{qv_\perp^2}{2}\,\frac{m}{qB_z}\right| \;=\; \left|\frac{\frac{1}{2}mv_\perp^2}{B_z}\right|$$

i.e., μ is the kinetic energy in gyro-motions, divided by the flux tube's field strength.

Thus, $$\boxed{m\frac{dv_z}{dt} = -\mu\frac{\partial B_z}{\partial z}}$$ So what? Who cares?

This does tell us that

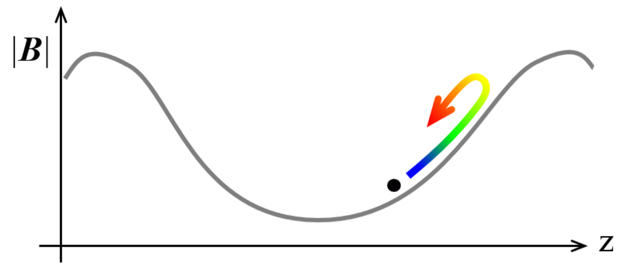
$$\text{charged particles} \left\{\begin{array}{c}\text{accelerate}\\ \text{decelerate}\end{array}\right\} \text{along the direction of} \left\{\begin{array}{c}\text{weakening } \mathbf{B}\\ \text{strengthening } \mathbf{B}\end{array}\right\}$$

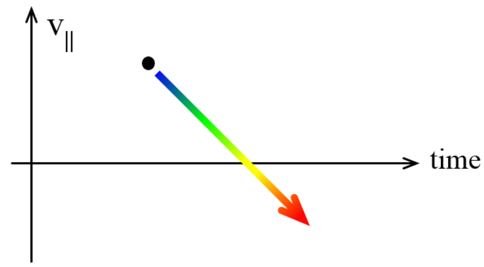
We will see that single particles obey $\boxed{\mu = \text{constant}}$ along a field line. If that's the case, then $v_\perp^2 \propto B_z$, so

$$\text{going from} \left\{\begin{array}{c}\text{weaker} \rightarrow \text{stronger } \mathbf{B}\\ \text{stronger} \rightarrow \text{weaker } \mathbf{B}\end{array}\right\} \text{means} \left\{\begin{array}{ccc}v_\parallel \downarrow & \text{and} & v_\perp \uparrow\\ v_\parallel \uparrow & \text{and} & v_\perp \downarrow\end{array}\right\}$$

Eventually, a decreasing $v_\parallel$ will hit zero, and there's nothing to stop it *continuing to decrease* into negative values!

The particle is thus reflected, and this kind of configuration is called a **magnetic mirror** or (if there are mirrors on both ends) a **magnetic bottle:**





Derivation: Why is $\mu =$ constant?

Let's start by writing the v_z equation of motion, and multiply both sides by $v_z = dz/dt$:

$$\begin{aligned} m v_z \frac{dv_z}{dt} &= -\mu \frac{dB_z}{dz}\frac{dz}{dt} \\ \frac{d}{dt}\left(\frac{1}{2} m v_z^2\right) &= -\mu \frac{dB_z}{dt} \end{aligned}$$

(The cancellation of dz is tricky... this has to be a 'Lagrangian' derivative, which follows the particle.)

On the left side above, we see the z-component of the kinetic energy $\mathcal{E}_{\rm K}$. In full,

$$\mathcal{E}_{\rm K} = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} m (v_\perp^2 + v_z^2)$$

However, we know that the magnetic Lorentz force does no work on a particle, so $\mathcal{E}_{\rm K}$ should be *constant.* This means

$$\frac{d\mathcal{E}_{\rm K}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) + \frac{d}{dt}\left(\frac{1}{2} m v_z^2\right) = 0 \ .$$

But we can use the definition of μ, and the above version of the equation of motion, to write

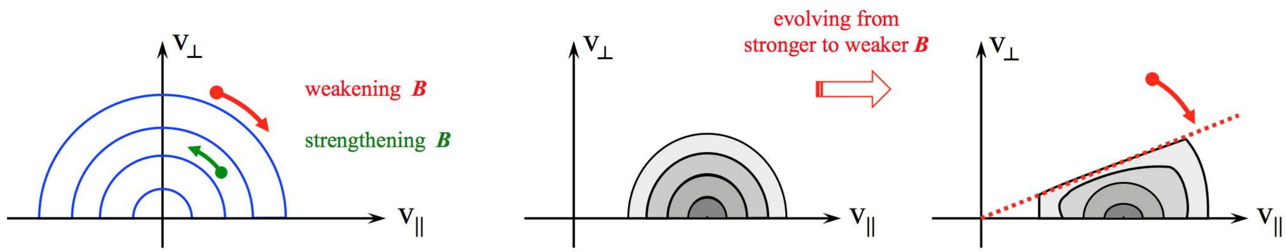
$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) + \frac{d}{dt}\left(\frac{1}{2} m v_z^2\right) \\ &= \frac{d}{dt}(\mu B_z) - \mu \frac{dB_z}{dt} \\ &= \left(\mu \cancel{\frac{dB_z}{dt}} + B_z \frac{d\mu}{dt}\right) - \mu \cancel{\frac{dB_z}{dt}} \end{aligned}$$

$$\text{i.e., since } B_z \neq 0, \text{ then } \quad \frac{d\mu}{dt} = 0 \qquad \text{or} \quad \mu = \text{constant} \ .$$

The magnetic moment μ is called an *adiabatic invariant.*

How does μ-conservation affect the MHD/fluid-like nature of a system as we transition from a collision-dominated to collisionless plasma?

Particles move on circles in velocity space:



Many insights about single-particle motion generalize to $f(\mathbf{p})$'s...

$$\left.\begin{aligned} \langle \tfrac{1}{2}mv^2\rangle &= \tfrac{3}{2}k_{\rm B}T \\ \langle \tfrac{1}{2}mv_\parallel^2\rangle &= \tfrac{1}{2}k_{\rm B}T_\parallel \quad \text{(1 degree of freedom)} \\ \langle \tfrac{1}{2}mv_\perp^2\rangle &= k_{\rm B}T_\perp \quad \text{(2 degrees of freedom)} \end{aligned}\right\} \qquad T = \frac{T_\parallel + 2T_\perp}{3}$$

Thus, if an initially isotropic ($T_\parallel = T_\perp = T$) distribution evolves from strong to weak $\mathbf{B}$, it develops $\boxed{T_\parallel > T_\perp}$.

It's possible to modify the MHD equations to account for anisotropy in temperature (and/or pressure). Assuming **bi-Maxwellian** distrubtions (i.e., elliptical contours in v-space),

$$f(\mathbf{p}) = \frac{n}{(2\pi m k_{\rm B})^{3/2}(T_\parallel^{1/2}T_\perp)} \exp\left[-\frac{(v_\parallel - u_\parallel)^2}{2k_{\rm B}T_\parallel/m} - \frac{v_\perp^2}{2k_{\rm B}T_\perp/m}\right]$$

is equivalent to a 3×3 stress tensor with

$$\mathbb{P} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix} \qquad \text{where} \quad P_{\parallel,\perp} = n k_{\rm B} T_{\parallel,\perp}\,.$$

We assume a "gyrotropic" distribution; i.e., the Larmor motions are so rapid that the 2 transverse directions (x, y) are statistically equivalent to one another.

In the **momentum equation,** the pressure-gradient force is modified:

$$\begin{aligned}\rho\frac{D\mathbf{u}}{Dt} &= -\nabla\cdot\mathbb{P} + \cdots \\ &= -\nabla P_\perp + (\mathbf{B}\cdot\nabla)\left[(P_\perp - P_\parallel)\frac{\mathbf{B}}{B^2}\right] + \cdots\end{aligned}$$

For 1D field-aligned flows ($\mathbf{u}, \mathbf{B} \parallel \hat{\mathbf{e}}_z$), it simplifies to

$$\rho\frac{Du_z}{Dt} = -\frac{\partial P_\parallel}{\partial z} - \left(\frac{P_\perp - P_\parallel}{B_z}\right)\frac{\partial B_z}{\partial z} + \cdots$$

where the 2nd term on the RHS is an effective **magnetic mirror force** that depends on how "squashed" $f(\mathbf{p})$ becomes in a field varying along z.

The mirror force provides extra acceleration when $T_\perp > T_\parallel$, i.e., lots of particles at high $v_\perp$ ready to be "folded down" into the $+v_\parallel$ direction.

. . . .

The **thermal energy equation** is also modified. Consider just the adiabatic limit. For a parcel of isotropic (Maxwellian) ideal gas,

$$\frac{D}{Dt}\left(\frac{P}{n^\gamma}\right) = \frac{D}{Dt}\left(\frac{T}{n^{\gamma-1}}\right) = 0 \qquad \longrightarrow \qquad \boxed{T \propto n^{2/3}} \qquad (\text{for } \gamma = 5/3)$$

Chew, Goldberger, and Low (1956) ["CGL"] plugged in the the modified $\mathbb{P}$ to develop **double-adiabatic** equations for $T_\parallel$ and $T_\perp$ that are often used in space physics.

Simple/approximate motivation for CGL equations:

(1) One finds that μ-conservation holds for distributions, not just for single particles:

$$\frac{v_\perp^2}{B} = \text{constant} \qquad \longrightarrow \qquad \frac{\langle v_\perp^2\rangle}{B} = \text{constant}$$

$$\text{Thus,} \qquad \frac{D}{Dt}\left(\frac{T_\perp}{B}\right) = 0 \qquad \longrightarrow \qquad \boxed{T_\perp \propto B}$$

(2) There is a 2nd adiabatic invariant: Consider a "magnetic bottle" as above, and particles bounce back and forth between the two ends.

If the length L of the bottle decreases in time, the particles will bounce off *approaching walls*, and increase their speed. In this system, one can show that $J = v_\parallel L =$ constant.

Now think of this bottle as a cylinder with length L, cross-section area A, and magnetic field $\mathbf{B}$ pointing along the axis.

If L shrinks, the number of particles in the tube should remain the same: $N = nV = nLA =$ constant.

We also know magnetic flux conservation requires $BA =$ constant. Thus,

$$\frac{nL}{B} = \text{constant} \quad , \qquad \text{i.e.,} \qquad L \propto \frac{B}{n} \ .$$

However, J conservation says $\qquad v_\parallel \propto \frac{1}{L} \propto \frac{n}{B}$

and making the same assumption that $v_\parallel^2$ behaves similarly to $\langle v_\parallel^2 \rangle$, we see that

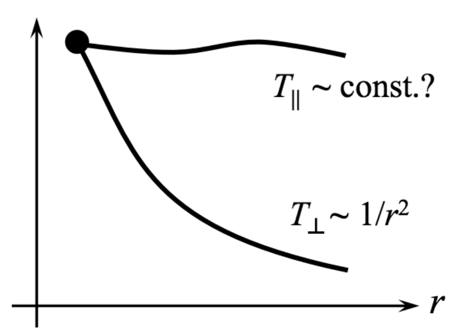
$$\frac{D}{Dt}\left(\frac{T_\parallel B^2}{n^2}\right) = 0 \qquad \longrightarrow \qquad \boxed{T_\parallel \propto \frac{n^2}{B^2}}$$

In the isotropic limit, the CGL relations (kind of) reduce back to the standard adiabatic law:

$$T \ \sim \ \left(T_\parallel T_\perp^2\right)^{1/3} \ \propto \ \left[\frac{n^2}{B^2}\ B^2\right]^{1/3} \ \propto \ n^{2/3} \ .$$

In space plasmas, CGL theory is useful, but expected trends in $T_\parallel \neq T_\perp$ are often perturbed by many other *non-adiabatic* effects.

Example: spherical solar wind:
n, B both $\sim 1/r^2$, so in the absence of other effects, we expect $\Longrightarrow \quad \Longrightarrow$



CGL would predict $T_\parallel/T_\perp \approx 100$ at 1 AU, but observations show $T_\parallel/T_\perp \sim 1$.

What other effects are at play? Next topic...

. . . .

Before moving on, I should mention one other anisotropic modification to the energy equation: **heat conduction** is much more efficient along a **B** field than across it: $\boxed{\kappa_\parallel \gg \kappa_\perp}$

(First, let's go back to thinking of **B** as a constant. One effect at a time!)

Why would heat conduction be anisotropic? Remember that collisions take particles on a **random walk,** with nominal step size $\ell_{\rm mfp}$.

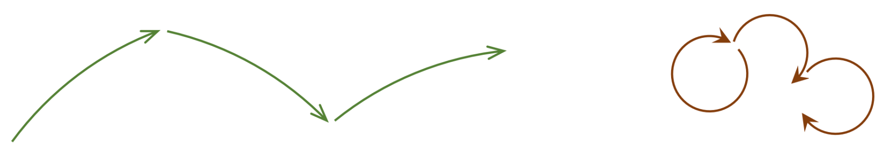
Recall from our study of single particles that motion along **B** is unimpeded by the magnetic Lorentz force.

$\Longrightarrow$ Collisions along the field act like there's no field.

However, in the $\perp$ direction, particles gyrate with mean radius $r_\perp = v_{\rm th}/\Omega$. Also, most magnetized plasmas tend to have

$$\frac{r_\perp}{\ell_{\rm mfp}} \ll 1 \quad , \qquad \text{i.e.,} \qquad \frac{\ell_{\rm mfp}}{r_\perp} = \tau_{\rm coll}\Omega = \left\{ \begin{array}{c} \text{magnetization} \\ \text{parameter} \end{array} \right\} \gg 1 \quad .$$

Thus, when "random walking" *across* the field, the effective **step sizes** are limited by $r_\perp$ rather than the larger $\ell_{\rm mfp}$ the particles would have wanted.



If the motion over a given $\perp$ distance was ballistic, it would take $N = \ell_{\rm mfp}/r_\perp$ times *more steps* to go a given distance than if there was no field.

However, random walks are usually **diffusive,** so it really takes $N^2 = (\ell_{\rm mfp}/r_\perp)^2 = (\tau_{\rm coll}\Omega)^2$ times more steps to go a given perpendicular distance.

Thus, it's not surprising that a more rigorous calculation gives

$$\boxed{\kappa_\parallel \approx \frac{\tau n k_{\rm B}^2 T}{m}} \qquad \text{and} \qquad \boxed{\kappa_\perp \approx \frac{\kappa_\parallel}{(\tau\Omega)^2} \ll \kappa_\parallel} \quad .$$

---

(2) Kinetic-Scale Plasma Waves

We've seen how **particle–particle** collisions can:

- transport momentum & energy
- drive velocity distributions to isotropic Maxwellian equilibrium
- damp out oscillations
- give finite thickness to shocks (not in this course)

However, all of these things occur in the absence of collisions, too.

$$\text{Collisionless effects like:} \quad \left\{ \begin{array}{rl} \text{(linear)} & \textbf{wave–particle} \text{ interactions} \\ \text{(nonlinear)} & \textbf{wave–wave} \text{ interactions} \end{array} \right\}$$

become important when the fluctuations take place on small enough (kinetic) space & time scales.

In the "Jello" problem, we considered electrostatic oscillations in of n_e (assuming n_p = fixed), and discovered $\omega = \omega_{pe}$.

However, we used MHD-ish conservation equations, which presumed $f_e(\mathbf{p})$ always remained a drifting Maxwellian.

Let's look again at these fluctuations. Assume $\mathbf{B}_0 = 0$, $\mathbf{E}_0 = 0$.
Also assume that all spatial variations are $\parallel$ to $\mathbf{E}_1$.

The kinetic approach is to linearize $f_e = f_{0,e} + f_{1,e}$ istelf, like in the Chapman–Enskog problem (and assume $f_p = f_{0,p}$, charged-balanced with $f_{0,e}$), then the Vlasov equation & Gauss' law can be solved for kinetic oscillations:

$$\text{In 1D, } \mathbf{k} = k\hat{\mathbf{e}}_x \text{, so:} \quad \begin{aligned} f_{1,e} &= \widetilde{f}_{1,e} \exp(ikx - i\omega t) \\ E_{1x} &= \widetilde{E}_{1x} \exp(ikx - i\omega t) \ , \quad E_{1y} = E_{1z} = 0 \end{aligned}$$

Faraday's law: $\mathbf{k} \times \mathbf{E}_1 = (\omega/c)\mathbf{B}_1 = 0$, so waves are purely electrostatic.

Thus,

$$\frac{\partial f_{1,e}}{\partial t} + v_x \frac{\partial f_{1,e}}{\partial x} - eE_{1x} \frac{\partial f_{0,e}}{\partial p_x} = 0$$

$$\nabla \cdot \mathbf{E} = 4\pi e(n_p - n_e) \qquad \Longrightarrow \qquad \frac{\partial E_{1x}}{\partial x} = -4\pi e \int d^3\mathbf{p} \; f_{1,e}(\mathbf{p})$$

In the jello problem, we imposed Δn, and both $\mathbf{E}$ and $\mathbf{v}$ responded. Here, we don't need to worry about what drives what... we just need to figure out what kinds of fluctuations are "allowed" in the system.

Vlasov & Gauss combine to produce the **dispersion relation** for $\omega(k)$:

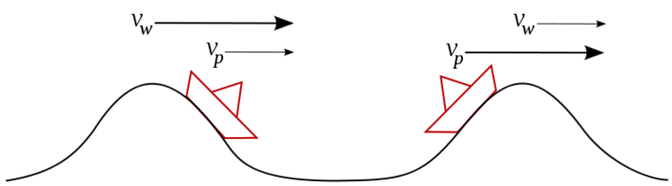
$$\widetilde{E}_{1x} \left[1 + \frac{4\pi e^2}{m_e k} \int d^3\mathbf{p} \, \frac{\partial f_{0,e}/\partial p_x}{\omega - k v_x} \right] = 0 \ .$$

I won't delve any deeper with the math, but note that:

- The dispersion relation depends on the **shape** of $f_{0,e}(\mathbf{p})$.

  If it's Maxwellian, $\quad \omega^2 \approx \omega_{pe}^2 + 3k^2 v_{\mathrm{th},e}^2 \qquad$ **(Langmuir waves) .**

- However, note the "resonant" term in the denominator. The integral *blows up* when $v_x \approx \omega/k$, i.e., when a particle in the distribution happens to be **"surfing"** along with the wave's phase velocity.

  Most particles "see" a sinusoidally oscillating $\mathbf{E}_1$ field, but these (few) resonant particles don't: they see a DC field (in their own frame).

  Thus, resonant particles can be rapidly **accelerated** by that DC field. Depending on phase, some are sped up, & some are slowed down.

  This yields **diffusion** in velocity space... much like collisions.

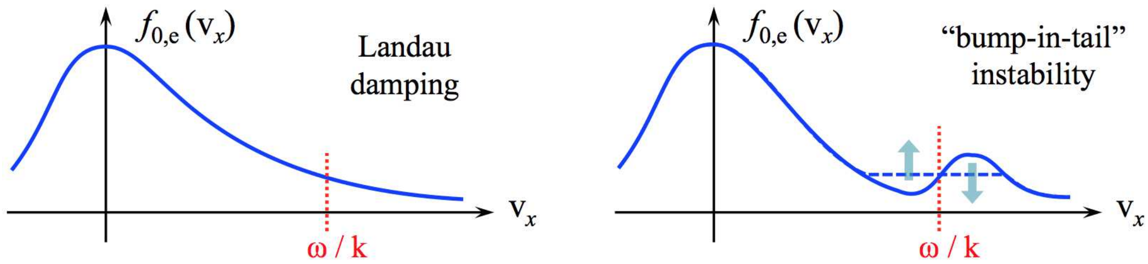
- The kinetic energy gained/lost by resonant particles must be balanced by something else: the energy in the wave oscillation itself.

  Thus, the presence of resonant particles acts like Coulomb collisions to create an imaginary part ($\omega = \omega_r + i\omega_i$). If $\omega_i < 0$, resonances produce wave damping (i.e., **Landau damping** in the above Langmuir wave example), or, if $\omega_i > 0$, they drive instabilities!

  The sign of ω_i depends on the shape of $f_{0,e}(\mathbf{p})$.

. . . .

What's really going on here? The sign of ω_i depends on the sign of $\partial f_{0,e}/\partial v_x$ at $v_x = \omega_r/k$:



It's clear that particles with $v_x \sim \omega_r/k$ interact **strongly** with waves. Think about this interaction as a kind of friction...

- Particles with $v_x = \omega_r/k$ surf exactly in phase with the wave. Their speeds are equal, so no friction.
- Particles with $v_x \gtrsim \omega_r/k$ (slightly faster than the wave) are "grabbed" by the wave and *slowed down* (i.e., particles **lose energy** to the waves).
- Particles with $v_x \lesssim \omega_r/k$ (slightly slower than the wave) are *"sped up"* by the quasi-friction (i.e., particles **take energy** from the waves).

Thus, the **net effect** depends on whether there are more particles of one kind or the other.

In a Maxwellian, there are more particles with $v_x \lesssim \omega_r/k$, so the net effect is wave damping & particle energization. For a bump-in-tail (beam), it's the opposite.

(Some complex systems oscillate back and forth between damping ($\omega_i < 0$) and instability ($\omega_i > 0$). After each "swing," the amplitude $|\omega_i|$ gets smaller, and it evolves to **marginal stability** ($\omega_i \approx 0$); i.e., kind of like equipartition of energy between waves & "free energy" in background $f_0(\mathbf{p})$.)

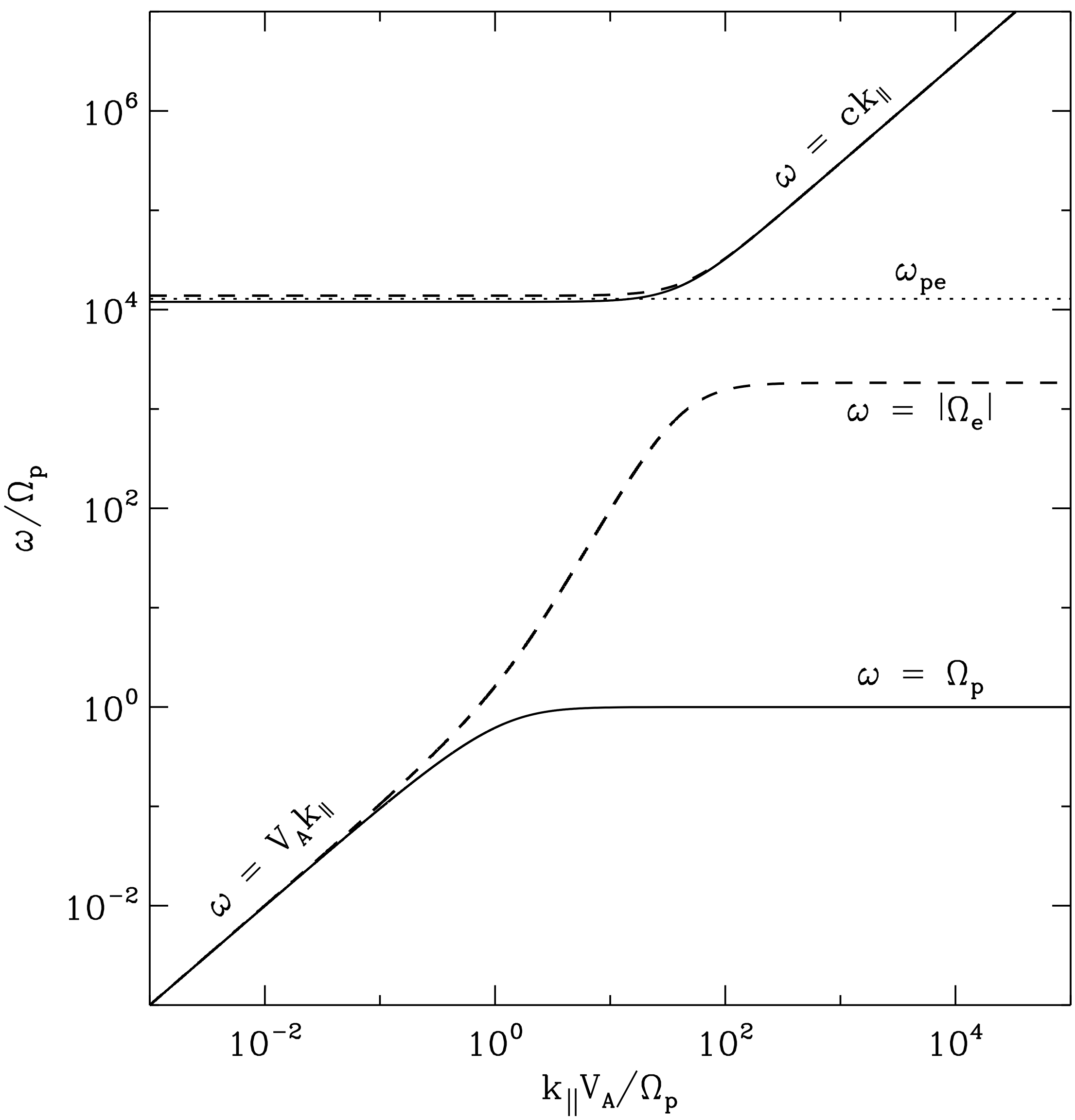
. . . .

There's so much "richness" in the physics of electrostatic kinetic fluctuations, and we didn't even include the magnetic field (neither a background $\mathbf{B}_0$ nor fluctuations $\mathbf{B}_1$).

When $\mathbf{B}$ is included, the Vlasov equation can be linearized, and combined with Faraday's law to write $\mathbf{B}_1$ in terms of $\mathbf{E}_1$.

Just like above, the perturbed f_1 is a function of both $\mathbf{E}_1$ and $\partial f_0/\partial \mathbf{p}$, and it all goes into a similar dispersion relation.

Solutions are *even richer* in terms of all the qualitatively different wave modes. Even with just $\mathbf{k} \parallel \mathbf{B}_0$ in the limit of $\beta \ll 1$, there's so much:



There are Alfvén-like waves, which eventually change into **cyclotron resonant waves** when ω gets as big as Ω. Particles in vicinity of these wave also undergo a type of "surfing" (like Landau damping), because the oscillating $\mathbf{E}_1$ is transverse to $\mathbf{B}_0$ and is circularly polarized. Thus, particles at the right frequency,

$$\omega - k_{\parallel} v_z = \pm\Omega_j \quad \text{(usually for } j = p, e\text{)}$$

see a DC electric field, and get rapidly spun-up or spun-down.

Application: high-speed solar wind!?

There's also a solution for $\omega(k_{\parallel})$ that is equivalent to **classical electromagnetic radiation.** Why not? We were looking for waves along some direction $\mathbf{B}_0$, for whom Faraday & Ampère say the $\mathbf{E}_1$ and $\mathbf{B}_1$ amplitudes must be transverse. It would be surprising if these waves *weren't* there.

---

(3) Collisionless Particle Drifts & Ambipolar Diffusion

Lastly, there are several non-ideal MHD effects that occur when collisions are *infrequent* enough that the bulk speeds $\mathbf{u}_s$ of various species are *unequal.*

When first deriving the MHD equations, we included ion–electron collisional "friction" on the RHS $\longrightarrow$ resistivity. When $\mathbf{u}_i \neq \mathbf{u}_e$, one sees additional **Hall terms** in the generalized Ohm's law.

More common to see in astrophysics, though, is the idea of friction between ions (i) & neutrals (n) in a **partially ionized plasma:**

$$\begin{aligned} m_i n_i \frac{D\mathbf{u}_i}{Dt} + \nabla P_i - \frac{\mathbf{J}\times\mathbf{B}}{c} &= m_i n_i \nu_{in} (\mathbf{u}_n - \mathbf{u}_i) \\ m_n n_n \frac{D\mathbf{u}_n}{Dt} + \nabla P_n &= m_n n_n \nu_{ni} (\mathbf{u}_i - \mathbf{u}_n) \end{aligned}$$

where $m_i n_i \nu_{in} \approx m_n n_n \nu_{ni}$, and we assume electrons are "mobile" enough to rapidly adjust their n_e and $\mathbf{u}_e$ to maintain charge neutrality.

Let's assume a **weakly ionized gas** (e.g., in a protoplanetary disk), so that

$$\frac{n_i}{n_n} \ll 1 \qquad \text{and} \qquad \text{the common fluid } \mathbf{u} \approx \mathbf{u}_n \ .$$

It's also true that $P_i \ll P_n$ because the neutrals dominate in density, so we can also be confident in saying $|\nabla P_i| \ll |\nabla P_n|$.

Let's consider a nearly time-steady system, so we can ignore the D/Dt terms in both equations.

Look at the momentum equation for the neutrals: the RHS is balanced only by ∇P_n. However, that *same* RHS is in the ion equation, and the ∇P_i term on the left is tiny in comparison to it. Thus, let's also ignore the ∇P_i term.

This means that the short-lived ions respond "instantaneously" to the collisional & magnetic forces only, so that the ion equation becomes

$$\frac{\mathbf{J} \times \mathbf{B}}{c} \approx m_i n_i \nu_{in} (\mathbf{u}_i - \mathbf{u}_n) \ .$$

We would like to know how $\mathbf{u}_i$ differs from the bulk $\mathbf{u} \approx \mathbf{u}_n$. Thus, we write

$$\mathbf{u}_i \ \approx \ \mathbf{u}_n + (\mathbf{u}_i - \mathbf{u}_n) \ \approx \ \mathbf{u} + \frac{\mathbf{J} \times \mathbf{B}}{m_i n_i \nu_{in} c}$$

2nd term: the **ambipolar drift velocity.**

Notation alert: Astronomers follow Mestel & Spitzer's (1956) adoption of the term "ambipolar" for this effect. However, some plasma physicists use this term for the ∇P_e term in Ohm's law (which was dealt with in astronomy by Pannekoek & Rosseland). Completely different usage!

Anyway, how does this drift term affect the system?

The magnetic induction equation really cares only about ion motion...

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_i \times \mathbf{B}) + D_{\mathrm{B}} \nabla^2 \mathbf{B}$$

$$= \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla \times \left[\frac{(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}}{m_i n_i \nu_{in} c} \right] + D_{\mathrm{B}} \nabla^2 \mathbf{B}$$

Thus, we have a new term on the right-hand side of the induction equation.

The new term is called ambipolar **diffusion.** Why?

Let's use Ampere's law without displacement current (i.e., $\mathbf{J} \propto \nabla \times \mathbf{B}$).

Also consider a simple geometry for a sheared field: $\mathbf{B} = B_z(x,t)\hat{\mathbf{e}}_z$, with $\mathbf{u} = 0$, $D_\mathrm{B} = 0$.

$$\frac{\partial B_z}{\partial t} = \left\{ \nabla \times \left[\frac{((\nabla \times \mathbf{B}) \times \mathbf{B}) \times \mathbf{B}}{4\pi m_i n_i \nu_{in}} \right] \right\}_z = \rightsquigarrow = \frac{\partial}{\partial x} \left[\left(\frac{B_z^2}{4\pi m_i n_i \nu_{in}} \right) \frac{\partial B_z}{\partial x} \right]$$

This is a (cross-field) diffusion equation, with an ambipolar diffusion coefficient given by

$$D_\mathrm{AD} = \frac{B_z^2}{4\pi m_i n_i \nu_{in}} = \frac{V_{\mathrm{A},i}^2}{\nu_{in}} = \frac{V_{\mathrm{A},n}^2}{\nu_{ni}}$$

and this effect is essentially a momentum redistribution mechanism that "allows" **B** to act on neutral particles, via the intermediary effect of ion-neutral collisions.

It doesn't formally "break" the field lines like the D_B resistivity term does. It just allows the field and the flow to become decoupled from one another. The field and flow **diffuse past one another.**

Ellen Zweibel introduced the idea of a corresponding ambipolar Reynolds number. We might as well call it

$$\mathrm{Zw} = \frac{u\,\ell}{D_\mathrm{AD}} .$$

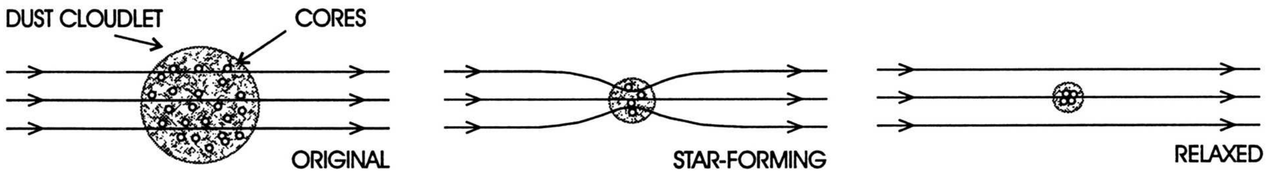
When $\mathrm{Zw} \gg 1$, all particles are frozen to the field lines, like in ideal MHD.

When $\mathrm{Zw} < 1$, ambipolar diffusion is important, and multi-fluid theory is needed to determine how much relative field/flow drift there will be.

. . . .

Application:

Mestel & Spitzer (1956) invoked ambipolar diffusion as a solution to low-mass star formation:



- An interstellar **B** exerts tension that may prevent gravitational collapse (especially $\perp$ to the field).
- In ideal MHD, a cloud can collapse only if its

$$\mu_\phi^2 \;=\; \left\{ \frac{\text{gravitational potential energy}}{\text{magnetic energy}} \right\} \;>\; 1\ .$$

  and H I gas observations show that $\mu_\phi \lesssim 0.2$ in the local ISM.

- Leaving out some constants, this quantity is

$$\mu_\phi^2 \approx \frac{GM^2/R}{B^2 R^3} \approx \frac{GM^2}{B^2 R^4} \approx \frac{GM^2}{\Phi^2}$$

  where the magnetic flux through the object is

$$\Phi \;=\; \int \mathbf{B} \cdot d\mathbf{A} \;\approx\; BR^2$$

- Thus, if an object's M and Φ are conserved as it collapses, in ideal MHD one would expect μ_ϕ to remain constant during the collapse, too.
- However, if neutrals can slip past those "tense" field lines, they may jump-start a collapse. Shu et al. (1987) showed that ambipolar diffusion can indeed evolve a GMC core from $\mu_\phi < 1$ to $\mu_\phi > 1$.
- The field distorts a bit, since bulk/neutral collapse does drag along the ions & electrons, and the friction also **heats** the gas.
