IDEAL MHD: What can we do with it?

We will discuss three general results of ideal MHD $(D_B = 0)$:

1. **Equilibrium states:** potential fields, force-free fields.
2. **MHD waves:** propagating perturbations about equilibrium states.
3. **MHD instabilities:** are equilibira stable or unstable?

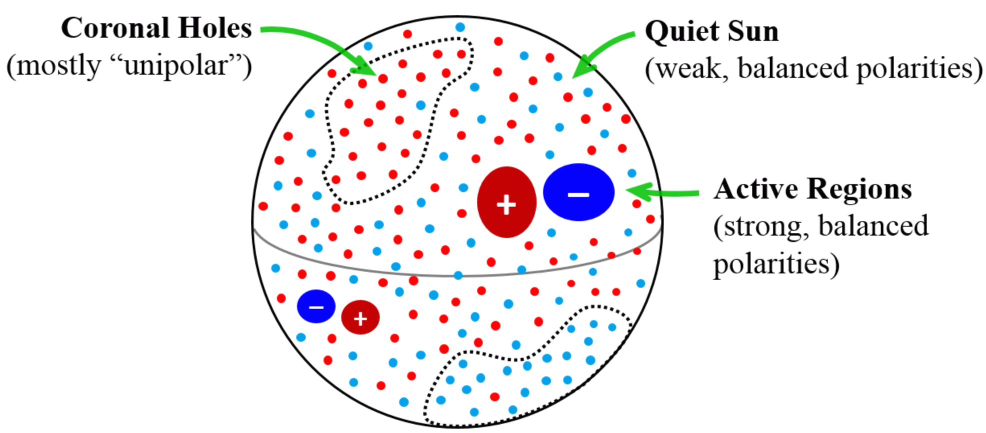
---

(1) MHD Equilibria

What kinds of **time-steady** magnetic fields are generated by astrophysical objects?

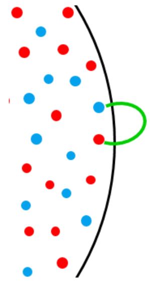
Fields and flows are interdependent (see the induction equation), but for now let's think about $\mathbf{u} = 0$ ("magnetostatic" equilibrium).

I'll use the Sun as an example *source* of magnetic fields, and we can ask how $\mathbf{B}(\mathbf{r})$ evolves above the surface...

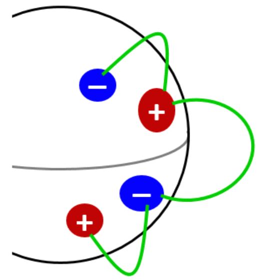


A spherical surface can be a complicated distribution of magnetic field. Even the apparently "field-free" regions in between strong active regions contain a distribution of salt-and-pepper tiny concentrations of **B.**

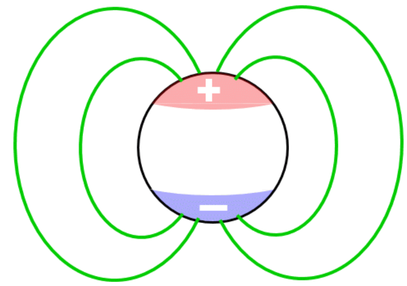
The geometry of the field lines that **connect** these regions depends on how far one has to go to "find" enough opposite polarity to connect with...



tiny bipoles
("ephemeral regions")



coronal loops
(often rooted in ARs)



largest-scale fields
(potential multipoles)

The near-Sun corona is magnetically dominated ($\beta \ll 1$).

Eventually, the solar wind accelerates, and you can see from eclipse images that the dipole-like field gets **stretched** out by the radially outflowing plasma (up there, $\beta > 1$ again).

But in the low corona ($r \lesssim 2\,R_\odot$), the flow speed $\mathbf{u} \approx 0$, so momentum conservation is hydrostatic:

$$-\nabla P_{\rm gas} + \rho\mathbf{g} + \frac{1}{c}\mathbf{J}\times\mathbf{B} = 0$$

For $\beta \ll 1$, the $\nabla P_{\rm gas}$ term is negligibly small compared to magnetic forces.

Also, when working out the numbers for coronal values of $\rho\mathbf{g}$ and $\mathbf{J}\times\mathbf{B}$, it turns out that (near the surface) gravity is often negligible, too:

$$\frac{|\rho\mathbf{g}|}{|\mathbf{J}\times\mathbf{B}|/c} \sim \frac{\rho\, GM_\odot/r^2}{(B^2/4\pi)\,/\,r} \sim \frac{V_{\rm esc}^2}{V_{\rm A}^2} \sim \left(\frac{< 600\ \text{km/s}}{1000\text{–}2000\ \text{km/s}}\right)^2 < 1\ .$$

Thus, in the corona, $\boxed{\mathbf{J}\times\mathbf{B} \approx 0}$ **("force-free fields")**

There are 2 ways to make $\mathbf{J}\times\mathbf{B} = 0$ (with a nonzero $\mathbf{B}$):

$\boxed{\mathbf{J} = 0}$ or $\boxed{\mathbf{J}\ \text{parallel to}\ \mathbf{B}}$

We will see that $\mathbf{J} \parallel \mathbf{B}$ occurs for **twisted** strands of magnetic field ("flux ropes"), which occur in prominences, filaments, and CMEs.

Most of the volume of the corona obeys the first condition, $\mathbf{J} = 0$, which in MHD means that

$$\frac{4\pi \mathbf{J}}{c} = \nabla \times \mathbf{B} = 0 \ .$$

Remember that $\nabla \times \nabla\psi = 0$ for all scalar functions ψ, so this means that we can express this kind of magnetic field as a gradient of a potential function, $\boxed{\mathbf{B} = -\nabla\psi}$.

(The minus sign is arbitrary, but it's the usual convention.)

This is called a **potential field,** (or a zero-current field, or a vacuum field) and it's seen to be sort of a "ground state" (lowest magnetic-energy state) of a magnetic field. Add any nonzero $\mathbf{J}$ to the system – say, by twisting it up – and you add to the total magnetic energy.

. . . .

Proofs:

(a) Hand-wavy: Remember the equation for evolution of magnetic energy:

$$\frac{\partial U_{\mathrm{B}}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} = -\eta |\mathbf{J}|^2 \qquad \text{where} \qquad U_{\mathrm{B}} = \frac{|\mathbf{B}|^2}{8\pi} \ .$$

So, as long as $\mathbf{J} \neq 0$ in a "real plasma" ($\eta \neq 0$), one can still *lose* magnetic energy. It can drain away until we reach a final state where both $\mathbf{J} = 0$ and U_{B} is at a minimum value.

(b) More rigorous: We want to minimize the magnetic energy in a given parcel of volume V

$$W = \int_V dV \ (B^2/8\pi)$$

and we also know that, since $\nabla \cdot \mathbf{B} = 0$ everywhere, Gauss' divergence theorem

$$\text{gives} \quad \int_V dV \ \nabla \cdot \mathbf{B} = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{over the closed surface } S,$$

and we can specify a fixed & known boundary condition for $\mathbf{B}$ at the surface S.

Let's specify the field inside V as $\mathbf{B} = \mathbf{B}_p + \mathbf{b}$, where $\mathbf{B}_p = -\nabla\psi$, and $\mathbf{b}$ is an arbitrary non-potential perturbation.

However, we've already specified the boundary condition, and it can be shown to be consistent with $\mathbf{B}_p$ (this is a classical Neumann-type boundary condition that satisfies Laplace's equation).

Thus, we're NOT free to allow $\mathbf{b}$ to mess with this known boundary condition. Thus, we must specify that $[\mathbf{b} \cdot d\mathbf{S}]_S = 0$ (i.e., the component of $\mathbf{b}$ normal to the surface is always zero).

The magnetic energy inside the parcel is

$$W = \frac{1}{8\pi} \int_V dV \ \left(B_p^2 + 2\mathbf{B}_p \cdot \mathbf{b} + b^2\right)$$

and we'd like to know whether it can ever be *smaller* than the energy of the corresponding potential-field parcel,

$$W_p = \frac{1}{8\pi} \int_V dV \ B_p^2 \quad .$$

Since the b^2 term is additive, clearly we can only have $W < W_p$ if the cross-term is sufficiently strong and *negative.*

However, if we can show that the cross-term is ZERO, we'll have proven that W_p is the minimum!

$$\int_V dV \ (\mathbf{B}_p \cdot \mathbf{b}) = -\int_V dV \ (\nabla\psi \cdot \mathbf{b}) = -\int_V dV \ [\nabla \cdot (\psi\mathbf{b}) - \cancel{\psi(\nabla \cdot \mathbf{b})}]$$

since $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{B}_p = 0$, then $\nabla \cdot \mathbf{b} = 0$ too.

But what remains can be transformed with Gauss' divergence theorem:

$$-\int_V dV \ \nabla \cdot (\psi\mathbf{b}) = -\oint_S (\psi\mathbf{b}) \cdot d\mathbf{S}$$

and since the component of $\mathbf{b}$ normal to S is always zero, this surface integral is zero, too. Q.E.D.

. . . .

What do potential fields look like? Remember that $\nabla \cdot \mathbf{B} = 0$ for all magnetic fields, so this means that $\mathbf{B}_p$ obeys

$$\nabla \cdot \nabla\psi = \nabla^2\psi = 0 \qquad \text{(Laplace's equation)}$$

The solution to Laplace's equation for the spherical domain $r > R_\odot$, when we specify $\mathbf{B}(\theta, \phi)$ at the surface boundary $r = R_\odot$, is well known.

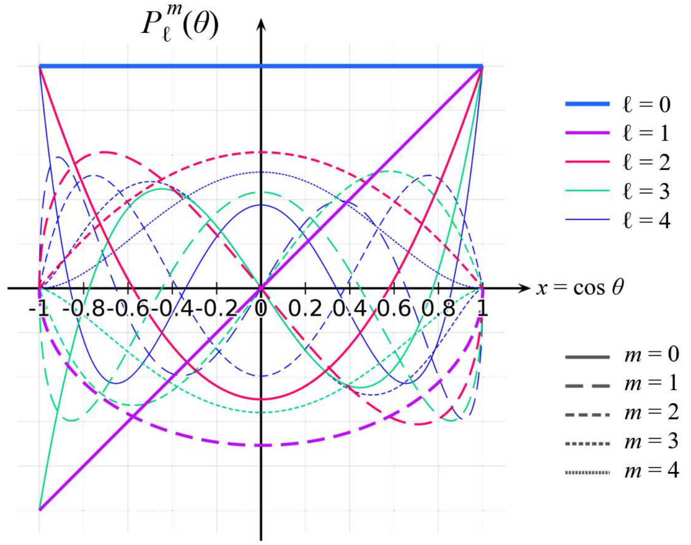
In spherical coordinates, one can solve Laplace's equation using the trick of "separation of variables" to assume ψ is the product of 3 functions of each spatial variable:

$$\psi(r,\theta,\phi) \;=\; R(r)\,\Theta(\theta)\,\Phi(\phi) \;\; .$$

If we substitute this back into Laplace's equation, we can separate it into 3 ordinary differential equations.

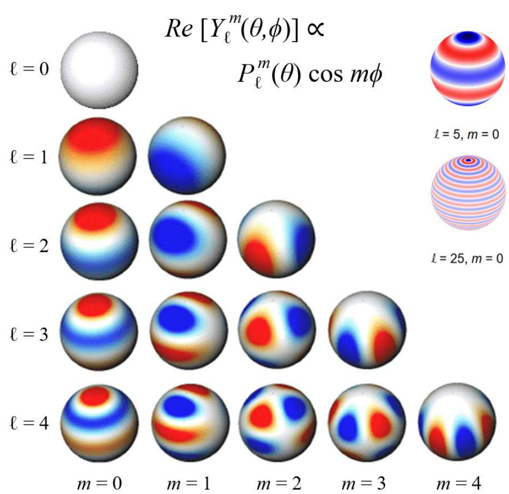
The azimuthal part ends up sinusoidal: $\Phi(\phi) \propto e^{im\phi}$, for $m = 0, \pm 1, \pm 2, \ldots$

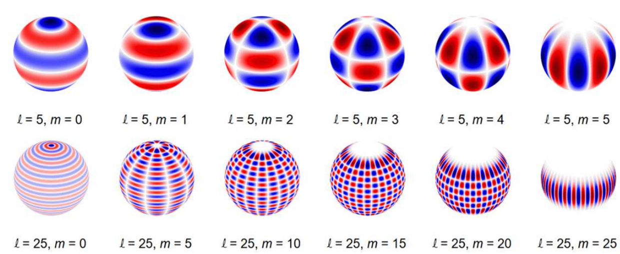
The meridional part $\Theta(\theta)$ is satisfied by **associated Legendre polynomials** $P_\ell^m(\theta)$, for $\ell = 0$ or a positive integer, and $-\ell \le m \le +\ell$.

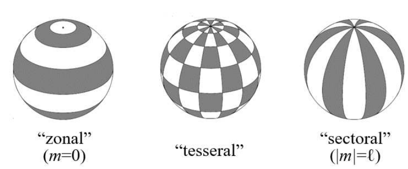


$$P_0^0(\cos\theta) = 1$$
$$P_1^0(\cos\theta) = \cos\theta$$
$$P_1^1(\cos\theta) = -\sin\theta$$
$$P_2^0(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$
$$P_2^1(\cos\theta) = -3\cos\theta\sin\theta$$
$$P_2^2(\cos\theta) = 3\sin^2\theta$$
$$P_3^0(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$
$$P_3^1(\cos\theta) = -\tfrac{3}{2}(5\cos^2\theta - 1)\sin\theta$$
$$P_3^2(\cos\theta) = 15\cos\theta\sin^2\theta$$
$$P_3^3(\cos\theta) = -15\sin^3\theta$$
$$P_4^0(\cos\theta) = \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$$
$$P_4^1(\cos\theta) = -\tfrac{5}{2}(7\cos^3\theta - 3\cos\theta)\sin\theta$$
$$P_4^2(\cos\theta) = \tfrac{15}{2}(7\cos^2\theta - 1)\sin^2\theta$$
$$P_4^3(\cos\theta) = -105\cos\theta\sin^3\theta$$
$$P_4^4(\cos\theta) = 105\sin^4\theta$$

The combined angular solutions $Y_\ell^m(\theta,\phi)$ are the **spherical harmonics,** and they're used in many fields...





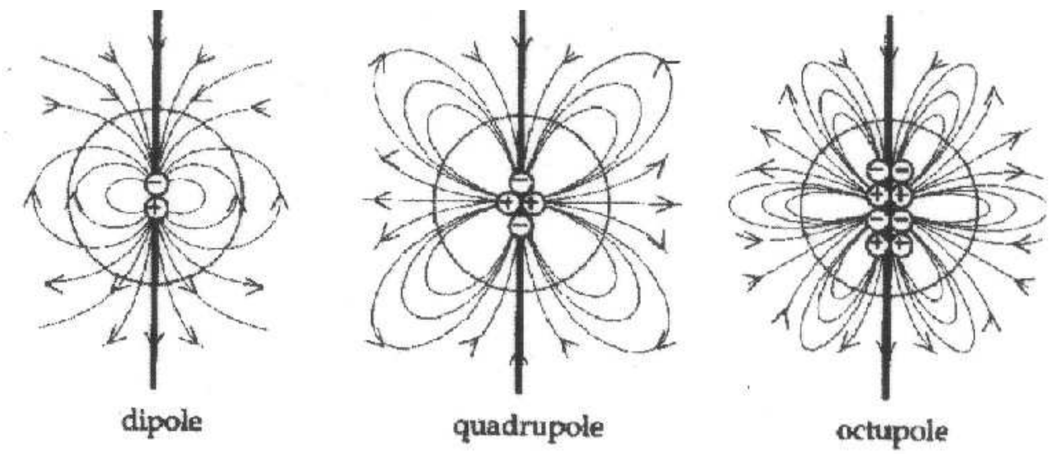


Lastly, the radial dependence of the potential $\psi(r, \theta, \phi)$ is given by power-laws, so the full solution is

$$\psi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left[a_{\ell m} \left(\frac{R_\odot}{r} \right)^{\ell+1} + b_{\ell m} \left(\frac{r}{R_\odot} \right)^{\ell} \right] Y_{\ell m}(\theta, \phi)$$

where the $a_{\ell m}$ coefficients are computed from the lower boundary condition, and most physical systems also have $b_{\ell m} = 0$ in order for $\psi \to 0$ at $r \to \infty$. When we have this solution, we then take $\mathbf{B} = -\nabla\psi$ to get the actual field.

$\ell = 0$	**monopole** field	$\psi \propto r^{-1}$	$\mathbf{B}$ drops off as $1/r^2$
$\ell = 1$	**dipole** field	$\psi \propto r^{-2}$	$\mathbf{B}$ drops off as $1/r^3$
$\ell = 2$	**quadrupole** field	$\psi \propto r^{-3}$	$\mathbf{B}$ drops off as $1/r^4$
$\ell = 3$	**octupole** field	$\psi \propto r^{-4}$	$\mathbf{B}$ drops off as $1/r^5$



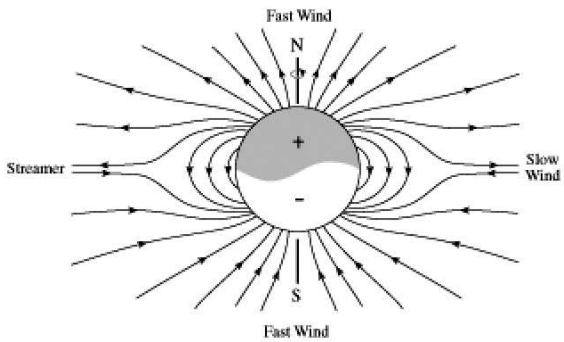
Higher ℓ $\to$ more complex the structure in θ, ϕ
$\to$ faster drop-off with increasing height.

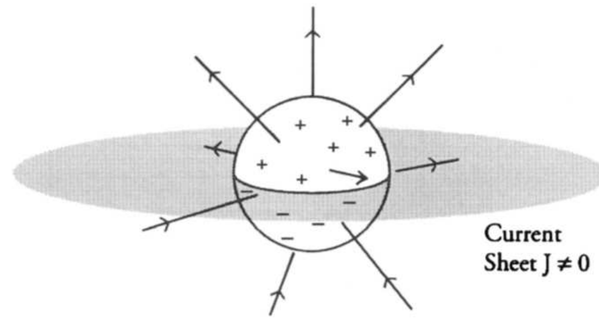
Thus, the star's dipole component "survives" to largest distances. Note that Laplace's equation allows there to be a monopole component (a_{00}), but we know that true magnetic monopoles don't exist.

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{implies that} \qquad \oint \mathbf{B} \cdot d\mathbf{A} = \oint d\Omega \; B_r(\theta, \phi) = 0$$

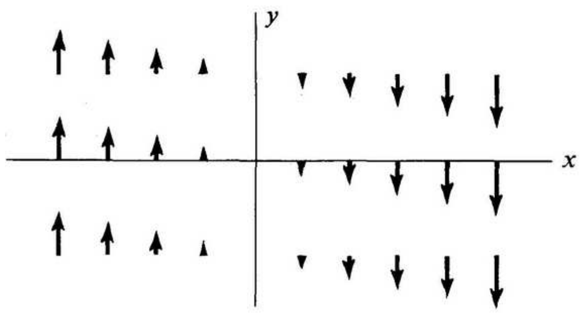
so inward & outward flux over the surface must balance, and $a_{00} = 0$.

However, a **split-monopole** can exist: plus polarity in one hemisphere, minus polarity in the other... and with $|\mathbf{B}| \propto 1/r^2$. Any kind of radial flow can stretch out the surviving dipole into something like this:





The penalty is that at the equator, $\nabla \times \mathbf{B} \neq 0$. Why? $\mathbf{B}$ is highly sheared in this plane. You don't need curved/swirling vectors for a nonzero curl...



For this field, $\nabla \times \mathbf{B} \neq 0$.

Thus, $\mathbf{J} \neq 0$ in this thin

current sheet region; $J_\phi \neq 0$.

In the case of the Sun, we know that at larger radii the *solar wind* accelerates, so all other terms in the momentum equation (including $D\mathbf{u}/Dt$) become important again.

However, if $\mathbf{u} \parallel \mathbf{B}$ (i.e., particle flow follows the field), then the ideal induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \cancel{(\mathbf{u} \times \mathbf{B})} = 0$$

and $\mathbf{B}$ doesn't evolve. If this remains a force-free situation, then everything can remain **time-steady.**

. . . .

Is it always the case that a system evolves to the minimum-energy ($\mathbf{J} = 0$) state? No!

A plasma can get "stuck" in a force-free situation where (by definition) the forces acting on it are zero, but there is a nonzero $\mathbf{J} \parallel \mathbf{B}$.

In that case, we can write

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}$$

where, in general, the scalar α can be a function of position. However, there is an additional constraint on α. If we take the divergence, we get

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\alpha \mathbf{B}) = 0 \qquad \text{(from vector identity)}$$

$$\text{and} \qquad \nabla \cdot (\alpha \mathbf{B}) = \cancel{\alpha \nabla \cdot \mathbf{B}} + \mathbf{B} \cdot \nabla \alpha = 0$$

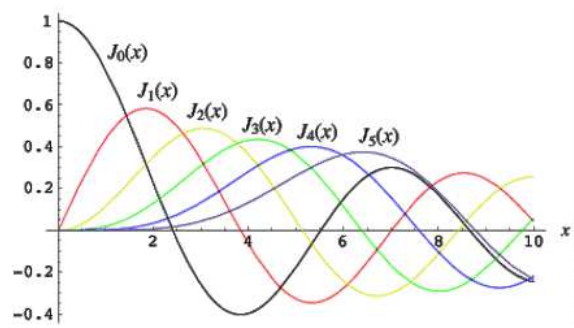
which means that the projected component of $\nabla \alpha$ along the field is zero.
$\boxed{\alpha \text{ is constant along field lines.}}$

The constant α is a **magnetic torsion parameter** that describes how twisted-up the field is. $\alpha = 0$ is a potential field.

Many systems like to *relax* to the so-called LFFF (linear force-free field) that occurs when $\alpha =$ constant, throughout the whole volume.

Lundquist (1950) worked out an analytic solution in cylindrical geometry, using Bessel functions $J_n(x)$,

$$\begin{aligned} B_r(r) &= 0 \\ B_\phi(r) &= \pm B_0 \, J_1(\alpha r) \\ B_z(r) &= B_0 \, J_0(\alpha r) \end{aligned}$$



The "outer edge" of the cylinder is the first zero of $J_0(x)$, which occurs at $x = \alpha R_{\text{edge}} \approx 2.405$.

The Lundquist solution works very well as a fitting formula for twisted "magnetic clouds" encountered by spacecraft in the heliosphere. These are the interplanetary remnants of **coronal mass ejections (CMEs).**

In laboratories, cylindrical plasmas with helical, twisted fields are often created via the **pinch effect:**

$$\left\{ \begin{array}{c} \text{axial} \\ \text{azimuthal} \end{array} \right\} \text{ fields are created by applying } \left\{ \begin{array}{c} \text{azimuthal} \\ \text{axial} \end{array} \right\} \text{ currents .}$$

---

(2) Linear MHD Waves

Why study waves? They help us understand the "simplest" ways that:

- a plasma responds to changes in external forces (e.g., ∇P, gravity, $\mathbf{J} \times \mathbf{B}$);
- information propagates through a system in space & time.

Let's start with the ideal MHD equations (no gravity or resistivity):

$$\text{Mass:} \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\text{Momentum:} \qquad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}$$

$$\text{Induction:} \qquad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\text{No monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

Note there are 2 "restoring forces" on right-hand side of the momentum equation. Thus, there will be 2 qualitatively distinct flavors of waves.

There's also the energy conservation equation, which we'll put aside for now. (Later we'll assume **adiabatic** perturbations.)

So, the standard trick is to limit ourselves to LINEAR oscillations; i.e., small (1st order) perturbations on top of a large-scale (0th order) homogeneous background state.

$$\text{Thus, we assume} \qquad \left\{ \begin{array}{rcccl} \rho(\mathbf{r},t) & = & \rho_0 & + & \rho_1(\mathbf{r},t) \\ P(\mathbf{r},t) & = & P_0 & + & P_1(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) & = & \mathbf{B}_0 & + & \mathbf{B}_1(\mathbf{r},t) \\ \mathbf{u}(\mathbf{r},t) & = & & & \mathbf{u}_1(\mathbf{r},t) \end{array} \right\}$$

where subscript-0 quantitites are constants, and we want to solve for how all 8 (scalar) subscript-1 quantities vary in space & time.

Also, note that $\mathbf{u}_0 = 0$, implying the background is **at rest** (or that we're in the reference frame of a known flow).

We assume $|\rho_1| \ll |\rho_0|$, and so on, so that the equations can be linearized:

- Insert the full (0th + 1st) terms into the equations, and expand.
- Cancel out all 0th order terms, since the 0th order terms satisfy the conservation equations exactly (trivially?). All $\partial/\partial t$ and ∇ are zero.
- Ignore 2nd order (and higher) terms (products of two or more 1st order terms; e.g., $\rho_1 u_1$, $u_1 B_1$) because they're so small in magnitude.
- For the 1st order terms, assume they vary as $\sim \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$.
- Boil the system down to a single linear equation, in which each term ought to contain just one 1st order quantity. Cancelling out that quantity in each term (OK because its amplitude is $\neq 0$) gives the **dispersion relation.**

. . . .

Let's try it. Start with the simplest ones...

$$\text{No monopoles:} \qquad \boxed{\nabla \cdot \mathbf{B}_1 = 0}$$

Mass conservation:

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u}_1) = 0 \qquad (\text{recalling that } \mathbf{u}_0 = 0)$$

$$\boxed{\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0} \qquad (\text{since } \rho_0 = \text{constant})$$

Magnetic induction:

$$\boxed{\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0)} \qquad (\text{again because } \mathbf{u}_0 = 0)$$

For the momentum equation, evaluating $(\nabla \times \mathbf{B}) \times \mathbf{B}$ takes some thought. Only 1 term survives:

$(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = 0$	because $\mathbf{B}_0$ is constant
$(\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 = 0$	also because $\mathbf{B}_0$ is constant
$(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \neq 0$	**the one we keep (see below)**
$(\nabla \times \mathbf{B}_1) \times \mathbf{B}_1 = 0$	negligible because it's 2nd order.

Thus, the momentum equation is: $$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla P_1 + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi}$$

However, we're not done with it yet. Consider adiabatic changes in an ideal gas. You've seen that

$$\frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right) = 0 \qquad \text{or} \qquad P \propto \rho^\gamma$$

for a parcel in adiabatic equilibrium (no heat added or removed).

This is a constraint that must be in place if changes in P and ρ are to maintain constant entropy; i.e.,

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \qquad \text{so} \qquad \frac{dP}{d\rho} = P_0 \frac{\gamma \rho^{\gamma-1}}{\rho_0^\gamma} = \frac{\gamma P_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^\gamma .$$

For small perturbations, we can call $P_1 = dP$, $\rho_1 = d\rho$.

Also, $\left(\frac{\rho}{\rho_0}\right)^\gamma \approx 1$ for small perturbations.

Thus, for 1st order adiabatic oscillations,

$$\frac{P_1}{P_0} = \gamma \frac{\rho_1}{\rho_0} \qquad \text{and we define} \qquad c_{s0} = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

as the 0th-order **adiabatic sound speed.** Thus, $P_1 = c_{s0}^2 \rho_1$, and we get

the linearized momentum eqn: $$\boxed{\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -c_{s0}^2 \nabla \rho_1 + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi}}$$

We've now removed P from the system of equations.

....

The next step to linearization is to assume a **sinusoidal** dependence for the 1st order quantities.

As we saw before, expressions like

$$\rho_1(\mathbf{r}, t) = \widetilde{\rho}_1 \exp\left[i\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right)\right]$$

are good & separable solutions to classical wave equations, which we can show the MHD equations to be equivalent to.

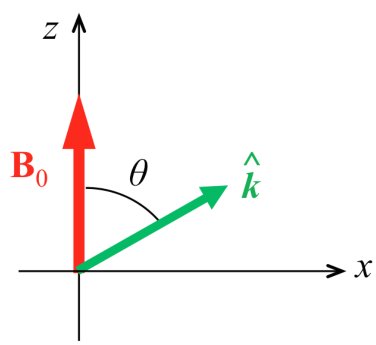
Note that the amplitude $\widetilde{\rho}_1$ can be complex, but we assume it's constant in space & time.

We'll assume the same kind of solutions apply to the components of $\mathbf{u}_1$ and $\mathbf{B}_1$.

$\mathbf{k}$ is the **wavevector,** pointing in the direction of propagation of the oscillatory wave.

$$\text{In general,} \qquad \mathbf{k} \cdot \mathbf{r} \;=\; k_x x \;+\; k_y y \;+\; k_z z$$

but let's simplify our model geometry.



Assume $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_z$

and that $\mathbf{k}$ is in the x–z plane;
i.e., $k_y = 0$, but $k_x, k_z \neq 0$.

We can write $\quad k_x = k \sin\theta \;, \quad k_z = k \cos\theta \;.$

Of course, $\mathbf{u}_1$ and $\mathbf{B}_1$ can have nonzero components in all three directions (including y).

The sinusoidal variations make derivatives easy to write;

$$\text{e.g.,} \qquad \frac{\partial \rho_1}{\partial t} \;=\; -i\omega\rho_1 \qquad\qquad \frac{\partial \rho_1}{\partial x} \;=\; i k_x \rho_1$$

and we can thus replace: $\nabla \cdot \mathbf{F}$ by $i\mathbf{k} \cdot \mathbf{F}$, $\nabla \times \mathbf{F}$ by $i\mathbf{k} \times \mathbf{F}$, ∇f by $i f \mathbf{k}$.

. . . .

We can turn the crank on the equations to see what we get. For example,

$$\nabla \cdot \mathbf{B}_1 = 0 \qquad \Longrightarrow \qquad \mathbf{k} \cdot \mathbf{B}_1 = 0 \;\;.$$

This means that the magnetic perturbation $\mathbf{B}_1$ is always **perpendicular to k.**

The other equations contain a mix of ρ_1, $\mathbf{u}_1$, & $\mathbf{B}_1$ terms... but each term has only *one of each* (truly linear).

Thus, by combining them, we can end up with just one equation with all terms proportional to just one variable, and collect it all together like, say,

$$\Big\{ \text{stuff} + \text{stuff} + \text{stuff} + \text{stuff} \Big\}\, \mathbf{u}_1 \;=\; 0$$

Since $\mathbf{u}_1$ is our assumed "amplitude," we don't want it to be zero, so the other $\{$stuff$\} = 0$ is the **dispersion relation** (i.e., a relationship between ω and $\mathbf{k}$).

Preview: We'll solve

$$\left\{ \begin{array}{c} \text{mass conservation} \\ \text{induction} \end{array} \right\} \text{ equation for } \left\{ \begin{array}{c} \rho_1 \\ \mathbf{B}_1 \end{array} \right\} \text{ as a function of } \mathbf{u}_1 \quad ,$$

then plug both back into the momentum equation.

Mass:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0$$

$$-i\omega\rho_1 + i\rho_0(\mathbf{k} \cdot \mathbf{u}_1) = 0 \qquad \Longrightarrow \qquad \boxed{\rho_1 = \frac{\rho_0}{\omega}(k_x u_{1x} + k_z u_{1z})} \ .$$

Induction:

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \qquad \Longrightarrow \qquad \mathbf{B}_1 = -\frac{\mathbf{k} \times (\mathbf{u}_1 \times \mathbf{B}_0)}{\omega}$$

which verifies that $\mathbf{B}_1$ must be $\perp$ to $\mathbf{k}$. ($\nabla \cdot \mathbf{B}_1 = 0$ was redundant info.)

Working out the cross products,

$$\boxed{B_{1x} = -\frac{k_z u_{1x} B_0}{\omega} \qquad B_{1y} = -\frac{k_z u_{1y} B_0}{\omega} \qquad B_{1z} = +\frac{k_x u_{1x} B_0}{\omega}}$$

If ω, $\mathbf{k}$, and $\mathbf{u}_1$ are all real, then ρ_1 and $\mathbf{B}_1$ are real, too.

Doing the derivatives in the momentum equation,

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -c_{s0}^2 \nabla \rho_1 + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi} \quad \Rightarrow \quad \omega\rho_0\mathbf{u}_1 = \rho_1 c_{s0}^2 \mathbf{k} - \frac{i(\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi}$$

Notice a few things:

- If $\mathbf{B}_0 \to 0$, then $\mathbf{u}_1 \parallel \mathbf{k}$. Acoustic waves are longitudinal. ($\beta \gg 1$)
- If $c_{s0} \to 0$, then $\mathbf{u}_1 \perp \mathbf{B}_0$. Purely "cold" MHD waves oscillate transverse to the background magnetic field. ($\beta \ll 1$)

These illustrate the 2 extremes, in which the "restoring force" is either *all* ∇P_{gas} or *all* $\mathbf{B}$ tension.

I'll spare you the vector cross products, but it's useful to keep going step by step. Multiply each term by ω/ρ_0, and we get three component equations in terms of $\mathbf{u}_1$ only:

$$\left\{ \begin{array}{lcl} \omega^2 u_{1x} &=& c_{s0}^2 k_x (k_x u_{1x} + k_z u_{1z}) \quad + \quad \frac{B_0^2}{4\pi\rho_0} u_{1x}(k_x^2 + k_z^2) \\ \omega^2 u_{1y} &=& \qquad\qquad\qquad\qquad\qquad\qquad \frac{B_0^2}{4\pi\rho_0} u_{1y} k_z^2 \\ \omega^2 u_{1z} &=& c_{s0}^2 k_z (k_x u_{1x} + k_z u_{1z}) \end{array} \right\}$$

The c_s^2 term is gone in the y equation because it depends on $\mathbf{k}$, and $k_y = 0$.

The B_0^2 term is gone in the z equation because of that final $() \times \mathbf{B}_0$ in the Lorentz force.

The x & z equations are coupled to one another; the y equation stands apart. Thus, there are 2 separate, **decoupled** kinds of MHD waves.

Let's just look at the y equation:

The linear u_{1y} terms cancel, and $\qquad \omega^2 = \dfrac{B_0^2}{4\pi\rho_0} k_z^2$

Thus, we define the Alfvén speed, $\boxed{V_\mathrm{A} \equiv \dfrac{B_0}{\sqrt{4\pi\rho_0}}}$ so $\omega = \pm V_\mathrm{A} k_z$.

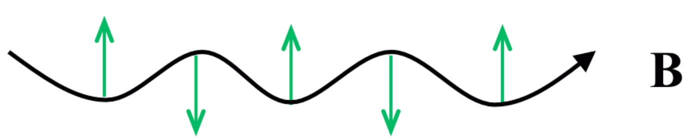
There are 2 solutions, corresponding to waves propagating in 2 directions.

These are **Alfvén waves,** for which:

- They induce fluctuations in only u_{1y} and B_{1y}, since the induction equation gives

$$B_{1y} = -\frac{B_0}{\omega} k_z u_{1y} \qquad \Longrightarrow \qquad \boxed{\left|\frac{B_{1y}}{B_0}\right| = \left|\frac{u_{1y}}{V_\mathrm{A}}\right| \ll 1}$$

  and u_{1y} & B_{1y} are 180° out of phase. Alfvénic perturbations are **transverse** to the background field...



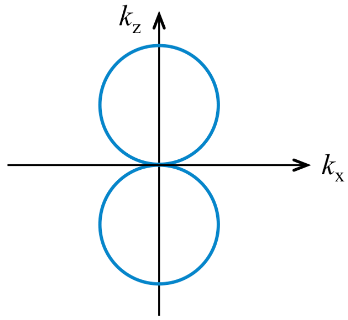
  ...and also transverse to the plane containing $\mathbf{B}_0$ & $\mathbf{k}$.

  Tension is the restoring force, like in a taut, plucked wire.

- They are **incompressible,** since this mode has

$$u_{1x} = u_{1z} = 0 \ , \quad \text{so that} \quad \rho_1 = B_{1x} = B_{1z} = 0 \ , \quad \text{as well.}$$

- Their scalar *phase speed* (i.e., speed at which wave crests & troughs propagate through space) is



$$V_{\rm ph} \equiv \frac{\omega}{|\mathbf{k}|} = \pm V_{\rm A} \cos\theta$$

and note that $V_{\rm ph} = 0$ for $\theta = \pi/2$.

- Their vector *group velocity* (i.e., speed at which they transmit energy through the plasma) is

$$\mathbf{V}_{\rm gr} \equiv \nabla_k \omega = \frac{\partial \omega}{\partial \mathbf{k}} = \hat{\mathbf{e}}_x \frac{\partial \omega}{\partial k_x} + \hat{\mathbf{e}}_z \frac{\partial \omega}{\partial k_z}$$

$$\text{and thus,} \quad \mathbf{V}_{\rm gr} = \pm V_{\rm A} \hat{\mathbf{e}}_z \ .$$

Thus, Alfvén waves always carry energy parallel to $\mathbf{B}_0$, even if the wave itself is propagating obliquely to the field.

. . . .

Now, let's look at the coupled x & z equations. We solve the z equation for

$$u_{1z} = \left(\frac{c_{s0}^2 k_x k_z}{\omega^2 - c_{s0}^2 k_z^2} \right) u_{1x}$$

then plug that back into the x equation.

I won't go through the algebra. Each term is linearly proportional to u_{1x}, so those terms can be cancelled out. The resulting dispersion relation is

$$\omega^4 - \omega^2 k^2 (c_{s0}^2 + V_{\rm A}^2) + c_{s0}^2 V_{\rm A}^2 k^2 k_z^2 = 0 \ .$$

Very different from the Alfvén mode! It's quadratic in ω^2, but note that if we divide by k^4, we get a quadratic in $V_{\rm ph}^2 = \omega^2/k^2$,

$$V_{\rm ph}^4 - V_{\rm ph}^2 (c_{s0}^2 + V_{\rm A}^2) + c_{s0}^2 V_{\rm A}^2 \cos^2\theta = 0$$

and the nice outcome that the ω's and k's completely disappear means these waves are *"dispersionless"* (i.e., phase speeds independent of frequency).

The solution to the quadratic is

$$\boxed{V_{\rm ph}^2 \;=\; \frac{\omega^2}{k^2} \;=\; \frac{1}{2}\left[\left(V_{\rm A}^2 + c_{s0}^2\right) \pm \sqrt{\left(V_{\rm A}^2 + c_{s0}^2\right)^2 \;-\; 4c_{s0}^2 V_{\rm A}^2 \cos^2\theta}\,\right]}$$

and there are **4** possible real solutions for $V_{\rm ph}$:

$\pm$[with upper sign] and $\pm$[with lower sign].

Upper sign gives a larger $|V_{\rm ph}|$, so call it the **fast-mode MHD wave.**
Lower sign gives a smaller $|V_{\rm ph}|$, so call it the **slow-mode MHD wave.**

Restoring forces are a combination of magnetic & ∇P forces, so these are also called **magnetosonic waves.** First worked out by Herlofson (1950, *Nature,* 165, 1020). To understand these modes better, let's look at 4 limiting cases:

(1) Weak fields: $B_0 \to 0$, or $c_{s0} \gg V_{\rm A}$. $(\beta \gg 1)$

The Alfvén wave goes away completely, but the magnetosonic wave has

$$V_{\rm ph}^2 \;=\; \frac{1}{2}\left[c_{s0}^2 \pm \sqrt{c_{s0}^4}\right] \qquad \text{(isotropic in } \mathbf{k}\text{)}$$

$$\text{Thus,} \quad \left\{ \begin{array}{lll} \text{the slow solution:} & V_{\rm ph} = 0 & \\ \text{the fast solution:} & V_{\rm ph} = \pm c_{s0} & \text{(the sound wave!)} \end{array} \right\}$$

and thus we think of magnetosonic waves as "magnetic modifications" of the sound wave.

$$\text{Mass conservation gives} \quad \frac{\rho_1}{\rho_0} = \frac{\mathbf{k}\cdot\mathbf{u}_1}{\omega} = \frac{u_1\,k}{\omega} \quad \Rightarrow \quad \boxed{\left|\frac{\rho_1}{\rho_0}\right| = \left|\frac{u_1}{c_{s0}}\right|}$$

(2) Strong fields: $c_{s0} \to 0$, or $V_{\rm A} \gg c_{s0}$. $(\beta \ll 1)$

$$\text{Similarly,} \quad \left\{ \begin{array}{ll} \text{the slow solution:} & V_{\rm ph} = 0 \\ \text{the fast solution:} & V_{\rm ph} = \pm V_{\rm A} \end{array} \right\}$$

but this is kind of "isotropic" in space. Not like the Alfvén mode, which has $V_{\rm ph} = \pm V_{\rm A}\cos\theta$.

For $\theta = 0$, the strong-field fast mode is degenerate with the Alfvén mode. (Makes sense; for $\theta = 0$ there's nothing to distinguish x from y). It's incompressible:

$$\left|\frac{B_{1x}}{B_0}\right| = \left|\frac{u_{1x}}{V_{\rm A}}\right| \qquad \text{(orthogonal to Alfvén)}$$

(3) Parallel propagation: $\theta = 0$, but arbitrary c_{s0} and V_A .

The term under the square root is

$$\pm\sqrt{(V_A^4 + 2c_{s0}^2 V_A^2 + c_{s0}^4) - 4c_{s0}^2 V_A^2} \;=\; \pm\sqrt{V_A^4 - 2c_{s0}^2 V_A^2 + c_{s0}^4} \;=\; \pm(V_A^2 - c_{s0}^2)$$

$$\text{Thus,} \qquad V_{\text{ph}}^2 \;=\; \frac{1}{2}\left[\left(V_A^2 + c_{s0}^2\right) \pm \left(V_A^2 - c_{s0}^2\right)\right]$$

One sign choice gives $\boxed{V_{\text{ph}} = \pm V_A}$. Other sign choice gives $\boxed{V_{\text{ph}} = \pm c_{s0}}$.

Our naming of "fast" vs. "slow" depends on which one is bigger.

Thus, for waves propagating along the field, there are only acoustic waves and Alfvén waves. No mixing.

(Consistency check: for $\theta = 0$, the system *ought* to behave the same in x as it does in y). It does! ✓)

(4) Perpendicular propagation: $\theta = \pi/2$, but arbitrary c_{s0} and V_A .

I won't go through the math, but it's straightforward to show that

$$\text{Slow:} \quad \boxed{V_{\text{ph}} = 0} \quad , \qquad \text{Fast:} \quad \boxed{V_{\text{ph}} = \pm\sqrt{V_A^2 + c_{s0}^2} \quad , \qquad \equiv \pm V_M}$$

where V_M is sometimes called "the" magnetosonic speed.

For waves propagating $\perp$ to the field, the only restoring forces are ∇P_{gas} and ∇P_{mag}, and they join forces with one another to produce this mixed mode.

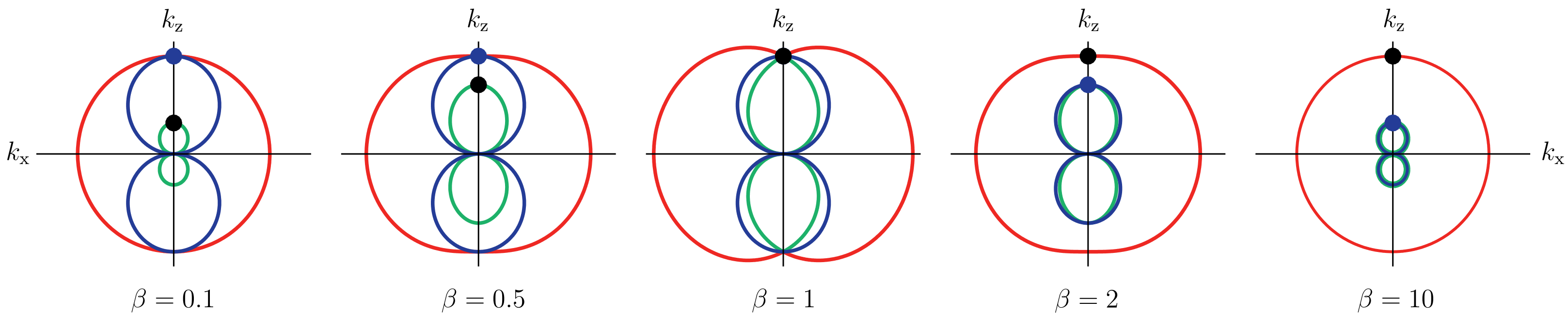
. . . .

Handout: Here we use the plasma beta ratio to organize our knowledge about the background properties:

$$\beta \;=\; \frac{P_{\text{gas}}}{P_{\text{mag}}} \;=\; \frac{8\pi P_0}{B_0^2} \;=\; \frac{2}{\gamma}\left(\frac{c_{s0}}{V_A}\right)^2$$

though note that *space physicists* often define $\beta \equiv (c_{s0}/V_A)^2$ for simplicity (for $\gamma = 5/3$, it's smaller by a factor of 1.2). Be mindful of definitions!

The Alfvén mode is sometimes called the "intermediate mode MHD wave" because it's phase speed is always between fast & slow.



"Friedrichs diagrams" for MHD waves: Phase speed plotted as radial distance, with the angle between **k** and $\mathbf{B}_0$ shown as the angle away from the y–axis. Here, $\beta = (c_s/V_A)^2$. Blue point: Alfvén speed. Black point: sound speed. Curve color-codes shown below.

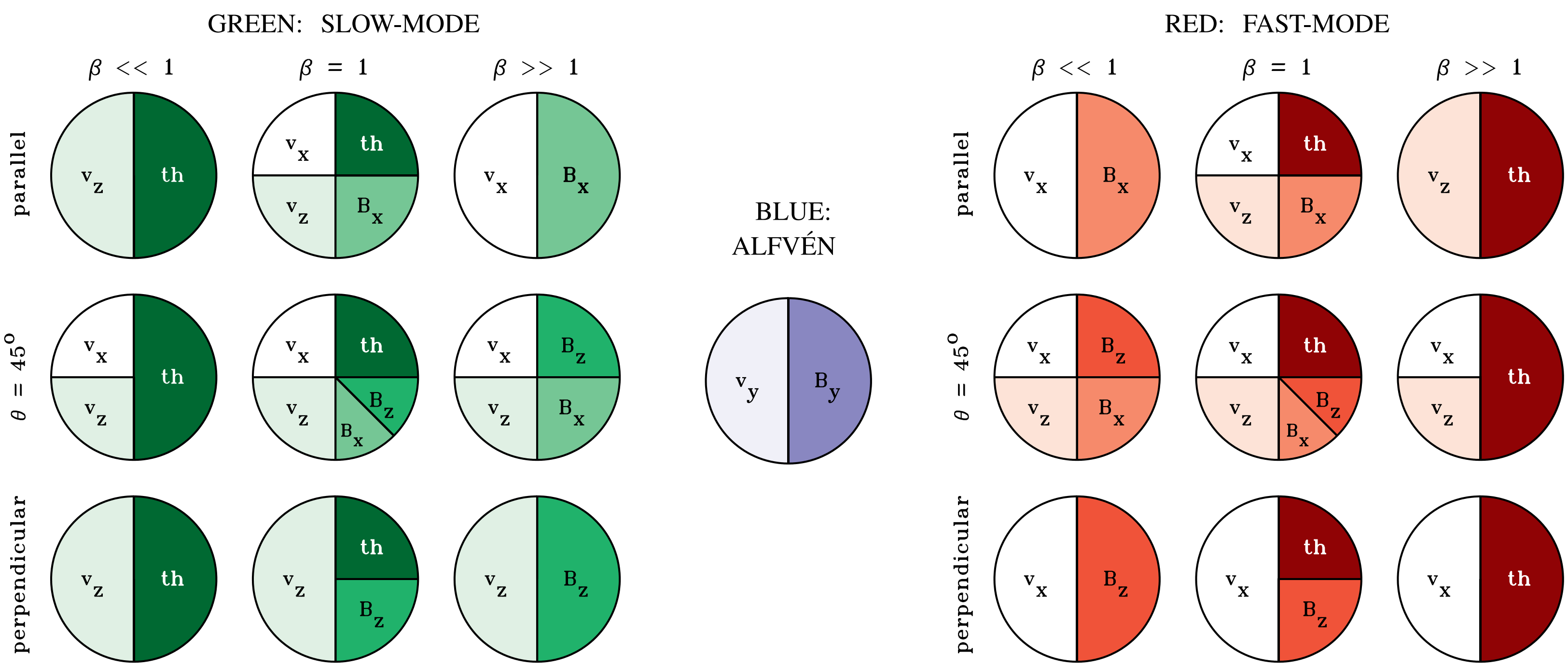


Illustration of how MHD waves **partition their total fluctuation energy into kinetic, magnetic, and thermal energy** in various regimes: wavevectors parallel to $\mathbf{B}_0$ (top row), an isotropic distribution of wavevectors (middle row), wavevectors perpendicular to $\mathbf{B}_0$ (bottom row); columns denote plasma β regimes. Kinetic energy fractions are denoted v_i, magnetic energy fractions are denoted B_i, and the thermal energy fraction is denoted 'th'.

The handout describes how the various types of waves are "partitioned" in terms of their **energy density** fractions.

Waves transport energy. First, I'll show you how the energy components are written, then I've got to convince you they're really meaningful quantities!

Energy densities are **2nd order** products of fluctuations:

$$\text{Kinetic:} \quad U_{\rm K} = \frac{1}{2}\rho_0 |\mathbf{u}_1|^2 \qquad\qquad \text{Magnetic:} \quad U_{\rm B} = \frac{|\mathbf{B}_1|^2}{8\pi}$$

$$\text{Thermal:} \quad U_{\rm th} = \frac{P_1^2}{2\gamma P_0} = \frac{1}{2}\rho_0\, c_{s0}^2 \left(\frac{\rho_1}{\rho_0}\right)^2 \qquad \text{(adiabatic)}$$

What about electric $U_{\rm E} = |\mathbf{E}_1|^2/8\pi$? In ideal MHD, we can ignore it.

$$\text{Recall Faraday's law:} \quad \mathbf{k}\times\mathbf{E}_1 = \frac{\omega\mathbf{B}_1}{c} \qquad \text{so} \qquad |\mathbf{E}_1| \sim \frac{V_{\rm ph}}{c}|\mathbf{B}_1| \ll |\mathbf{B}_1|$$

Of course, for electromagnetic radiation, $V_{\rm ph} = c$, so $|\mathbf{E}_1| = |\mathbf{B}_1|$.

The handout shows how the MHD wave components are partitioned.

For Alfvén waves (and other MHD waves when $\beta \ll 1$),

$$U_{\rm K} = U_{\rm B} \quad \text{and} \quad U_{\rm th} = 0 \qquad \left(\frac{|\mathbf{u}_1|}{V_{\rm A}} = \frac{|\mathbf{B}_1|}{B_0}\right)$$

For acoustic waves (i.e., magnetosonic waves when $\beta \gg 1$),

$$U_{\rm K} = U_{\rm th} \quad \text{and} \quad U_{\rm B} = 0 \qquad \left(\frac{|\mathbf{u}_1|}{c_{s0}} = \frac{\rho_1}{\rho_0}\right)$$

It's also possible to show that $U_{\rm K}$ always takes up exactly half of the "pie" (in ideal MHD).

But what is really going on with these 2nd order fluctuations?
If they're really meaningful, they've got to be <u>something more</u> than just "even tinier" fluctuations.

A true "energy transport" (which makes a difference to the plasma as a whole) must have a **nonzero mean value,** when integrated over times > 1 wave period.

However, we know the 1st order quantities are **sinusoidal;** i.e., they average to zero over 1 or more periods.

Thought experiment: Consider 2 sinusoidal variables:

$$f(\mathbf{r},t) \;=\; f_0 \,+\, f_1(\mathbf{r},t) \;=\; f_0 \,+\, Fe^{i\theta} \qquad (\theta = \mathbf{k}\cdot\mathbf{r} - \omega t)$$

Also, $g(\mathbf{r},t) \;=\; g_0 \,+\, Ge^{i(\theta+\psi)}$ and we say that F and G are real.

We allow g to be offset in phase from f, but note that the MHD waves we derived above all have $\psi = 0$.

Our energy densities contained products like $(f_1 g_1)$. However, *physically relevant* energies must be real.

$$\text{Thus, let us only consider} \qquad U \;=\; \Re(f_1)\,\Re(g_1)$$

U oscillates, but what is the average value taken over 1 period?

$$\langle U \rangle \;=\; \frac{\int d\theta\; U}{\int d\theta} \;=\; \frac{1}{2\pi}\int_0^{2\pi} d\theta\; \Re(f_1)\,\Re(g_1)$$

$$=\; \frac{FG}{2\pi}\int_0^{2\pi} d\theta\;\; \cos\theta\;\; \underbrace{\cos(\theta+\psi)}_{\cos\theta\cos\psi - \sin\theta\sin\psi}$$

We know the integral of $\sin\theta\cos\theta$ is 0, and integral of $\cos^2\theta$ is π. Thus,

$$\langle U \rangle \;=\; \frac{1}{2} FG\, \cos\psi$$

which is just $FG/2$ for MHD waves with $\psi = 0$.

(Other kinds of waves, like internal gravity waves in stellar atmospheres, can have $\psi = 90^\circ$, so $\langle U \rangle = 0$.)

Where were the 2nd order "cross terms?" (i.e., $\rho_1 u_1$)
They show up embedded inside the full **energy conservation equation** for linear waves:

$$\frac{\partial U}{\partial t} \,+\, \nabla\cdot\mathbf{F} \;=\; 0$$

where (in ideal MHD) it ends up that the energy flux $\mathbf{F} = U\mathbf{V}_{\rm gr}$, but its components contain all those cross terms.

Lastly, I should remind you that it's really kind of artificial to think about fluctuations in terms of:

0th order	background state
1st order	oscillations
2nd order	wave energy

In reality, it's **one** system. Some time-averaged 2nd order terms feed back into the 0th order conservation equations:

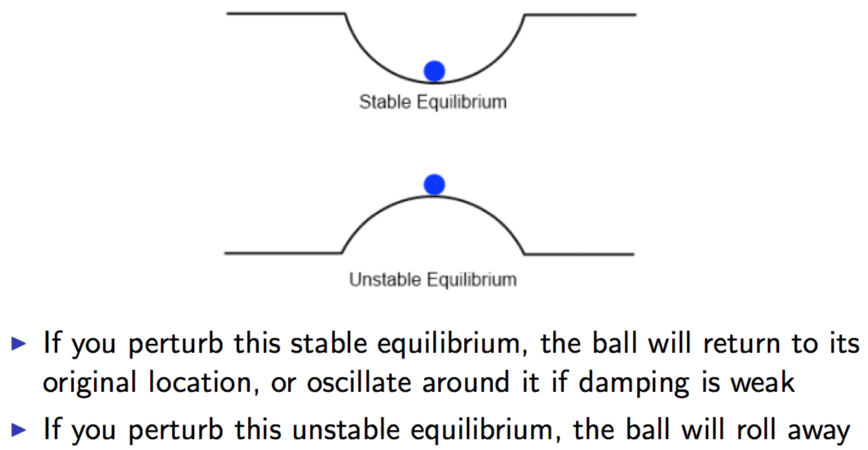
- Wave damping $\longrightarrow$ heats the gas (**energy equation**).
- Even "passive" wave propagation (through an inhomogeneous background state) can produce 2nd order terms in the **momentum equation:** "ponderomotive wave-pressure forces."

Those are topics for other courses....

(3) MHD instabilities

Plasmas are host to many possible instabilities. They matter *a lot* for fusion plasmas!

When the 0th order "background" state is no longer spatially homogeneous, it's difficult to tell whether it is a stable or an unstable equilibrium:



The idea is to perturb the equilibrium with 1st order oscillations ($e^{-i\omega t}$) then check to see if ω is real (stable), imaginary & negative (also stable), or imaginary & positive (**unstable:** exponential growth).

Of course the linear growth phase is just the beginning. Most instabilities change character when they enter the **nonlinear** phase (often saturating), but by that time the "damage has been done."

We will limit ourselves to studying instabilities in **ideal MHD.** According to Andrei Simakov (LANL):

- *"if Ideal MHD predicts instability you are dead;*
- *if Ideal MHD predicts stability some other unpleasant instabilities can still exist but you might have a chance."*

(This doesn't include **kinetic** "micro-instabilities," which we'll talk about briefly, later.)

. . . .

We won't delve too far into the mathematical details, but I'll illustrate one popular linearization strategy.

As with MHD waves, separate the plasma properites into 0th and 1st order parts, but now allowing background to be spatially varying:

$$\begin{array}{rcccl} \rho(\mathbf{r},t) & = & \rho_0(\mathbf{r}) & + & \rho_1(\mathbf{r},t) \\ \mathbf{u}(\mathbf{r},t) & = & & & \mathbf{u}_1(\mathbf{r},t) \\ P(\mathbf{r},t) & = & P_0(\mathbf{r}) & + & P_1(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) & = & \mathbf{B}_0(\mathbf{r}) & + & \mathbf{B}_1(\mathbf{r},t) \end{array}$$

Also, define the fluid **displacement** vector $\boldsymbol{\xi}$ (i.e., the oscillating position vector pointing to the parcel):

$$\mathbf{u}_1 \equiv \frac{\partial \boldsymbol{\xi}}{\partial t} \quad , \qquad \boldsymbol{\xi}(\mathbf{r},t) = \int_0^t dt' \; \mathbf{u}_1(\mathbf{r},t') \ .$$

Linearize the conservation equations, but then **integrate them in time** (assuming all perturbations = 0 at $t = 0$), to get expressions for:

$$\rho_1(\mathbf{r},t) = -\nabla \cdot (\rho_0 \boldsymbol{\xi})$$

$$P_1(\mathbf{r},t) = -\boldsymbol{\xi} \cdot \nabla P_0 - \gamma P_0 \nabla \cdot \boldsymbol{\xi}$$

$$\mathbf{B}_1(\mathbf{r},t) = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)$$

where, on the RHS, there are only 0th order quantities and $\boldsymbol{\xi}$.

Using the above, the momentum equation turns into a true *equation of motion* for a small linear parcel,

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}\left[\boldsymbol{\xi}(\mathbf{r},t)\right]$$

where the ideal MHD force operator is

$$\mathbf{F}[\boldsymbol{\xi}] = -\nabla P_1 + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi} + \frac{(\nabla \times \mathbf{B}_0) \times \mathbf{B}_1}{4\pi}$$

which is a function of only $\boldsymbol{\xi}$, P_0, and $\mathbf{B}_0$ and their spatial derivatives.

(Note that this $\mathbf{F}$ has units of force per unit volume.)

We can glean some basic "intuition" about the physics...

- If $\boldsymbol{\xi} \cdot \mathbf{F} < 0$:
  - The displacement and force are in opposite directions
  - The force opposes displacements
  - The system will typically oscillate around the equilibrium
  - This corresponds to stability (for this displacement, anyway)
- If $\boldsymbol{\xi} \cdot \mathbf{F} > 0$:
  - The displacement and force are in the same direction
  - The force encourages displacements
  - We would expect the perturbation to grow
  - This corresponds to instability
- If $\boldsymbol{\xi} \cdot \mathbf{F} = 0$, then this perturbation is neutrally stable

Also, people often assume $\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\zeta}(\mathbf{r}) e^{-i\omega t}$, so the equation of motion becomes an eigenvalue equation for ω^2.

They often also prefer to work with the potential energy (due to work done by the displacement), which can be written for a volume V as

$$\delta W \ = -\frac{1}{2} \int_V dV \ \boldsymbol{\xi} \cdot \mathbf{F}[\boldsymbol{\xi}] \qquad \text{(and if } \delta W < 0\text{, it's unstable!)}$$

We will mainly examine several situations with different forces ($\mathbf{F}$) on the right-hand side, and see what kinds of instability occur.

. . . .

Brief overview of 4 types of instabilities:

A. **Buoyancy instabilities** act when the background state is gravitationally stratified.

B. **Shear instabilities** act when "nearby" flows have different speeds; what happens at the boundary between them?

C. **Pinch instabilities** act when guiding fields vary to disrupt confined "flux tubes" (important in lab plasmas!)

D. **Resistive instabilities:** act only when $\eta \neq 0$; possibly important in magnetic reconnection regions.

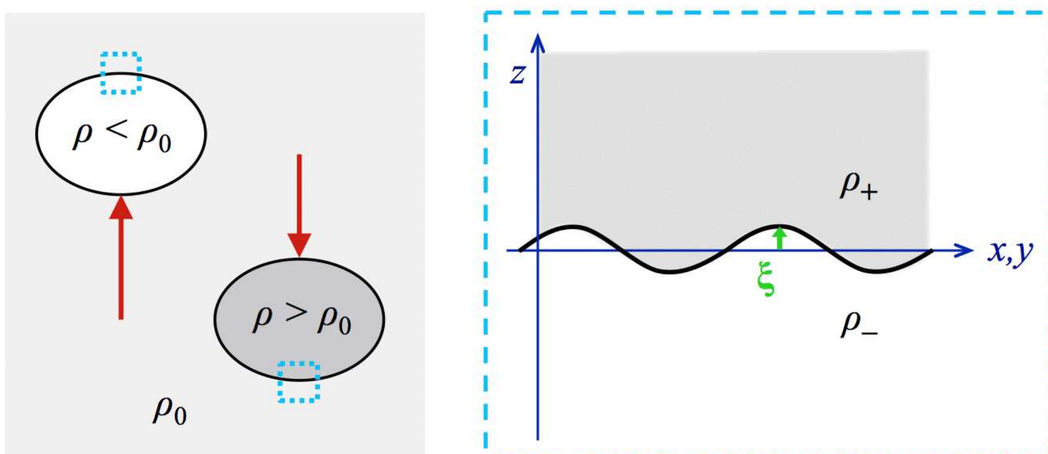
(A) Buoyancy Instabilities

I'm sure you've seen the Schwarzschild convective instability before.

In a hydrostatic stellar interior ($\partial\rho_0/\partial r < 0$), a bubble of gas may become displaced up or down by Δr from its initial height r. If it stays in **pressure equilibrium** with its surroundings, and its evolution is **adiabatic,** then its new density $\rho_{\rm B}$ may be either $>$ or $<$ than the surrounding density $\rho_{\rm S}$ at its new height.

$$
\begin{aligned}
\rho_{\rm B}(r+\Delta r) &> \rho_{\rm S}(r+\Delta r) \ , \qquad \text{the bubble will drop} \quad \text{(stable)} \ , \\
\rho_{\rm B}(r+\Delta r) &< \rho_{\rm S}(r+\Delta r) \ , \qquad \text{the bubble will keep rising} \quad \text{(unstable)} \ ,
\end{aligned}
$$

See my stellar astrophysics (ASTR-5700) lecture notes for more!



However, let's *zoom in* on places at the bubble boundary where there is high-density gas "on top of" low-density gas; i.e., consider a local environment in which $\boxed{\mathbf{g}\cdot\nabla\rho_0 < 0.}$ Other examples:

- Water sitting on top of oil in a beaker
- An explosion, in which hot (low-ρ) gas is at the "center" and is plowing into cool (dense) gas "higher up." (e.g., supernova... or atomic bomb!)
- A strong-**B** piece of plasma embedded in a weak-**B** atmosphere, which is in total pressure equilibrium.

With no magnetic field, this is the classical **Rayleigh–Taylor instability.**

If a heavy parcel drops down, while a lighter parcel rises up buoyantly to replace it, the gravitational potential energy of the system *decreases.*

Thus, the system will always "want" to evolve this way, and thus this heavy-on-top-of-light system ($\rho_+ > \rho_-$) is always unstable.

You'll find that, for $\mathbf{B} = 0$, the resulting linear frequency of the perturbed system is

$$\omega^2 = -\left(\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}\right) g\, k_\perp$$

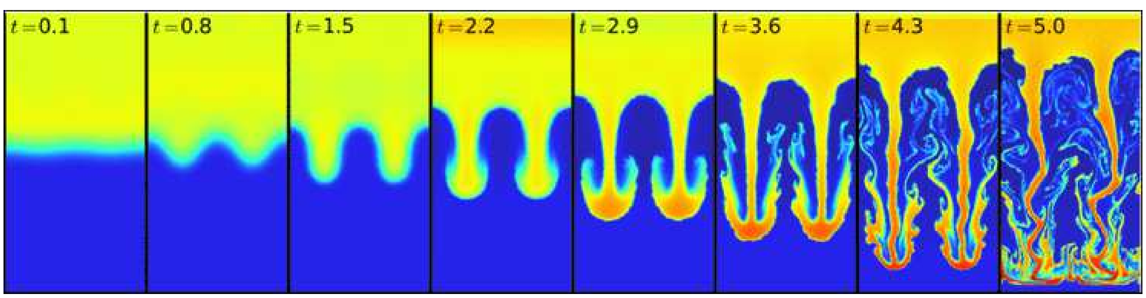
where $g > 0$ and $k_\perp$ is the imposed horizontal wavenumber of the vertical displacement vector (at the interface),

$$\xi_z \propto \exp(-i\omega t + k_x x + k_y y) \qquad k_\perp = \sqrt{k_x^2 + k_y^2}\ .$$

Thus, for $\rho_+ > \rho_-$, ω is imaginary. Sometimes the *growth rate* γ is defined as $\omega = i\gamma$, so that $\gamma > 0$ corresponds to instability.

On the ocean surface, $\rho_- \gg \rho_+$, so the interface can host stable **surface gravity waves** (f-modes) with $\omega \approx \sqrt{gk_\perp}$.

The nonlinear phase of the growth departs from sinusoidal ξ,



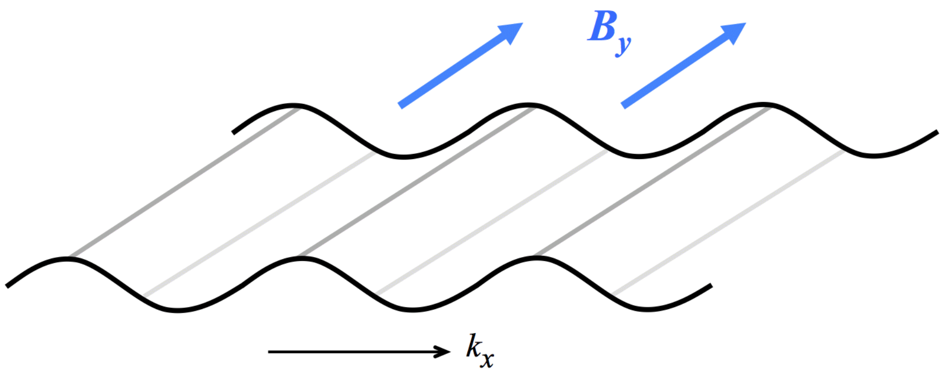
Of interest in this course is what happens when a magnetic field is added. In general, this is the "magnetic R–T instability."

For simplicity, consider a horizontal $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_y$ in the LOWER region, and no field in the upper region.

Because the y direction is now different from the x direction, there are 2 qualitatively different types of magnetic buoyancy instability:

(A1) Interchange Mode for $\mathbf{k} \perp \mathbf{B}_0$, i.e., k_x only

Also called the "fluting" instability, or the Kruskal–Schwarzschild (1954) instability... but some assert Tserkovnikov (1960) discovered it earlier?!



Field lines don't bend, but they are shuffling around in space. Magnetic **pressure** can change, but tension doesn't.

Above & below, there's stratification in $P_{\text{tot}} = P_{\text{gas}} + P_{\text{mag}}$ in the same way that the R–T instability had stratification in just P_{gas}.

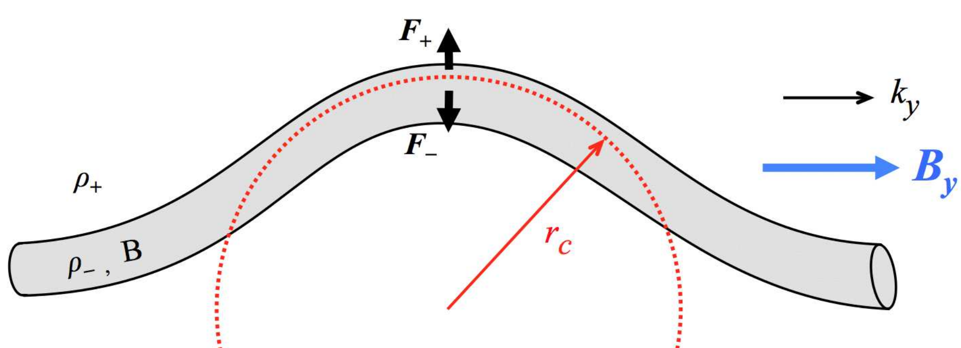
$\Longrightarrow$ The growth rate is comparable to R–T.

(A2) Undular Mode for $\mathbf{k} \parallel \mathbf{B}_0$, i.e., k_y only

Astro & space physics types usually call this the Parker (1966) instability... even though Newcomb (1961) found it first?!

In this case, $\boldsymbol{\xi}$ will induce field-line curvature; i.e., changes the magnetic **tension** force. Stability is possible if an upward $\boldsymbol{\xi}$ corresponds to a net downward total $\mathbf{F}$.

Think about the magnetic interface as being part of a "flux tube:"



$$F_+ \approx g\,\Delta\rho \approx g\,(\rho_+ - \rho_-) \qquad\qquad F_- \approx \frac{|\mathbf{B}\cdot\nabla\mathbf{B}|}{4\pi} \sim \frac{B^2}{4\pi r_{\rm curv}}$$

If ρ_- is low, the tube will be buoyant. Assume the 2 regions are in total pressure equilibrium & thermal equilibrium:

$$P_+ = P_- + \frac{B^2}{8\pi} \qquad\qquad T_+ = T_-$$

Thus, $$\frac{\rho_+ k_{\rm B}T}{\mu m_{\rm H}} = \frac{\rho_- k_{\rm B}T}{\mu m_{\rm H}} + \frac{B^2}{8\pi} \qquad \Longrightarrow \qquad \Delta\rho = \rho_+ - \rho_- = \frac{B^2}{8\pi}\left(\frac{\rho_+}{P_+}\right)$$

And the tube is held down (i.e., is stable to perturbations) if

$$F_- > F_+ \qquad \text{i.e., if} \qquad \frac{B^2}{4\pi r_{\rm curv}} > \frac{B^2}{8\pi}\left(\frac{\rho_+ g}{P_+}\right)$$

Recall the isothermal scale height $$H = \frac{k_{\rm B}T}{\mu m_{\rm H} g} = \frac{P_+}{\rho_+ g}$$

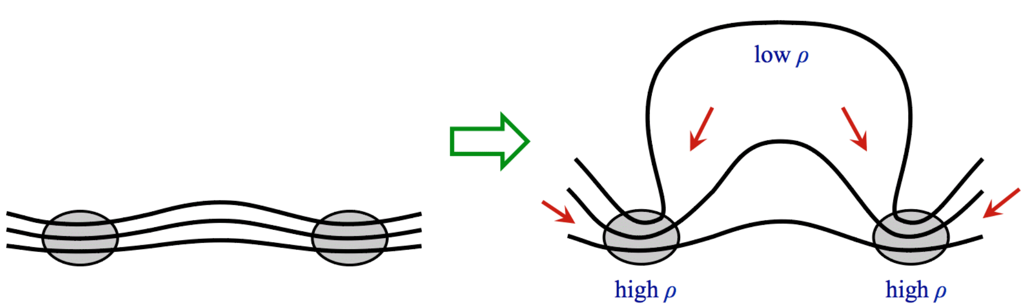
so the system is stable if $\boxed{r_{\rm curv} < 2H}$.

"Tighter curves" = more tension; *long wavelength* undulations most unstable.

(This was all very qualitative. We ignored the stratification that must exist between the equilibrium height and the "plucked" height! See worked-example handout for a more exact approach.)

Note: There is also a related **ballooning instability** that has several different definitions (depending on the author/book).

In my understanding, it involves what happens to 2 fixed "sources" of strong $\mathbf{B}$ connected by horizontal fields in a *stratified* medium:



If there's a buoyancy instability, 3 additional things will happen:

- Lower regions have smaller r_{curv} than upper regions, so the field lines will preferentially **"balloon out"** in upper regions.
- This may occur too rapidly to maintain P_{tot} equilibrium, so the upper (low-$\mathbf{B}$) regions may get **evacuated** (low-ρ).
- Thus, gas may "drain down" to the sources and induce strong **compressive fluctuations.**

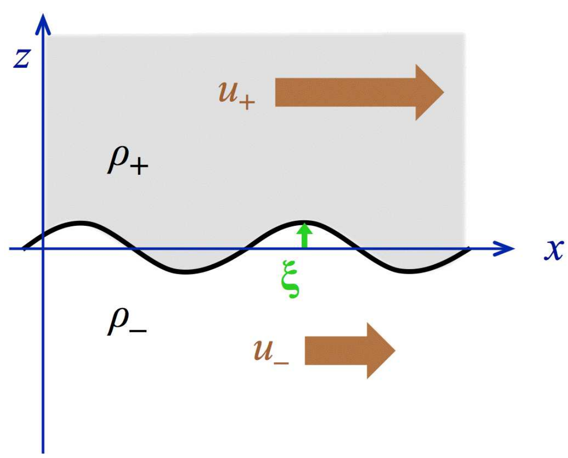
This may be an important process for GMC formation & accretion in the ISM.

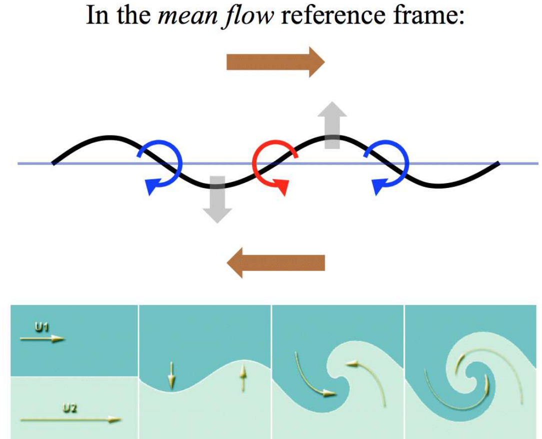
. . . .

(B) Shear Instabilities

What happens when there is a 0th-order velocity field $\mathbf{u}_0$ that is not uniform in space? Two examples:

(B1) Kelvin–Helmholtz Instability occurs for parallel flows with different speeds:





With no magnetic field, a sinusoidal perturbation at the interface creates a tiny amount of **vorticity**. Flows induced in the surrounding regions push "up" on the crests, and "down" on the troughs, thus amplifying the instability.

One can show that the instability occurs when

$$k_x > \frac{(\rho_-^2 - \rho_+^2)g}{\rho_+\rho_-|\Delta u|^2} \qquad \text{where} \qquad \Delta u = u_+ - u_-$$

(i.e., using x as the direction parallel to the $\mathbf{u}_0$ vectors).

- When gravity is unimportant ($g \to 0$), any amount of shear is unstable.
- When stratification is R–T stable ($\rho_- > \rho_+$), gravity stabilizes low-k (long wavelength) modes, and it's high-k (short wavelengths) that's most unstable.
- When Δu grows, the threshold $k_\perp$ drops, so more shear means it's *easier* to excite more modes.

What happens when we add a magnetic field?

If $\mathbf{B}_0$ points along y and $\mathbf{u}_0$ points along x, it's similar to the interchange mode: no change in stability. (For *this* setup with an interface!)

If $\mathbf{B}_0$ is parallel to $\mathbf{u}_0$, the instability is suppressed by strong magnetic tension, similar to the Parker instability. It's unstable only when

$$|\Delta u|^2 > \frac{(B_+^2 + B_-^2)(\rho_+ + \rho_-)}{4\pi\, \rho_+ \rho_-}$$

Having $B \neq 0$ on one side stabilizes the system, but having $B \neq 0$ on *both sides* stabilizes it even more.

(B2) Magnetorotational Instability (MRI) (Balbus & Hawley 1991) is triggered when $\mathbf{B}$ starts out $\perp$ to a plane containing rotational shear; e.g., a Keplerian disk.

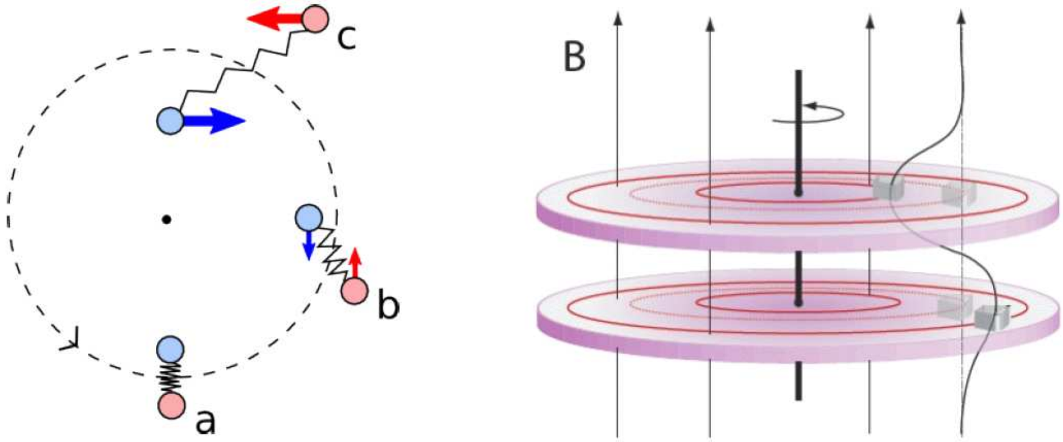
MRI is the dominant idea for what makes accretion possible in accretion disks. (By the way, Wikipedia's page on MRI is surprisingly good!)

A Keplerian disk contains parcels of gas that rotate around a central object of mass M_c with

$$\frac{v_\phi^2}{r} = \frac{GM_c}{r^2} \qquad \Longrightarrow \qquad \Omega(r) = \frac{v_\phi}{r} = \sqrt{\frac{GM_c}{r^3}} \qquad \Longrightarrow \qquad J \sim mr^2\Omega(r) \propto r^{+1/2}$$

Each "ring" is **sheared** with respect to surrounding rings. Even though Ω decreases as $r \uparrow$, angular momentum increases outward.

Magnetic tension provides "springiness" between neighboring parcels, that connects them and generates friction. Consider the following chain of events:



- Initial perturbation: inner parcel m_i rotates faster than an outer parcel m_o.
- If **B** had its way, the disk would be rotating rigidly! Magnetic tension tries to pull back on m_i, and tries to drag m_o forward.
- Thus, m_i experiences a retarding torque, loses angular momentum, and must fall *inward* to be where it "should." Similarly m_o moves outward.
- Greater separation means *more tension force...* which induces larger torques... and the perturbation grows in an **unstable** way.

Long-wavelength modes are most unstable, because tension is weak $\longrightarrow$ shear "wins."

As long as the disk has $\partial\Omega/\partial r < 0$, and there's enough *ionization* for the magnetic field to exert a force on the gas... shear generates transverse "plucks" in the high-tension field lines: i.e., Alfvén waves: $V_{\rm A} = \omega/k_z$.

Small wave-like perturbations **grow,** so J-transport becomes **global** ($\omega \sim \Omega$).

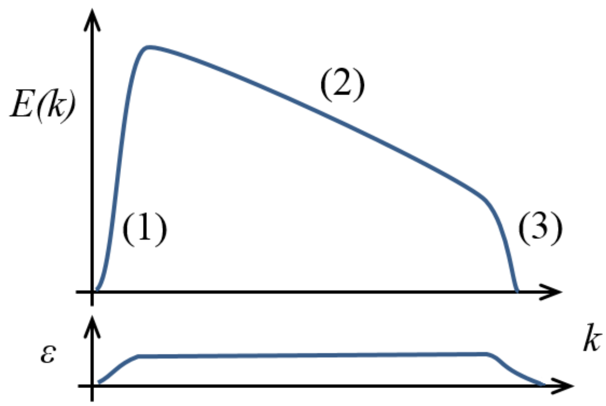
$$\text{MRI maximum growth rate occurs for} \quad k_{z,\max} \approx \frac{\Omega}{V_{\rm A}}$$

where $V_{\rm A} = B/\sqrt{4\pi\rho}$ is the familiar Alfvén speed.

$$\text{Perturbations grow like } e^{\gamma t} \quad \text{with } \gamma \approx \frac{3}{4}\Omega \ .$$

Note: wave period ω is $\approx$ growth rate γ.

If the perturbations driven by MRI "stir up" the gas in the disk, they may excite Kolmogorov-like turbulence...



(1) Energy injected at input rate $\gamma = 3\Omega/4$.
(2) Energy cascades down the pipe at rate $\tau_{\rm nl}^{-1} = v_0/\ell_0 = k_0 v_0$.

Thus, if $k_0 = k_{z,\max}$, and if the rates are **equal** (for steady-state... what goes in must come out!), then we can solve for the driving-scale eddy velocity

$$v_0 \approx \frac{\gamma}{k_0} \approx \frac{3}{4} V_{\rm A}$$

and the "turbulent viscosity" is

$$\nu_{\rm turb} \approx v_0 \ell_0 \approx \frac{v_0}{k_0} \approx \rightsquigarrow \rightsquigarrow \approx \underbrace{\left(\frac{3V_{\rm A}^2}{4c_s^2}\right)}_{\alpha} \frac{c_s^2}{\Omega}$$

and $\alpha = 3/(4\beta)$, where $\beta = (c_s/V_{\rm A})^2 \sim 10$ gives $\alpha \sim 0.075$.

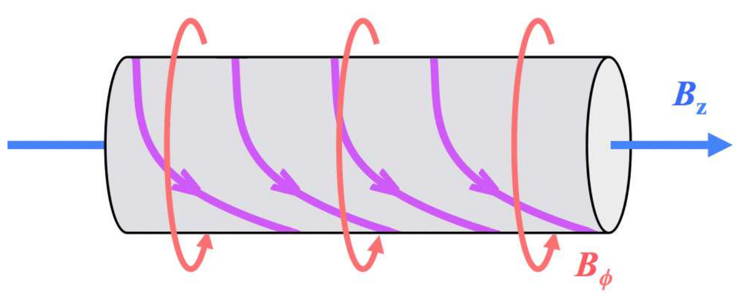
Bai & Stone (2011, ApJ, 736, 144) gave numerical evidence for $\alpha \approx 0.5/\beta$, too.

Once we know ν, there are theories (e.g., Shakura & Sunayev 1973) that let one turn the crank to derive the **accretion rate** $\dot{M}_{\rm acc}$.

. . . .

(C) Pinch Instabilities

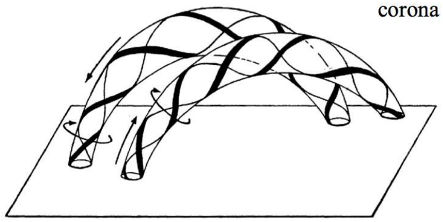
Consider a cylinder of plasma with both axial (B_z) and azimuthal (B_ϕ) fields:

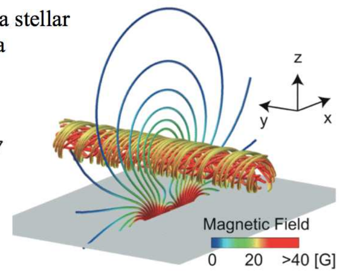


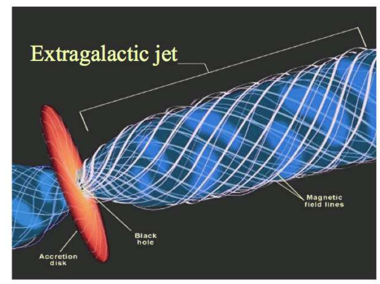
In laboratory devices, the tension force from B_ϕ points inward to "pinch" (i.e., confine) plasma with high $P_{\rm gas}$ along the central axis. The $\nabla P_{\rm gas}$ force points outward to balance.

Lab plasma B_ϕ is generated by J_z, so the subsequent evolutions are often called **current-driven instabilities (CDI).**

In astrophysics, helical-field "flux ropes" occur in many places:

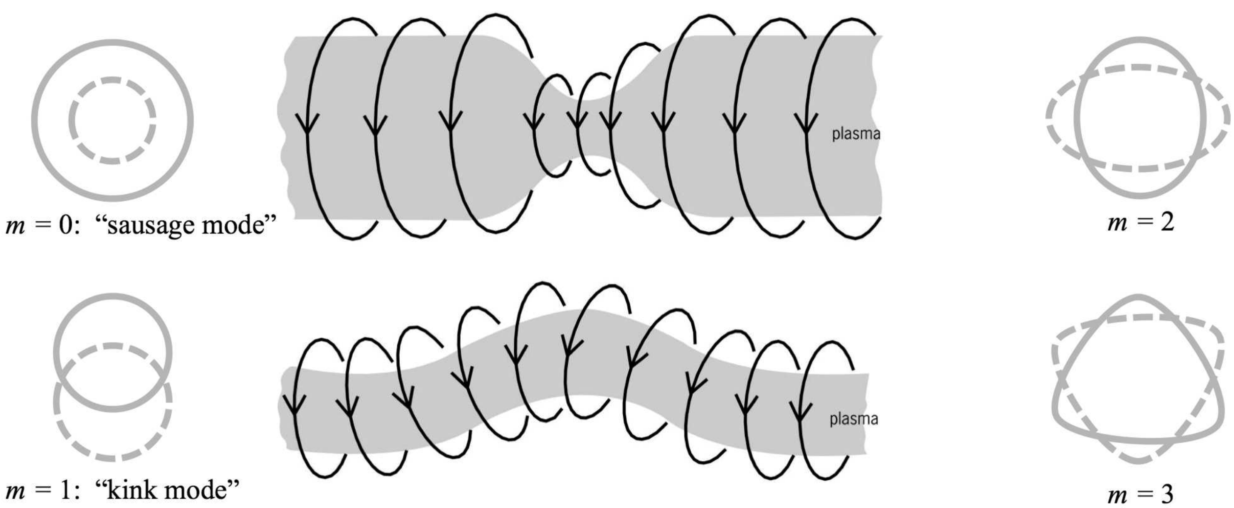






Tayler (1957) first considered small perturbations in the cylindrical surface, with ripples in the k_ϕ direction.

Because the ϕ direction is bounded, there are resonances: $e^{im\phi}$ with integer m:



Sausage mode ($m = 0$): Consider a cylinder dominated by an externally imposed B_ϕ, with oscillations in radius.

If time variations are *slow,* volume is conserved, and $P_{\rm gas}$ remains constant.

However, in the constricted regions, the overall $\mathbf{J} \times \mathbf{B}$ force (due to B_ϕ) increases, and

$$\left| \frac{1}{c} \mathbf{J} \times \mathbf{B} \right| \text{ (inward) } > |\nabla P_{\text{gas}}| \text{ (outward)} \quad \text{so constrictions keep constricting!}$$

However, if there's enough B_z threading the interior of the tube, there can be extra *outward* ∇P_{mag}, and the above the sausage-mode instability is stabilized for

$$B_z > B_\phi / \sqrt{2} \ .$$

Kink mode ($m = 1$): If the above cylinder undergoes lateral displacements, field lines are compressed together on the concave side, and the external $\mathbf{J} \times \mathbf{B}$ force (due to B_ϕ) increases.

Like above, constrictions keep constricting, and it's a growing (kink-mode) instability.

Like above, more B_z can inhibit the instability, this time because of the *tension* due to the axial field.

. . . .

(D) Resistive Instabilities

If there's time, we'll cover them when we discuss magnetic reconnection.