

## FLUID CONSERVATION EQUATIONS $\implies$ MHD

Often, we don't care about all the microscopic details of  $f_s(\mathbf{p})$ .

We want to squint our eyes and examine how the macroscopic **moments** change with position and time:

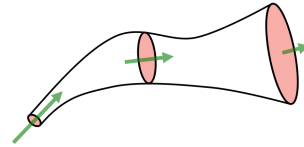
$$n_s, \quad \mathbf{u}_s, \quad T_s \text{ (also pressure)}, \quad \text{and of course also } \mathbf{E} \text{ \& } \mathbf{B}.$$

We can take moments of the entire Vlasov/Boltzmann equation by multiplying each term by some power of  $\mathbf{p}$  and integrating each term over  $d^3\mathbf{p}$ .

It's worthwhile to go through the first few moments in detail. Start with just Vlasov (no collisions), and we get:

### 0th moment: Mass

- Guarantees particles aren't created or destroyed (unless there's ionization or recombination)
- What flows in = what flows out!



### 1st moment: Momentum

- If you push on something, it accelerates ( $F = ma$ )
- If the forces are balanced, we're in hydrostatic equilibrium. (But is it a stable or **unstable** equilibrium?)

### 2nd moment: Energy

- Total energy is conserved, but it can mutate into different forms...
- If heat sources & sinks are balanced, we're in thermal equilibrium. (But is it a stable or **unstable** equilibrium?)

**Mass conservation:** Just integrate each term over momentum space...

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{F} \cdot \nabla_p f = 0$$

Recall: the  $\mathbf{r}$ ,  $\mathbf{p}$ , and  $t$ , coordinates are all independent of one another... e.g., put aside "knowledge" that  $p_x = m(dx/dt)$  for individual particles.

$$\text{First term: } \int d^3\mathbf{p} \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int d^3\mathbf{p} f = \boxed{\frac{\partial n}{\partial t}}$$

Second term:

$$\int d^3\mathbf{p} (\mathbf{v} \cdot \nabla f) = \int d^3\mathbf{p} [\nabla \cdot (f\mathbf{v}) - f(\nabla \cdot \mathbf{v})] \quad \left( \begin{array}{l} \text{by vector} \\ \text{chain rule} \end{array} \right)$$

*(because  $\mathbf{r}$  and  $\mathbf{v}$  coordinates are linearly independent!)*

$$= \nabla \cdot \int d^3\mathbf{p} f \mathbf{v} = \boxed{\nabla \cdot (n\mathbf{u})} \quad (\text{recalling definition of } \mathbf{u})$$

Third term:

$$\int d^3\mathbf{p} (\mathbf{F} \cdot \nabla_p f) = \mathbf{F} \cdot \int d^3\mathbf{p} \nabla_p f \quad \left( \begin{array}{l} \text{okay as long as } \mathbf{F} \text{ depends on } \mathbf{r} \text{ \& } t \text{ only, or if} \\ F_i \text{ depends on } p_j, \text{ with } i \neq j \text{ (true for Lorentz)} \end{array} \right)$$

But we don't have to make that step unilaterally. There are 3 terms in that dot product. Let's just look at the  $x$  component for now...

$$\int d^3\mathbf{p} F_x \frac{\partial f}{\partial p_x} = \int_{-\infty}^{+\infty} dp_z \int_{-\infty}^{+\infty} dp_y F_x \int_{-\infty}^{+\infty} dp_x \frac{\partial f}{\partial p_x}$$

since  $F_x$  may depend on  $p_y$  &  $p_z$ , but not on  $p_x$ . (That's true for the magnetic part of the Lorentz force. For other forces like gravity or the electric Lorentz force,  $\mathbf{F}$  doesn't depend on  $\mathbf{p}$  at all.)

The fundamental theorem of calculus says that the right-most part (i.e., the integral of a derivative of  $f$ ) is just  $f$  itself.

$$\text{Thus, the integral over } dp_x \text{ gives } f(+\infty) - f(-\infty) \rightarrow 0$$

$$\text{as long as we can assume that } f \rightarrow 0 \text{ as } |p_x| \rightarrow \infty$$

Thus, the third term vanishes, and the full equation of mass conservation is:

$$\boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0}$$

Don't forget that  $n$  and  $\mathbf{u}$  are really  $n_s$  and  $\mathbf{u}_s$  (particle species  $s$ ).

Note that the 0th moment equation contains  $\mathbf{u}$ , which is the 1st moment of  $f$ . Each  $N^{\text{th}}$  moment equation will contain the  $(N + 1)^{\text{th}}$  moment of  $f$ .

Thus, any finite set of fluid-moment equations is INCOMPLETE. To get to a useful set of equations, we need "closure" (i.e., a way to stop the eternal need for the next  $N + 1$  moment). Stay tuned.

Also, note that the RHS would be 0 even if the Boltzmann collision term would have been included. We saw that collisions don't create or destroy mass.

The form of the mass conservation equation given above is how we often see it in textbooks. However, there is a more intuitive way to conceptualize it. Using one of the vector 'product rules' given in the useful-math document,

$$\begin{aligned} \frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n + n \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n &= -n \nabla \cdot \mathbf{u} \\ \frac{Dn}{Dt} &= -n \nabla \cdot \mathbf{u} \end{aligned}$$

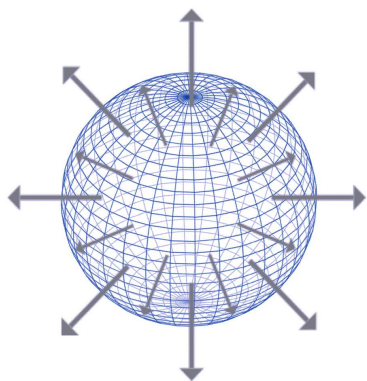
where we should recall the definition of the **advective derivative**. For a fluid parcel, there are multiple ways of changing the local density:

- There's a change in time at a single, fixed location.
- The parcel moves into a region of different density (like the trees!).
- The parcel can be compressed or expanded:
  - $\nabla \cdot \mathbf{u} < 0$ : converging flow (compression).  $n$  in parcel **increases**.
  - $\nabla \cdot \mathbf{u} > 0$ : diverging flow (expansion).  $n$  in parcel **decreases**.
  - $\nabla \cdot \mathbf{u} = 0$ : the fluid is **incompressible** (like water on Earth).

The mass conservation equation says that these 3 ways are linked together. "There's no free lunch" (i.e., mass can neither be created nor destroyed).

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One application often seen in astronomy is the study of **spherical outflows** (and/or inflows!) around stars, galaxies, or compact objects.



In spherical symmetry,

$$\nabla \cdot (n\mathbf{u}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n u_r)$$

and if  $\partial n / \partial t = 0$ , then mass conservation says that

$$r^2 n(r) u_r(r) = \text{constant.}$$

Also, if the gas is “coasting out” with  $u_r = \text{constant}$ , this means the density is dropping as  $n \propto 1/r^2$ . This is, in fact, what we observe for the gas that makes up the **solar wind** throughout the solar system.

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**Momentum conservation:** Multiply each term by  $\mathbf{v}$ ,

$$\mathbf{v} \frac{\partial f}{\partial t} + \mathbf{v} (\mathbf{v} \cdot \nabla f) + \mathbf{v} (\mathbf{F} \cdot \nabla_p f) = 0$$

(it’s now a vector equation), then integrate over  $d^3\mathbf{p}$ .

First term: goes like in the mass equation:

$$\int d^3\mathbf{p} \mathbf{v} \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int d^3\mathbf{p} f \mathbf{v} = \boxed{\frac{\partial}{\partial t} (n\mathbf{u})}$$

Second term: trickier.

I’m going to make the bold assertion that, because the  $\mathbf{p}$  and  $\mathbf{r}$  coordinates are independent of one another, one can *pull out* the spatial gradient in the 2nd term all the way... i.e.,

$$\int d^3\mathbf{p} \mathbf{v} (\mathbf{v} \cdot \nabla f) = \nabla \cdot \int d^3\mathbf{p} \mathbf{v} \mathbf{v} f$$

$\mathbf{v} \mathbf{v}$  is a dyadic tensor, so taking its divergence gives back a vector. Formally,

$$\mathbf{v} \mathbf{v} = \begin{pmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{pmatrix}$$

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Let’s prove the above assertion. First, use the vector chain rule to rewrite...

$$\mathbf{v} (\mathbf{v} \cdot \nabla f) = \mathbf{v} [\nabla \cdot (f\mathbf{v}) - f(\nabla \cdot \mathbf{v})]$$

(again,  $\mathbf{r}$  and  $\mathbf{v}$  are linearly independent)

So, we’ve pulled out the  $\nabla$  part of the way. I’m not sure I can find a general dyadic identity to get you the rest of the way, but in *our* case, we need to show that  $\mathbf{A} = \mathbf{B}$ , where

$$\mathbf{A} = \nabla \cdot (f\mathbf{v}\mathbf{v}) , \quad \mathbf{B} = \mathbf{v} \nabla \cdot (f\mathbf{v}) .$$

Using Cartesian coordinates, the  $x$  component is

$$A_x = \frac{\partial}{\partial x} (fv_x v_x) + \frac{\partial}{\partial y} (fv_x v_y) + \frac{\partial}{\partial z} (fv_x v_z)$$

and this is

$$A_x = v_x \left[ \frac{\partial}{\partial x} (fv_x) + \frac{\partial}{\partial y} (fv_y) + \frac{\partial}{\partial z} (fv_z) \right] + f v_x \frac{\partial v_x}{\partial x} + f v_y \frac{\partial v_x}{\partial y} + f v_z \frac{\partial v_x}{\partial z}$$

and since (in our case) the blue terms **all = 0**, we have  $A_x = v_x \nabla \cdot (f\mathbf{v}) = B_x$ . Similar arguments hold for the  $y$  and  $z$  components.

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Thus, the second term in the momentum conservation equation is

$$\boxed{\nabla \cdot (n\langle \mathbf{v}\mathbf{v} \rangle)}$$

where we use the same angle-bracket moment notation as above.

Let's pause a second to write out in full what this dyadic "thing" really looks like. Divergence of a tensor gives back a vector:

$$[\nabla \cdot (n\langle \mathbf{v}\mathbf{v} \rangle)]_x = \frac{\partial}{\partial x} (n\langle v_x v_x \rangle) + \frac{\partial}{\partial y} (n\langle v_x v_y \rangle) + \frac{\partial}{\partial z} (n\langle v_x v_z \rangle)$$

$$[\nabla \cdot (n\langle \mathbf{v}\mathbf{v} \rangle)]_y = \frac{\partial}{\partial x} (n\langle v_y v_x \rangle) + \frac{\partial}{\partial y} (n\langle v_y v_y \rangle) + \frac{\partial}{\partial z} (n\langle v_y v_z \rangle)$$

$$[\nabla \cdot (n\langle \mathbf{v}\mathbf{v} \rangle)]_z = \frac{\partial}{\partial x} (n\langle v_z v_x \rangle) + \frac{\partial}{\partial y} (n\langle v_z v_y \rangle) + \frac{\partial}{\partial z} (n\langle v_z v_z \rangle)$$

and each of those moments is straightforwardly defined.

Third term: for the case of an  $\mathbf{F}$  that doesn't depend on  $\mathbf{p}$ , we can remove it from the integral...

$$\int \mathbf{v} (\mathbf{F} \cdot \nabla_p f) d^3\mathbf{p} = \mathbf{F} \cdot \int \mathbf{v} (\nabla_p f) d^3\mathbf{p}$$

What's left under the integral is a tensor, and it can be rewritten using a similar chain rule as discussed above:

$$\mathbf{v} (\nabla_p f) = \nabla_p (f\mathbf{v}) - f(\nabla_p \mathbf{v})$$

1st term on RHS: like above, the integral of a  $\mathbf{p}$ -derivative over all  $\mathbf{p}$  space will be **zero**, as long as  $\mathbf{v}f \rightarrow 0$  as all  $|\mathbf{p}| \rightarrow \infty$ .

2nd term on RHS: the thing in parentheses is a simple tensor:

$$\nabla_p \mathbf{v} = \begin{pmatrix} \partial v_x / \partial p_x & \partial v_x / \partial p_y & \partial v_x / \partial p_z \\ \partial v_y / \partial p_x & \partial v_y / \partial p_y & \partial v_y / \partial p_z \\ \partial v_z / \partial p_x & \partial v_z / \partial p_y & \partial v_z / \partial p_z \end{pmatrix} = \begin{pmatrix} 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{pmatrix} \quad \begin{array}{l} \text{assuming} \\ v \ll c \end{array}$$

Thus, this term is

$$-\frac{1}{m} \mathbf{F} \int f d^3 \mathbf{p} = -n \frac{\mathbf{F}}{m} = \boxed{-n \mathbf{a}}$$

So, the full equation of momentum conservation is

$$\frac{\partial}{\partial t} (n \mathbf{u}) + \nabla \cdot (n \langle \mathbf{v} \mathbf{v} \rangle) - n \mathbf{a} = 0$$

Not quite in the *most useful* form yet, though...

That first term on the LHS can be broken up into

$$\begin{aligned} n \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial n}{\partial t} \\ n \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} [\nabla \cdot (n \mathbf{u})] & \quad (\text{using mass conservation}) \\ n \frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (n \mathbf{u} \mathbf{u}) + n \mathbf{u} \cdot \nabla \mathbf{u} & \quad (\text{chain rule yet again}) \end{aligned}$$

Lastly, we can multiply each term by  $m$  (mass of particle) and use  $\rho = nm$ , to write the “final” momentum conservation equation as

$$\boxed{\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \cdot \mathbb{P} - \rho \mathbf{a} = 0}$$

where the **pressure tensor** is given by

$$\mathbb{P} = \rho [\langle \mathbf{v} \mathbf{v} \rangle - \mathbf{u} \mathbf{u}] = \rho \langle \mathbf{v} \mathbf{v} - \mathbf{u} \mathbf{u} \rangle$$

It was okay to put  $\mathbf{u} \mathbf{u}$  inside the brackets since it’s *already* a moment.

(Note that 1st two terms in momentum equation can be simplified as  $\rho D\mathbf{u}/Dt$ .)

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**NOTE:** Acceleration  $\mathbf{a}$  should really be written as a mean, moment-like  $\langle \mathbf{a} \rangle$ .

Are the forces on single particles *modified* when integrated over  $d^3\mathbf{p}$ ?

- Gravity is unchanged. It doesn't depend on velocity.
- Lorentz  $\mathbf{E}$  force is unchanged. Doesn't depend on velocity either.
- Lorentz  $\mathbf{B}$  force depends linearly on  $\mathbf{v}$ . Averaging replaces  $\mathbf{v}$  with  $\langle \mathbf{v} \rangle = \mathbf{u}$ .

$$\text{Thus, } \langle \mathbf{a} \rangle = \mathbf{g} + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right)$$

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Note also that the  $\mathbf{u} \cdot \nabla \mathbf{u}$  term is weird. Gradient of a vector? Formally,  $\nabla \mathbf{u}$  gives a  $3 \times 3$  tensor, and taking the dot product with  $\mathbf{u}$  gives back a vector.

However, you can also think about it as  $(\mathbf{u} \cdot \nabla) \mathbf{u}$ , where it's a scalar "operator" acting on a vector. It tells us how much  $\mathbf{u}$  varies along the direction that the  $\mathbf{u}$  vector points.

In Cartesian coordinates, it's straightforward to work out:

$$\begin{aligned} \frac{\partial u_x}{\partial t} + \left( u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) u_x + \dots \\ \frac{\partial u_y}{\partial t} + \left( u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) u_y + \dots \end{aligned}$$

and similarly for the  $z$  component.

In spherical coordinates, we'll most often want to evaluate the *radial* component of the momentum equation. It's got extra terms, since the coordinates depend on one another...

$$\frac{\partial u_r}{\partial t} + \left( u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) u_r - \frac{u_\theta^2 + u_\phi^2}{r} + \dots$$

Don't worry... usually we'll assume a  $\sim$ symmetric system in which many of these other derivatives are zero.

The second moment of the Boltzmann equation is the **energy conservation equation**. In its most general form, it ought to contain  $3 \times 3 \times 3$  tensors related to  $\mathbf{v}\mathbf{v}\mathbf{v}$ . I'll give some more useful versions of the full equation later, but let's first get a feel for what we'll expect, using a Maxwellian  $f(\mathbf{p})$  for the sake of familiarity.

In a reference frame where  $\mathbf{u} = 0$ , the Maxwellian distribution is

$$f_{\text{MB}}(\mathbf{p}) \propto e^{-E_{\text{K}}/k_{\text{B}}T} \quad \text{where} \quad E_{\text{K}} = \frac{p^2}{2m} .$$

Basic meaning: more particles can occupy lower-energy "states."

Note that it's extremely rare/unlikely to find particles with  $E_{\text{K}} \gg k_{\text{B}}T$ .

This neglects other types of energy besides kinetic (e.g., various kinds of **potential energy**), but right now just consider the gas as simple point-masses flying around freely in a vacuum (i.e., right now, consider *only* energy as  $E_{\text{K}}$ ).

Let's work out how  $f_{\text{MB}}$  is normalized, using the zeroth moment. Limit ourselves to the non-relativistic definition of kinetic energy, and express it in the frame of the bulk motion...

$$f = C \exp\left(-\frac{p^2}{2mk_{\text{B}}T}\right)$$

$$n = \int d^3\mathbf{p} f = \int_0^\infty dp (4\pi p^2) f \quad \left\{ \begin{array}{l} \text{since } f \text{ depends only} \\ \text{on magnitude of } \mathbf{p} \end{array} \right\}$$

Change variables...

$$x = \frac{p}{\sqrt{2mk_{\text{B}}T}} \quad \text{so, we can then write} \quad n = 4\pi C (2mk_{\text{B}}T)^{3/2} \int_0^\infty dx x^2 e^{-x^2}$$

The integral is dimensionless, and has a value of  $\sqrt{\pi}/4$  (see tables). Thus...

$$n = C (2\pi mk_{\text{B}}T)^{3/2} \quad \text{so we can solve for } C \text{ and use it to write}$$

$$\boxed{f_{\text{MB}} = \frac{n}{(2\pi mk_{\text{B}}T)^{3/2}} \exp\left(-\frac{p^2}{2mk_{\text{B}}T}\right)}$$

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## Additional properties of a M-B distribution

The mean thermal energy density is defined as

$$U = \int d^3\mathbf{p} E_K f = n \langle E_K \rangle \approx n \left\langle \frac{p^2}{2m} \right\rangle \quad \left\{ \begin{array}{l} \text{which for M-B,} \\ \text{as above,} \end{array} \right\} = \frac{3}{2} n k_B T.$$

This is a second moment of  $f$ .

It's also useful to write the mean kinetic energy *par particle* as

$$\langle E_K \rangle = \tilde{U} = \frac{U}{n} = \frac{3}{2} k_B T$$

(i.e.,  $\frac{1}{2} k_B T$  for each degree of freedom...  $p^2 = p_x^2 + p_y^2 + p_z^2$ ). Some books also define the internal energy *per unit mass* as

$$e = \frac{U}{\rho} = \frac{3k_B T}{2m} \quad (\text{units of } v^2).$$

Recall that we defined the  $3 \times 3$  pressure tensor as

$$\mathbb{P} = \rho \langle \mathbf{v}\mathbf{v} - \mathbf{u}\mathbf{u} \rangle \approx \rho \langle \mathbf{v}\mathbf{v} \rangle \quad \left\{ \begin{array}{l} \text{in a reference frame} \\ \text{where } \mathbf{u} = 0 \end{array} \right\}$$

and this is

$$\mathbb{P} = \rho \begin{pmatrix} \langle v_x v_x \rangle & \langle v_x v_y \rangle & \langle v_x v_z \rangle \\ \langle v_y v_x \rangle & \langle v_y v_y \rangle & \langle v_y v_z \rangle \\ \langle v_z v_x \rangle & \langle v_z v_y \rangle & \langle v_z v_z \rangle \end{pmatrix}$$

One can prove that for a M-B distribution,

$$\mathbb{P}_{ij} = \begin{cases} n k_B T, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

and one can see from the units that this energy-density-like quantity truly “acts” like a pressure (i.e., force per unit area).

Since the tensor is diagonal, then  $\nabla \cdot \mathbb{P} = \nabla P$ , which defines scalar pressure

$$P = n k_B T.$$

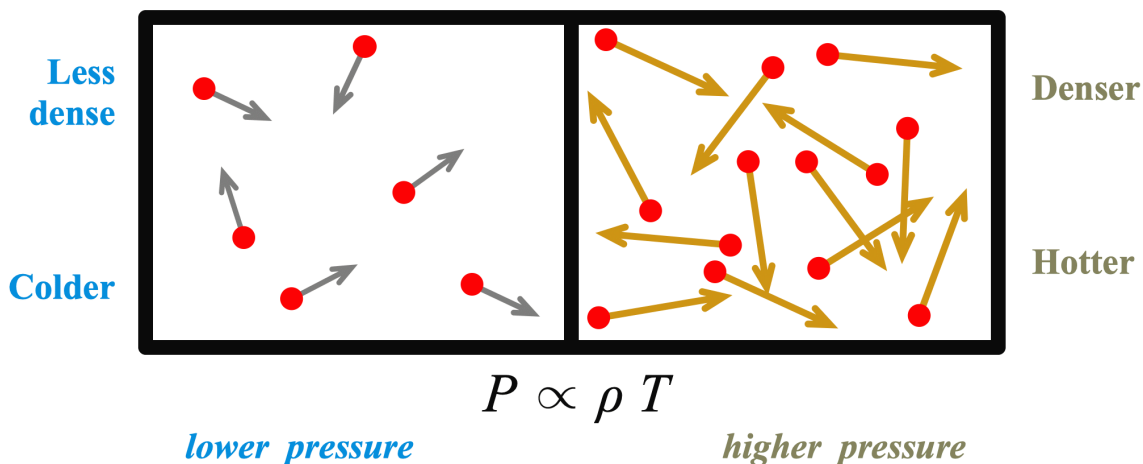
Note also that  $P = (2/3)U$ , which you should keep in your hip pocket. We'll revisit this proportionality later.

In the momentum equation, there is a **pressure gradient force**:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{a} - \frac{\nabla P}{\rho} .$$

Note this is a completely “collisionless” phenomenon. We often think of pressure as being exerted by particles colliding with a “wall” of some kind. However, no walls or physical collisions are necessary!

**Thought experiment:** Consider the case of two volumes of gas—one with a higher density & temperature than the other—that are brought into contact with one another. What happens when we remove the “wall” between them?



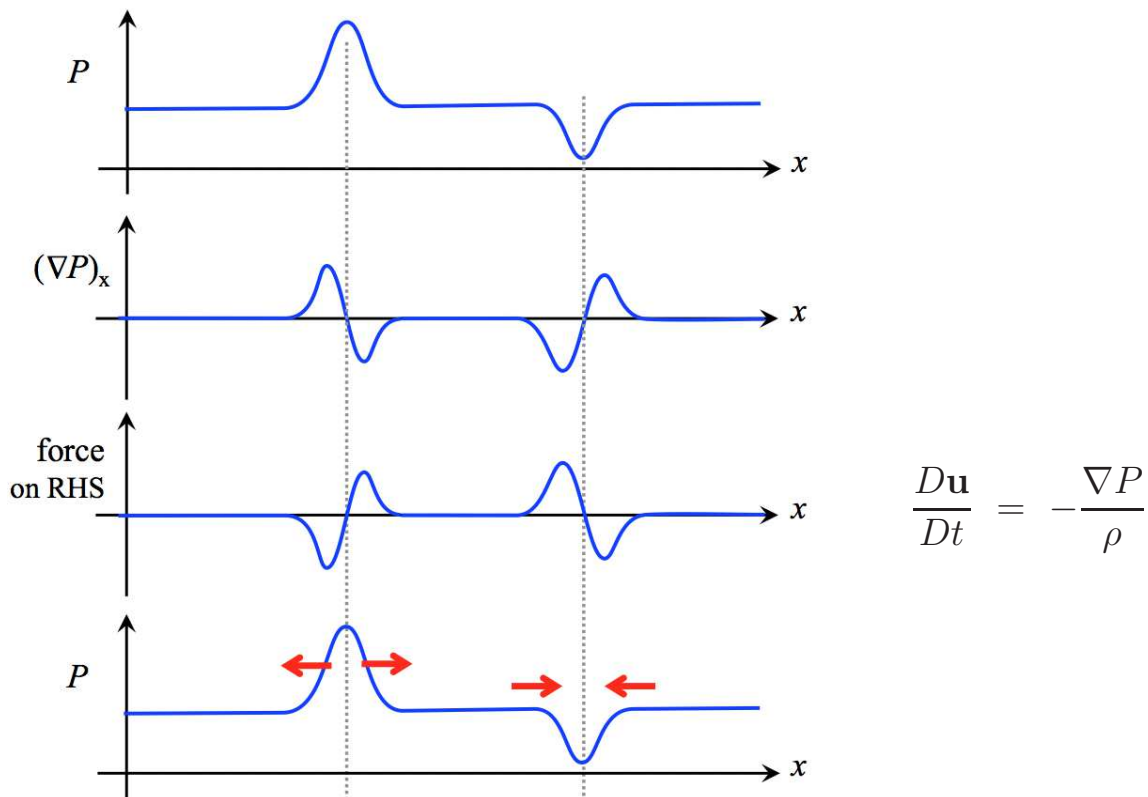
Purely random motions of the particles will result in more particles going from the denser/hotter volume into the other volume than go in the opposite direction.

Initially,  $\mathbf{u} = 0$ , but after the wall is removed,  $\mathbf{u} \neq 0$ .

This **acceleration** is the manifestation of the pressure gradient force, which required *NO* particle–particle collisions to occur.

High-pressure regions “want to” expand into low-pressure regions.

Another way to see this is in 1D. Here, the gradient is just a simple derivative in  $x$ . What happens to regions with higher and/or lower pressure?



Peaks will get smeared out (eroded away). Valleys will get filled in. Ultimately, the fluid reaches pressure equilibrium, in which the RHS is zero ( $\nabla P = 0$ , i.e., pressure is constant). Momentum conservation thus demands the LHS is zero, too (no parcels are accelerated; everything stays still).

*Aside:* The gas is made up of **multiple elements**. The above definitions need to be modified to take account of (at the very least) the co-existence of hydrogen, helium, “metals” ( $Z$ ), and free electrons.

When including the effects of multiple types of particles, we often want to use a single average value of, say, density ( $n$  or  $\rho$ ) or pressure.

Thus, we sum over all of the unique species  $s$  of particles:

$$\rho = mn \implies \sum_s m_s n_s$$

$$P = nk_B T \implies \sum_s n_s k_B T_s$$

where  $s$  accounts for both electrons and all atoms/ions.

When summing to get  $\rho$ , we often assume that  $m_e$  is **much tinier than** the masses of all nuclei, so free electrons don't contribute to  $\rho$ .

For  $P$ , however, electrons can be important because there's no (small)  $m_e$  factor in the terms being summed. However, in this sum we often assume all  $T_s$  are equal to a common (collisionally coupled)  $T$ .

By convention, we write

$$n_{\text{tot}} = \sum_s n_s \qquad \langle m \rangle = \frac{\sum_s m_s n_s}{\sum_s n_s}$$

thus,  $\rho = \langle m \rangle n_{\text{tot}}$       and       $P = n_{\text{tot}} k_B T = \frac{\rho k_B T}{\langle m \rangle}$

and our goal is to specify  $\langle m \rangle \equiv \mu m_H$  for an arbitrary mixture of elements & ionization states.

For a mixture,  **$\mu$  is the mean mass per particle** (in units of  $m_H$ ).

In astronomy, I'm sure you've often heard that the composition of the universe is jokingly broken down into 3 things: **hydrogen, helium, & "metals."**

We do often describe the mass density by breaking it into 3 mass fractions:

$$\rho = m_H n_H + m_{\text{He}} n_{\text{He}} + m_Z n_Z$$

where

$$X \equiv \frac{m_H n_H}{\rho} \quad , \quad Y \equiv \frac{m_{\text{He}} n_{\text{He}}}{\rho} \quad , \quad Z \equiv \frac{m_Z n_Z}{\rho} \quad , \quad X + Y + Z = 1$$

and the  $n$ 's are number densities of *all* nuclei of either H, He, or heavier elements, irrespective of ionization state. Thus, let's use them to solve for

$$\frac{1}{\langle m \rangle} = \frac{\sum_s n_s}{\sum_s m_s n_s}$$

and you'll see why I flipped it in a minute.

For a **NEUTRAL** gas, there are no free electrons, so

$$\frac{1}{\langle m \rangle} = \frac{n_{\text{H}} + n_{\text{He}} + n_{\text{Z}}}{\rho} = \frac{1}{\rho} \left[ \frac{\rho X}{m_{\text{H}}} + \frac{\rho Y}{4m_{\text{H}}} + \frac{\rho Z}{A_{\text{Z}}m_{\text{H}}} \right]$$

where, for solar abundances, the “weighted” atomic weight of all metal species is  $A_{\text{Z}} \approx 15.5$  (i.e., O dominates, C brings it down a bit). Thus,

$$\boxed{\frac{1}{\mu_{\text{neut}}} = X + \frac{1}{4}Y + \frac{1}{A_{\text{Z}}}Z}$$

For a fully **IONIZED** gas, free electrons do contribute to  $n_{\text{tot}}$ , and we have to keep track of how many electrons are “supplied” by each fully-stripped kind of nucleus...

$$\frac{1}{\langle m \rangle} = \frac{n_{\text{H}} + n_{\text{He}} + n_{\text{Z}} + n_{\text{e}}}{\rho}, \quad \text{and} \quad n_{\text{e}} \approx n_{\text{H}} + 2n_{\text{He}} + \left(\frac{A_{\text{Z}}}{2}\right)n_{\text{Z}}$$

where we assume a fully ionized heavy ion has a nucleus that’s  $\sim$  half protons & half neutrons. Doing the algebra . . .

$$\boxed{\frac{1}{\mu_{\text{ion}}} = 2X + \frac{3}{4}Y + \left(\frac{1}{2} + \frac{1}{A_{\text{Z}}}\right)Z}$$

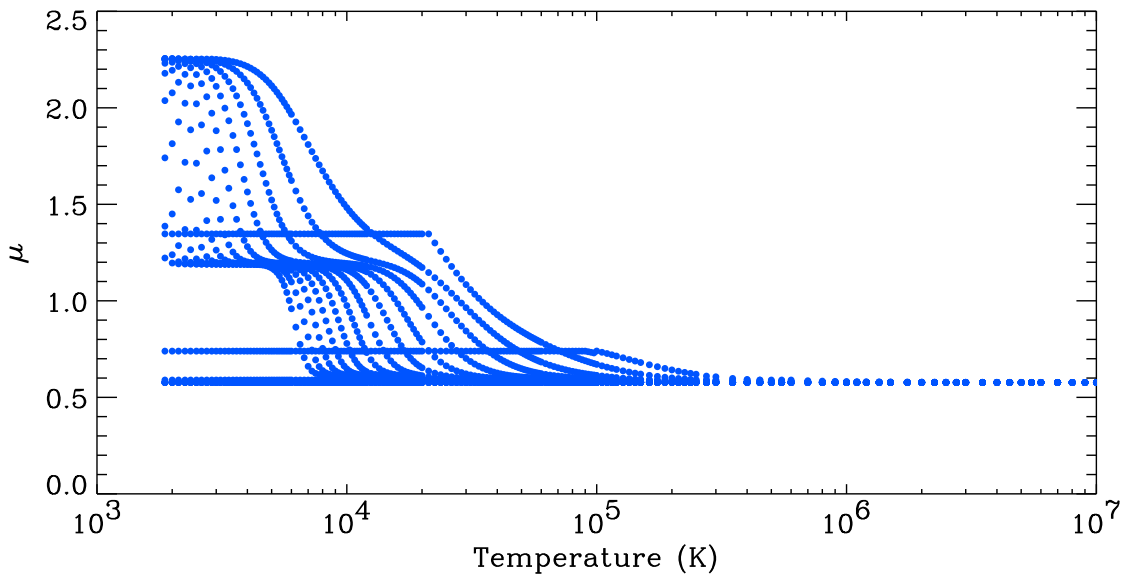
In the above expression, the last quantity in parentheses is often estimated as just 1/2 (since  $A_{\text{Z}} \gg 1$ ).

Example: composition of the present-day Sun:

$$\left\{ \begin{array}{l} X \approx 0.73 \\ Y \approx 0.25 \\ Z \approx 0.02 \end{array} \right\} \longrightarrow \begin{array}{l} \mu_{\text{neut}} \approx 1.26 \\ \mu_{\text{ion}} \approx 0.60 \end{array}$$

If it was just hydrogen, then  $\mu_{\text{neut}} = 1$  and  $\mu_{\text{ion}} = 0.5$ .

As  $T$  goes up, so does ionization, and  $\mu$  goes down. Ionization state also depends on density. Here's output from the OPAL atomic-physics database, for many different densities...



You can see clumps of points around our values of  $\sim 1.2$  and  $0.6$ , but why does  $\mu$  get so much larger at the coldest temperatures?

Molecules start to form...  $\text{H} + \text{H} \rightarrow \text{H}_2$ , and so on.

.....

Whatever the value of  $\mu$ , it's often valid to take the simple expressions for Maxwellian distributions and replace  $(n, m)$  by  $(n_{\text{tot}}, \langle m \rangle)$  to get an equation for the entire "coupled plasma."

Can we estimate  $\mu$  for the Earth's atmosphere, in this room?

Back-of-the envelope:  $\mu \approx 29$  [mostly  $\text{N}_2$  (28) with some  $\text{O}_2$  (32)].

Let's move on to think a bit more about **thermodynamics** (i.e., how things like  $\rho$ ,  $P$ ,  $T$ ,  $U$ , etc. *change* in astrophysical environments).

We now need to look at the **energy conservation equation**. There are many ways of writing it. If you just take the 2nd moment of the Vlasov equation, you get “total” energy conservation. Schematically,

$$\mathbf{u} \cdot \left\{ \begin{array}{l} \text{each term in the} \\ \text{momentum conservation eqn} \end{array} \right\} + \text{other stuff} = 0$$

and since the sum of the terms in brackets = 0, the only unique contribution is the “other stuff.” That's **internal** energy conservation.

For a Maxwell-Boltzmann distribution of particles in isolation, it boils down to the following equation for the thermal energy density:

$$\frac{DU}{Dt} + (U + P)\nabla \cdot \mathbf{u} = 0 .$$

Units of each term: energy/time/volume, or power/volume.

However, sometimes the right-hand side is  $\neq 0$ . This happens when

- $f(\mathbf{p})$  has non-Maxwellian skewness, which describes heat conduction,
- the particles don't all have the same temperature, so there can be additional “heat transfer” due to Coulomb collisions (see below),
- the particles are embedded in electric/magnetic fields, or are exposed to photon radiation... and these things can either heat or cool the gas.

Thus, sometimes these heat gains and/or losses are put on the RHS:

$$\frac{DU}{Dt} + (U + P)\nabla \cdot \mathbf{u} = \sum \frac{DQ}{Dt}$$

where the quantity of heat  $Q$  doesn't really exist by itself; only as a sum of rates of change, from multiple sources.

This equation may look new to you, but you may have seen it before in another guise: **the first law of thermodynamics**. Let's show how that works...

First, let's use mass conservation to replace:  $\nabla \cdot \mathbf{u} = -\frac{1}{n} \frac{Dn}{Dt} .$

We'll also replace  $D$  by  $d$ , for familiarity, so that

$$\frac{dU}{dt} - \frac{(U + P)}{n} \frac{dn}{dt} = \frac{dQ}{dt} .$$

Remember that  $U = n \langle \frac{1}{2} m v^2 \rangle$ , where we will now refer to the mean kinetic energy per particle as  $\tilde{U}$ .

$$\text{Since } U = n\tilde{U}, \quad \text{then} \quad \frac{dU}{dt} = n \frac{d\tilde{U}}{dt} + \tilde{U} \frac{dn}{dt} .$$

Putting that in, two terms cancel. Divide each term by  $n$ , and we get

$$\begin{aligned} \frac{d\tilde{U}}{dt} - \frac{P}{n^2} \frac{dn}{dt} &= \frac{1}{n} \frac{dQ}{dt} \\ \frac{d\tilde{U}}{dt} + P \frac{d}{dt} \left( \frac{1}{n} \right) &= \frac{1}{n} \frac{dQ}{dt} \end{aligned}$$

$1/n$  has units of **volume**, so call it  $V$ . Also, the amount of heat input **per particle** can be hand-wavingly written as

$$\frac{1}{n} \frac{dQ}{dt} = \frac{d\tilde{Q}}{dt} .$$

Thus, if we annoy the mathematicians and multiply each term by  $dt$ , we get the familiar form of the first law of thermodynamics:

$$\boxed{d\tilde{U} + PdV = d\tilde{Q}}$$

i.e.,

$$d\tilde{U} = d\tilde{Q} - PdV$$

change in  
the parcel's  
thermal energy

heat supplied  
**TO** the  
parcel

work done **BY**  
the parcel on  
surroundings

which shows there are two ways to change a parcel's thermal energy: either by *giving it* some heat directly, or by having the environment do work on it (i.e.,  $d\tilde{W} = PdV < 0$ ).

FYI, units of work check out:  $F dx = \frac{F}{\text{Area}} dx \times \text{Area} = P dV$  . ✓



To illustrate other processes, let's ignore  $DQ/Dt$ . Without heat being added to or removed from the “parcel,” we're talking about **adiabatic** energy conservation.

Using  $U = (3/2)P$ , we can write

$$\frac{3}{2} \frac{DP}{Dt} + \frac{5}{2} P (\nabla \cdot \mathbf{u}) = 0$$

and we can multiply by 2/3, and use mass conservation:  $\nabla \cdot \mathbf{u} = -\frac{1}{n} \frac{Dn}{Dt}$

This gives: 
$$\frac{DP}{Dt} - \frac{5}{3} \frac{P}{n} \frac{Dn}{Dt} = 0 \quad \rightsquigarrow \text{or } \rightsquigarrow \quad \frac{D}{Dt} \left( \frac{P}{n^{5/3}} \right) = 0$$

which means that  $\boxed{P/n^{5/3} = \text{constant}}$ , or  $P \propto n^{5/3}$  for adiabatic changes in a gas parcel.

This can be a source of confusion. It's important to emphasize the difference between the ideal-gas equation of state ( $P = nk_B T$ ) and the adiabatic relations ( $P \propto n^{5/3}$ ). They may seem mutually contradictory, but they're not.

$P = nk_B T$  is always true for an ideal gas.

$P \propto n^{5/3}$  is true for *relative* changes that (are constrained to) remain adiabatic.

Silly example: If you want to **double**  $P$ , there are many ways to do it:

$P =$	$n$	$k_B T$
$\times 2$	$\times 2$	1
$\times 2$	1	$\times 2$
$\times 2$	$\times \sqrt{2}$	$\times \sqrt{2}$
$\times 2$	$\times 20$	$\times 0.1$
$\times 2$	$\times 1.51572$	$\times 1.31951$

Nearly all choices involve either adding or removing heat from the system.

The adiabatic relations tell us the *one* unique way that keeps  $DQ/Dt = 0$ :

$$\frac{P_{\text{final}}}{P_{\text{initial}}} = \left( \frac{n_{\text{final}}}{n_{\text{initial}}} \right)^{5/3} = 2, \quad \text{so} \quad \frac{n_{\text{final}}}{n_{\text{initial}}} = 2^{3/5} \approx 1.51572,$$

$$\text{and} \quad \frac{T_{\text{final}}}{T_{\text{initial}}} = \left( \frac{n_{\text{final}}}{n_{\text{initial}}} \right)^{2/3} = 1.31951.$$

.....

Ideal gases are nice, but there's a simple way to generalize a bit to include many other types of gases:

Let's replace  $U = \frac{3}{2}P$  by  $U = \frac{P}{\gamma - 1}$

where we define the **adiabatic exponent**  $\gamma = 5/3$  for an ideal gas.

We'll explore what  $\gamma$  really means in a bit. For an adiabatic parcel,

$$P = nk_B T \propto n^\gamma \quad \text{so} \quad T \propto n^{\gamma-1} \quad . \quad \text{Also:} \quad P \propto T^{\gamma/(\gamma-1)} \quad .$$

With this more general way of describing adiabatic changes in gas, we can look at a few applications.

Many astrophysical systems that involve *expansion* also exhibit something called **adiabatic cooling**.

In other words, since  $T \propto n^{\gamma-1}$  (and usually  $\gamma > 1$ ), then if a parcel expands adiabatically and decreases in density, its temperature will decrease, too.

.....

Non-ideal gases can have different adiabatic values of  $\gamma$ . A useful way to think about it is to link it to the **3 degrees of freedom**.

Recall that the factor of 3/2 in  $\tilde{U}$  comes from each Cartesian dimension in which the particle is free to move:

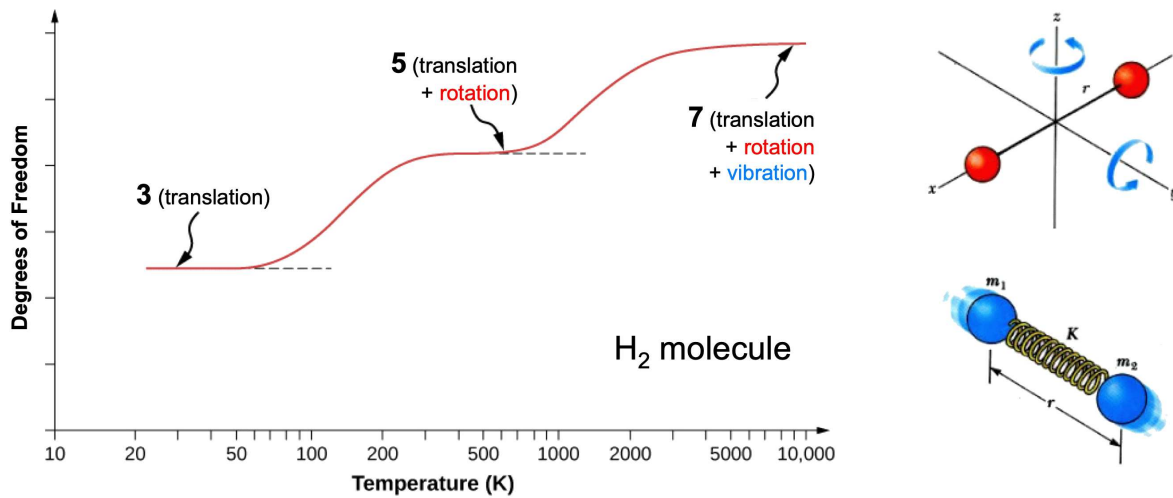
$$\left\langle \frac{1}{2}mv_x^2 \right\rangle = \frac{1}{2}k_B T \quad \left\langle \frac{1}{2}mv_y^2 \right\rangle = \frac{1}{2}k_B T \quad \left\langle \frac{1}{2}mv_z^2 \right\rangle = \frac{1}{2}k_B T$$

Thus for an ideal gas, the number of degrees of freedom is  $\delta = 3$ , and

$$\tilde{U} = \frac{3}{2}k_B T = \frac{k_B T}{\gamma - 1} \quad \text{but if } \delta \neq 3, \text{ then } \tilde{U} = \frac{\delta}{2}k_B T \quad , \quad \text{and } \gamma = 1 + \frac{2}{\delta} \quad .$$

Single-atom gases usually just have  $\delta = 3$ , but when atoms collect into **molecules**, they have more ways to be twitchy...

As  $T \uparrow$ , higher-energy “modes” of rotational & vibrational motion become available to molecules, and  $\delta$  increases. For example:



More degrees of freedom = more ways to “store” heat in the gas (in the form of internal motions), without raising the temperature.

In fact, as  $\delta \rightarrow \infty$ , you can do whatever you want to the particles – translate them, squeeze them, rotate them, etc. (in infinite ways) – and  $T$  never changes. (With all modes getting  $\sim$ equal amounts of energy, translations are just a drop in the bucket...)

Mathematically, this corresponds to  $\gamma = 1$ , and

$$T \propto n^{\gamma-1} \propto n^0 = \text{constant} \quad \text{and} \quad T \propto P^{(\gamma-1)/\gamma} \propto P^0 = \text{constant}.$$

Thus,  $\gamma = 1$  is called the “**isothermal**” adiabatic index. The outflows around the most massive stars (blue spectral types O & B) are dominated by radiation that keeps the gas cool ( $d\tilde{Q} < 0$ ) and effectively maintains a constant temperature. Theorists often use  $\gamma = 1$  in their hydrodynamics codes for these outflows.

## Magnetohydrodynamics (MHD)

In many situations, we don't want to follow the separate properties of each species. MHD is a one-fluid approximation in which we keep track of just a single  $\mathbf{u}$ ,  $T$ ,  $P$  that's assumed to be valid for all species.

Insight from Dr. Nick Murphy...

### When is MHD useful?

- ▶ MHD traditionally describes macroscopic force balance, equilibria, and dynamics
  - ▶ Describes dynamics reasonably well on large scales
- ▶ MHD is a good predictor of plasma stability
  - ▶ The most catastrophic instabilities are unstable in ideal MHD
  - ▶ Important in laboratory plasmas, solar atmosphere, etc.
- ▶ Systems that are described reasonably well by MHD include:
  - ▶ Solar wind, heliosphere, and Earth's magnetosphere<sup>1</sup>
  - ▶ Inertial range of plasma turbulence
  - ▶ Neutron star magnetospheres
- ▶ MHD is a reasonably good approximation in most astrophysical plasmas
  - ▶ However, extensions are often needed

$$L \gg \{\lambda_D, \ell_{\text{mfp}}, r_{\perp \text{gyro}}\}$$

---

<sup>1</sup>On large scales!

### When is MHD not useful?

- ▶ MHD has limited applicability when:
  - ▶ Non-fluid or kinetic effects are important
    - ▶ Dissipation in the turbulent solar wind
    - ▶ Magnetic reconnection
    - ▶ Small-scale dynamics in Earth's magnetosphere
  - ▶ The particle distribution functions are not Maxwellian
    - ▶ Cosmic rays
  - ▶ The plasma is weakly ionized
    - ▶ Solar photosphere/chromosphere, molecular clouds, protoplanetary disks, Earth's ionosphere, some laboratory plasmas
- ▶ MHD is mediocre at describing the dynamics of laboratory plasmas but remains a good predictor of stability

Let's work with large enough spatial scales that electric fields short out (Debye-scale) charge imbalances to keep the fluid quasi-neutral ( $\rho_c = 0$  everywhere).

To move forward, recall some “summed” definitions, and also estimate them for the simple case of an ionized hydrogen plasma...

$$\rho = \sum_s n_s m_s \approx (n_p m_p + n_e m_e) \sim n_p m_p \quad (\text{since } m_p \gg m_e)$$

$$\rho_c = \sum_s n_s q_s \approx 0 \quad \text{i.e., } n_p \approx n_e$$

$$\mathbf{u} = \frac{\sum_s n_s m_s \mathbf{u}_s}{\sum_s n_s m_s} \approx \frac{1}{\rho} (n_p m_p \mathbf{u}_p + n_e m_e \mathbf{u}_e) \sim \mathbf{u}_p$$

$$\mathbf{J} = \sum_s n_s q_s \mathbf{u}_s \approx (e n_p \mathbf{u}_p - e n_e \mathbf{u}_e) \sim e n_p (\mathbf{u}_p - \mathbf{u}_e) \quad (m \text{ doesn't appear!})$$

Mass conservation “sums up” neatly, as one would assume it should:

$$\sum_s m_s \left\{ \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0 \right\} \longrightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

However, if we re-write the momentum conservation equation for each species,

$$m_s n_s \frac{D\mathbf{u}_s}{Dt} + \nabla P_s - m_s n_s \mathbf{g} - n_s q_s \left( \mathbf{E} + \frac{\mathbf{u}_s}{c} \times \mathbf{B} \right) = 0$$

and sum over all species, we get

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} + \nabla P - \rho \mathbf{g} - \frac{1}{c} \mathbf{J} \times \mathbf{B} = 0}$$

noting that the  $\mathbf{E}$  term cancels out because  $\rho_c = 0$ .  $P$  = total gas pressure.

This is the MHD **equation of motion**. We still haven't morphed it completely into a “single-fluid” form, since there's still the possibility of  $\mathbf{J} \neq 0$  (i.e., relative flow between ions & electrons).

.....

Before working that out, though, let's make sure we understand all of the FORCES and how they balance one another.

.....

First, let's think about an unmagnetized plasma ( $\mathbf{B} = 0$ ) with no motions ( $\mathbf{u} = 0$ ). What we get is **hydrostatic equilibrium** where pressure gradient force balances gravity in an atmosphere,

$$\nabla P = \rho \mathbf{g} .$$

At the *surface* of a star or planet, let's look at "heights"  $z$  small compared to the radius  $R$ . On those scales,  $\mathbf{g} = -g\hat{\mathbf{z}} \approx \text{constant}$ . Also, for an ideal gas,

$$P = n_{\text{tot}} k_B T = \frac{\rho k_B T}{\mu m_H} .$$

If  $T \approx \text{constant}$ , then  $\left(\frac{k_B T}{\mu m_H}\right) \frac{d\rho}{dz} = -\rho g$

i.e.,  $\frac{d\rho}{\rho} = -\frac{\mu m_H g}{k_B T} dz$

$\rightsquigarrow \rightsquigarrow \quad \rho(z) = \rho_0 \exp\left(-\frac{z}{H}\right) \quad \text{where} \quad H = \frac{k_B T}{\mu m_H g}$

is the density scale height. For  $T = \text{const.}$ , it's also pressure scale height.

Usually,  $H \ll R$ , so  $\rho$  drops of *rapidly* compared to  $T$ , justifying the  $\sim$ isothermal approximation.

.....

If we look at static, magnetized plasmas in regions where  $\mathbf{g}$  is unimportant, then it's just pressure-gradient force balanced by the "Maxwell force,"

$$\nabla P = \frac{1}{c} \mathbf{J} \times \mathbf{B} .$$

When we think more carefully about this force, we'll understand more about how magnetic fields exert **pressure & tension** in a plasma.

Remember Ampère's & Faraday's laws,

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} - \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

In MHD, we often assume that the displacement current term is negligible.

Let's prove it... for non-relativistic fluid motions.

One can come up with an expression for the **ratio** of the 2 terms on the right-hand side of Ampère’s law. Factors of  $4\pi$  cancel...

$$\mathcal{R} = \frac{|\partial\mathbf{E}/\partial t|}{|c(\nabla \times \mathbf{B})|} . \quad \text{Is } \mathcal{R} \gg 1, \approx 1, \text{ or } \ll 1 ?$$

Assume the following:

(1) All time derivatives are roughly  $\frac{\partial}{\partial t} \approx \frac{1}{\tau}$

(2) All spatial derivatives are roughly  $\nabla \approx \frac{1}{\ell}$

(3) The speed of “stuff” moving through plasma is  $V \approx \frac{\ell}{\tau} \ll c$  .

With that, plus Faraday’s law for help, show that  $\mathcal{R} \ll 1$ .

Solution:

Faraday:  $\frac{E}{\ell} = \frac{B}{c\tau}$  i.e.,  $\frac{E}{B} = \frac{V}{c}$  .

Thus,  $\mathcal{R} = \frac{E/\tau}{cB/\ell} = \frac{\ell}{c\tau} \frac{E}{B} = \frac{V}{c} \frac{E}{B} = \frac{V^2}{c^2} \ll 1$  .

.....

Okay, with the displacement current neglected,

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \quad \text{and the Maxwell force is} \quad \frac{1}{c} \mathbf{J} \times \mathbf{B} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

and a chain-rule-like vector identity lets us write this as

$$\frac{1}{c} \mathbf{J} \times \mathbf{B} = \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \left( \frac{B^2}{8\pi} \right)$$

1st term: magnetic “tension” ..... 2nd term: magnetic “pressure gradient.”

Magnetic pressure: works similarly to gas pressure, so that we often group them together as

$$\nabla P_{\text{tot}} = \nabla (P_{\text{gas}} + P_{\text{mag}}) = \nabla \left( P + \frac{B^2}{8\pi} \right)$$

i.e., a bunched-up region of field lines acts like a high-pressure clump, and wants to *expand*.

We'll often define the **plasma beta** ratio,

$$\beta = \frac{P_{\text{gas}}}{P_{\text{mag}}}$$

$\beta \ll 1$ : magnetic field is important (i.e., it tells the fluid how to flow).

$\beta \gg 1$ : magnetic field is **unimportant** (i.e., it's pushed around by gas flows).

Magnetic Pressure cares only about spatial changes in the local **magnitude**  $B$ .  
Magnetic Tension cares about the **direction** (curvature) of the field.

$\mathbf{B} \cdot \nabla \mathbf{B}$  is a measure of how  $\mathbf{B}$  varies *along*  $\mathbf{B}$ . (See Spruit's "Essential MHD" notes for other useful/intuitive ways of writing this.)

**Simple example?** Consider cylindrical coordinates  $(r, \phi, z)$ , and a magnetic field  $\mathbf{B} = B_\phi \hat{\mathbf{e}}_\phi$ , with  $B_\phi$  constant in space.

$$(\mathbf{B} \cdot \nabla \mathbf{B})_r = B_r \frac{\partial B_r}{\partial r} + \frac{B_\phi}{r} \frac{\partial B_r}{\partial \phi} + B_z \frac{\partial B_r}{\partial z} - \frac{B_\phi^2}{r}$$

$$(\mathbf{B} \cdot \nabla \mathbf{B})_\phi = 0 \quad (\mathbf{B} \cdot \nabla \mathbf{B})_z = 0$$

The tension force vector points "inward" toward the center of curvature, and the magnitude is essentially  $B^2$  divided by the local *radius of curvature*.

Like taut wires that are "plucked," field lines are pulled back to being straight.

Okay, where do we stand, with respect to having a **closed set of equations** to describe this MHD system?

$$\frac{\partial \rho}{\partial t} = \text{function of } \rho, \mathbf{u}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \text{function of } \rho, \mathbf{u}, P, \mathbf{B}$$

$$\frac{\partial P}{\partial t} = \text{function of } \rho, \mathbf{u}, P$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$$

So how do we specify  $\mathbf{E}$ ?



To do this, we need to derive the so-called **generalized Ohm’s law**, which will join in with Maxwell’s equations to give us a fully closed set of MHD conservation equations.

What’s often done is to look at the momentum conservation equation for **just the electrons** by themselves.

We need to think again about the collisional source/sink terms on the right hand side...

Unfortunately, that means we have to go back to thinking about the Boltzmann collision term on the RHS of the Vlasov/Boltzmann equation.

Recall from previous lecture notes (**section d1**) where we followed a single particle (of species 1) as it collides with a large number of particles (of species 2) that obey a Maxwellian distribution? We got:

$$\frac{dv_{\parallel 1}}{dt} = -2\bar{v}_{S1} v_{th,2} G(x) \quad \text{where} \quad v_{th,2} = \sqrt{\frac{2k_B T_2}{m_2}}, \quad x = \frac{|\mathbf{v}_1 - \mathbf{u}_2|}{v_{th,2}}$$

We can now talk about the final step in the process of going from:

	single particle 1	→ colliding with →	single particle 2
to	single particle 1	→ colliding with →	distribution of particle 2’s
to	distribution of particle 1’s	→ colliding with →	distribution of particle 2’s .

i.e., we’re now coming back to **topic (d3)** on “multi-fluid equilibration.”

If we now estimate all  $f$ ’s as Maxwellians, but with  $T_1 \neq T_2$  and/or  $\mathbf{u}_1 \neq \mathbf{u}_2$ , we don’t need to keep worrying about the detailed form for  $(\partial f_1 / \partial t)_{coll,2}$ .

We’ve taken  $d^3\mathbf{p}_1$  fluid moments over the full Boltzmann equation, to obtain the fluid-moment conservation equations. In those, the collisional terms on the RHS will be in the form of the  $M_{12}$  moments discussed above.

(And, since the bulk of the particles in each distribution is “inside” the thermal core, we don’t need to worry much about the runaway effects associated with  $x \gg 1$ ... i.e., we can assume  $G(x) \propto x$ .)

The collisional source/sink term in the **mass** conservation equation is 0.

The collisional source/sink term in the **momentum** conservation equation looks like:

$$\frac{\partial \mathbf{u}_1}{\partial t} + \dots = -\bar{v}_{S1} (\mathbf{u}_1 - \mathbf{u}_2) \quad \text{where} \quad \bar{v}_{S1} = \frac{4\pi q_1^2 q_2^2 n_2 \ln \Lambda}{m_1 m_{12} v_{\text{eff}}^3}$$

and  $v_{\text{eff}}$  is the relative thermal spread between the two distribution functions,

$$v_{\text{eff}} = \sqrt{\frac{2k_B T_1}{m_1} + \frac{2k_B T_2}{m_2}}$$

(Why is it a plus? Recall how r.m.s. “errors” are summed in quadrature.)

Thus,  $\bar{v}_{S1}$  the rate at which momentum is *shared* between the two species. The collisions drive  $\mathbf{u}_1$  and  $\mathbf{u}_2$  together, like FRICTION.

As we saw earlier, there must then be a corresponding equation for  $\mathbf{u}_2$ , for which the total system conserves bulk momentum,

$$n_1 m_1 \frac{d\mathbf{u}_1}{dt} + n_2 m_2 \frac{d\mathbf{u}_2}{dt} = 0 .$$

Lastly, there’s the collisional source/sink term in the **energy** conservation equation. There are complicated versions of this for bi-Maxwellian distributions ( $T_{\parallel} \neq T_{\perp}$ ), but for now let us assume they always maintain isotropic Maxwellian shapes.

The internal (i.e., thermal) energy evolves collisionally like

$$\frac{\partial U_1}{\partial t} + \dots = \frac{\partial}{\partial t} \left( \frac{3}{2} n_1 k_B T_1 \right) = -\bar{v}_{E1} n_1 k_B (T_1 - T_2) \quad \text{where} \quad \bar{v}_{E1} = \frac{3m_{12}}{m_2} \bar{v}_{S1}$$

and this makes the system symmetric, so that thermal energy lost by one species is gained by the other,

$$\frac{dU_1}{dt} + \frac{dU_2}{dt} = 0 .$$

Those  $\bar{\nu}_{S1}$  rates are kind of complicated, but it will behoove us to think about their overall **magnitudes** in the simple case of a proton–electron plasma.

For  $n_p \approx n_e$  and  $T_p \approx T_e$ , let's look at all 4 combinations of the identities of species 1 and 2:

$$\bar{\nu}_{S1} = \underbrace{\frac{4\pi e^4 n \ln \Lambda}{(2k_B T)^{3/2} \sqrt{m_e}}}_{\text{constants}} \times \underbrace{\sqrt{\frac{m_e m_2}{m_1 (m_1 + m_2)}}}_{\text{dimensionless function } \equiv f_{12}}$$

Plugging in the numbers, we see that

particle 1	particle 2	$f_{12}$	
$e$	$e$	0.707	$(1/\sqrt{2})$
$e$	$p$	1	
$p$	$p$	1/60	$(\sqrt{m_e/2m_p})$
$p$	$e$	1/1836	$(m_e/m_p)$

Thus,

- Electrons collide “with themselves” (to establish their own Maxwellian) **very rapidly**,
- and they also force themselves into equilibrium with a pre-existing proton distribution **just as quickly**.
- Protons have higher masses (slower  $v_{th}$  & more inertia), so it **takes longer** for them to come into equilibrium with themselves.
- And it takes **even longer** for those heavy protons to finally feel the effects of the light, zippy electrons and come into equilibrium with them (i.e., like trying to turn a cruise ship by throwing baseballs at it).

Okay, back to the **generalized Ohm's law**. Let's write the momentum equation for just the electrons by themselves...

$$m_e n_e \left( \frac{D\mathbf{u}_e}{Dt} - \mathbf{g} \right) + \nabla P_e + e n_e \left( \mathbf{E} + \frac{\mathbf{u}_e}{c} \times \mathbf{B} \right) = 0 ?$$

$$= \sum_{i \neq e} n_e m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e)$$

where the RHS term represents collisional **friction** as outlined above.

As "usual" in physics, frictional force is proportional to  $\Delta \mathbf{u}$  between each pair of species with the electrons.

When  $\mathbf{u}_i$  is faster than  $\mathbf{u}_e$ , electrons are sped up.

When  $\mathbf{u}_i$  is slower than  $\mathbf{u}_e$ , electrons are dragged down.

We're simplifying life by just writing the collision rate as  $\nu_{ei}$ , without the over-bars or other specifiers from Coulomb collision theory.

Also, we know that

assuming  $\rho_c = 0$  means that  $n_e = \sum_{i \neq e} Z_i n_i$  (recall  $Z_i = \frac{q_i}{e}$ ).

Let's make a trial assumption that all  $\mathbf{u}_i$  are equal to one another (but *not* necessarily =  $\mathbf{u}_e$ ). In that case, the RHS is

$$\sum_{i \neq e} n_e m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e) = n_e m_e \left( \mathbf{u}_i \sum_{i \neq e} \nu_{ei} - \mathbf{u}_e \sum_{i \neq e} \nu_{ei} \right) = n_e m_e \nu_{\text{tot}} (\mathbf{u}_i - \mathbf{u}_e)$$

whereas  $\mathbf{J} = e \left( \sum_{i \neq e} Z_i n_i \mathbf{u}_i - n_e \mathbf{u}_e \right) = e n_e (\mathbf{u}_i - \mathbf{u}_e)$

and thus the RHS is

$$\sum_{i \neq e} n_e m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e) = \boxed{\frac{m_e \nu_{\text{tot}}}{e} \mathbf{J}} \quad \text{where} \quad \nu_{\text{tot}} = \sum_{i \neq e} \nu_{ei}$$

.....

We can now go back to the full electron momentum equation.

**Bold claim:** the magnitude of the first ( $m_e n_e$ ) term is negligibly small.

Why? There's also a *proton* momentum equation, and its RHS forces are of similar magnitude to the ones here. Those forces are balanced by an  $m_p n_p$  advection term! Thus, the  $m_e n_e$  term here is small potatoes.

Neglecting the  $m_e n_e$  term, the equation is

$$\nabla P_e + en_e \left( \mathbf{E} + \frac{\mathbf{u}_e}{c} \times \mathbf{B} \right) \approx \frac{m_e \nu_{\text{tot}}}{e} \mathbf{J}$$

which we can solve for  $\mathbf{E}$  to get a decent version of the generalized Ohm's law,

$$\mathbf{E} = -\frac{\mathbf{u}_e}{c} \times \mathbf{B} - \frac{\nabla P_e}{en_e} + \eta \mathbf{J} \quad \text{where resistivity } \eta \equiv \frac{m_e \nu_{\text{tot}}}{e^2 n_e} .$$

The expression for  $\eta$  is the basis of "Drude's theory of metals," whose dimensional analysis was worked out in 1900, prior to anyone knowing atoms were real!

Above,  $\mathbf{E}$  is the electric field needed to maintain charge neutrality ( $\rho_c = 0$ ).

Are we done yet? Not quite. Note the presence of  $\mathbf{u}_e$  above. Earlier, we saw that the bulk-MHD speed  $\mathbf{u} \approx \mathbf{u}_i$ , since it's weighted by mass.

Thus, it would be great to have a version of Ohm's law that uses  $\mathbf{u}$  and not  $\mathbf{u}_e$ . Why not just add "zero" to the above definition of  $\mathbf{E}$ ? One way to write zero is:

$$-\frac{\mathbf{u}_i}{c} \times \mathbf{B} + \frac{\mathbf{u}_i}{c} \times \mathbf{B}$$

so if  $\mathbf{u}_i \approx \mathbf{u}$ , the Ohm's law electric field can be written as

$$\boxed{\mathbf{E} = -\frac{\mathbf{u}}{c} \times \mathbf{B} + \frac{1}{en_e} \left[ -\nabla P_e + \frac{\mathbf{J} \times \mathbf{B}}{c} \right] + \eta \mathbf{J}}$$

since we know that  $\mathbf{J} = en_e(\mathbf{u}_i - \mathbf{u}_e)$ .

Let's go through the terms one by one:

- 1st term: **motional electric field**, associated with the reference frame of the electron fluid. (See below for a more intuitive way to think about it?)
- 2nd term: **electron pressure-gradient term**, which helps maintain overall charge neutrality when there's a pressure gradient (like in a hydrostatic atmosphere).
- 3rd term: **Hall term**, due to currents induced by electrons and ions having slightly different velocities.
- 4rd term: **Ohmic term** (due to collisional friction) that we'll see leads to currents getting dissipated and magnetic fields "diffusing away."

Many traditional surveys of MHD start out by ignoring the 2nd & 3rd terms, but there are applications (e.g., protoplanetary disks!) where they are important.

In any case, we often see Ohm's law written as

$$\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} = \mathbf{E}' \approx \eta \mathbf{J}$$

with  $\mathbf{E}'$  being the electric field in the plasma's co-moving frame of reference.

This is similar to  $V = IR...$  voltage ( $\mathbf{E}'$ ) = current ( $\mathbf{J}$ )  $\times$  resistance ( $\eta$ ).

Also, the *electrical conductivity*  $\sigma$  is defined as

$$\sigma = \frac{1}{\eta} \quad \text{and an alternate version of Ohm's law is} \quad \mathbf{J} = \sigma \mathbf{E}' .$$

Many MHD plasmas can be approximated as "good conductors," with  $\sigma \rightarrow \infty$  (i.e.,  $\eta = 0$ ) and thus  $\mathbf{E}' = 0$ . This infinite conductivity approximation is sometimes called the **hydromagnetic approximation**, or **Ideal MHD**. In this limit, the  $\mathbf{E}$ -field is always completely "shorted out."

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One more aside about the “induced” electric field,  $\mathbf{E} = -\mathbf{u} \times \mathbf{B}/c$ .

It shouldn’t be surprising that there’s a component to  $\mathbf{E}$  that goes away in the *co-moving frame of reference* of the plasma.

Einstein’s special relativity says that the partition of the electromagnetic field into  $\mathbf{E}$  &  $\mathbf{B}$  is frame-dependent. A force perceived by one “observer” as electric may be perceived by another observer as magnetic, or a mix.

In fact, we can understand this concept even *without* plasmas or MHD.

Consider a point-charge  $q$  in a vacuum, sitting near some collection of currents that generate a magnetic field  $\mathbf{B}$ .



If the charge is moving with velocity  $\mathbf{v}$ , it’s clear that it feels a magnetic Lorentz force of  $\mathbf{F} = q\mathbf{v} \times \mathbf{B}/c$ .

However, consider looking at the system in an inertial frame moving with the particle. In this frame, the particle is at rest, and the source of magnetic field is moving away...



The force on the charge *must* be the same, independent of frame, but in *this* frame the particle is not moving. The only other Lorentz force “available” to the particle is the electrostatic force,  $\mathbf{F} = q\mathbf{E}$ .

Thus, the only possible value for this induced electric field is  $\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$ .

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Anyway, we now know how to fill in the last major piece of the MHD puzzle: how do flows, currents, and field lines actually evolve in time together? To close the system, we need to derive the **magnetic induction equation**.

Start with Faraday's law, and insert  $\mathbf{E}$  from Ohm's law,

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \quad \text{with} \quad \mathbf{E} \approx -\frac{\mathbf{u}}{c} \times \mathbf{B} + \eta \mathbf{J} .$$

Thus, 
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - c \nabla \times (\eta \mathbf{J})$$

and we use Ampère's law again to replace  $\mathbf{J}$  (i.e.,  $\mathbf{J} = c \nabla \times \mathbf{B} / 4\pi$ ),

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times [D_B (\nabla \times \mathbf{B})]$$

where 
$$D_B = \frac{c^2 \eta}{4\pi} = \frac{c^2}{4\pi \sigma}$$
 is the magnetic diffusion coefficient.

Warning: some books use the symbol  $\eta$  for  $D_B$ . (In SI units,  $D_B = \eta / \mu_0$ .)

A common simplification is to assume  $D_B$  is constant in space, which lets us take it outside the derivative. Then, more vector identities help us rewrite the 2nd derivative of  $\mathbf{B}$  (taking into account that  $\nabla \cdot \mathbf{B} = 0$ ), to get

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + D_B \nabla^2 \mathbf{B}} , \quad \text{the (usual) magnetic induction equation.}$$

**The 1st term** tells us that the magnetic field lines are “frozen in” to the velocity/flow field.

Why? Let's look at it again in the **ideal** limit that  $D_B \rightarrow 0$ . We can go back to Faraday's law and combine it with the induction equation...

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \quad \rightsquigarrow \rightsquigarrow \quad \nabla \times (\mathbf{u} \times \mathbf{B}) = -c \nabla \times \mathbf{E}$$

$$\text{i.e.,} \quad \nabla \times \left( \mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right) = 0$$

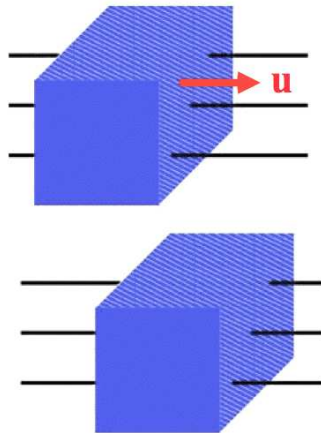
which means that  $\nabla \times \mathbf{E}' = 0$ , or that  $\partial \mathbf{B}' / \partial t = 0$  *in a frame moving with velocity  $\mathbf{u}$* . The vector field (i.e., the pattern of field lines) flows together with fluid parcels.



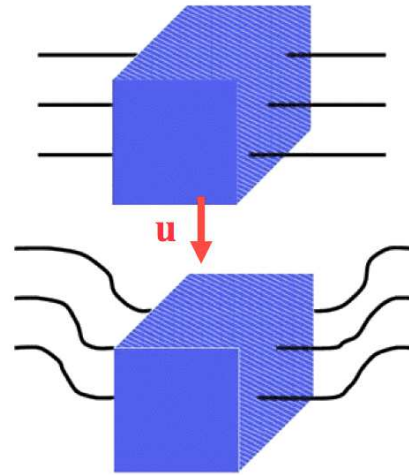
In this ideal limit ( $\eta \rightarrow 0$ ), the currents that make flux-freezing possible flow “instantly” (super-efficiently) wherever they want to go; nothing stops them. Nice cartoon from Cornelius Dullemond:

## Flux freezing: MHD in a nutshell

Strong field: matter can only move along given field lines (beads on a string):



Weak field: field lines are forced to move along with the gas:



Field lines aren't necessarily “fictitious” things like we're told in some E&M classes. They're identifiable by the plasma that's bound to them.

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**The 2nd term** tells us how magnetic field lines *diffuse* or dissipate away. When  $\mathbf{u} = 0$ , it's a diffusion equation.  $\mathbf{B}$  just dissipates away with time.

Note that  $D_B$  has units of a diffusion coefficient:  $\frac{\ell^2}{t}$  .

**How far** will this diffusion spread in a given time  $t$ ? Just use classical diffusion theory, to look at the *r.m.s.* (most probable) distance:

$$R \approx \sqrt{2D_B t} \quad \implies \quad t = t_{\text{diff}} \sim \frac{R^2}{D_B}$$

and  $t_{\text{diff}}$  is the time over which a magnetic field will diffuse *out* of an object of size  $R$ . Thus, if we see a strong  $B$ -field in an object of size  $R$  that is much OLDER than  $t_{\text{diff}}$ , we know something's up! There must be some kind of **dynamo** maintaining and/or regenerating the field. Otherwise it would have all diffused away.

When  $D_B \neq 0$ , then magnetic energy is continually “destroyed” by friction-like particle collisions. Where does that energy go? Into **Joule heating**. (Resistors in a circuit heat up.)

We've seen how the presence of  $\mathbf{J}$  gives rise to an irreversible loss of E&M energy in a system:

$$\frac{\partial}{\partial t} \left( \frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right) + \dots = -\mathbf{J} \cdot \mathbf{E} \approx -\eta |\mathbf{J}|^2 .$$

**Total** energy conservation demands that this be balanced by a net heating rate on the RHS of the thermal energy equation (i.e.,  $\partial U / \partial t + \dots = +\mathbf{J} \cdot \mathbf{E}$ ).

On a micro-scale, when  $D_B \neq 0$ , collisions kick particles away from their initial Larmor orbits, so field lines & particle streamlines “slip past one another.”

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We'll put aside  $\eta$  and  $D_B$  for now and think about **ideal MHD** for a while.

In the ideal MHD limit, we should note how flux-freezing allows for magnetic field **amplification**.

The ideal induction equation can be written

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{u}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{u}) + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}$$

The 1st term = 0, and the 4th term can be moved to the left side:

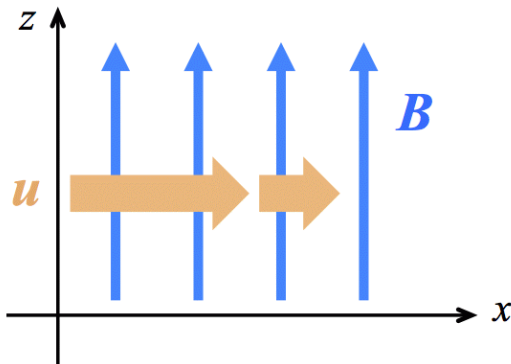
$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u})$$

Following a parcel in motion, there can be

**stretching/tilting** or **compression/rarefaction**.

Both can amplify  $\mathbf{B}$ ... i.e.,  $D\mathbf{B}/Dt > 0$ .

This leads to two interesting thought experiments. One I'll discuss here; the other one is for homework.



Consider initially straight field lines, with

$$\mathbf{B} = \{0, 0, B_0\} \quad (\text{init. cond.})$$

$$\mathbf{u} = \{u_x(x), 0, 0\} \quad (\text{true for all } t)$$

$(\mathbf{B} \cdot \nabla)\mathbf{u} = 0$  because  $B_x = 0$  and  $\mathbf{u}$  doesn't depend on  $z$ .

Thus, the 2nd term above survives:

$$\frac{D\mathbf{B}}{Dt} = -\mathbf{B} \frac{\partial u_x}{\partial x} \quad \text{with RHS}=0 \text{ for } x \text{ and } y \text{ components.}$$

Thus,

$$\frac{DB_x}{Dt} = 0 \quad \frac{DB_y}{Dt} = 0 \quad \frac{DB_z}{Dt} = -B_z \frac{\partial u_x}{\partial x}$$

$B_x$  and  $B_y$  start at 0 and remain at 0. For  $B_z$ , if  $\partial u_x / \partial x < 0$  (i.e., if  $u_x$  decreases as one moves to the right), then  $B_z$  grows.

Thus, **compression** (i.e., field lines getting “bunched up”) enhances  $|\mathbf{B}|$ .

This is one way of explaining why  $|\mathbf{B}|$  is often called the “magnetic flux density” (i.e., the density of field lines squeezed into a given parcel of space).

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We’ve now derived the MHD system of equations, which are interdependent in the following way:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \text{function of } \rho, \mathbf{u} \\ \frac{\partial \mathbf{u}}{\partial t} &= \text{function of } \rho, \mathbf{u}, P, \mathbf{B} \quad (\& \text{ viscosity}) \\ \frac{\partial P}{\partial t} &= \text{function of } \rho, \mathbf{u}, P \quad (\& \text{ conduction}) \\ \frac{\partial \mathbf{B}}{\partial t} &= \text{function of } \mathbf{u}, \mathbf{B} \quad (\& \text{ resistivity})\end{aligned}$$

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