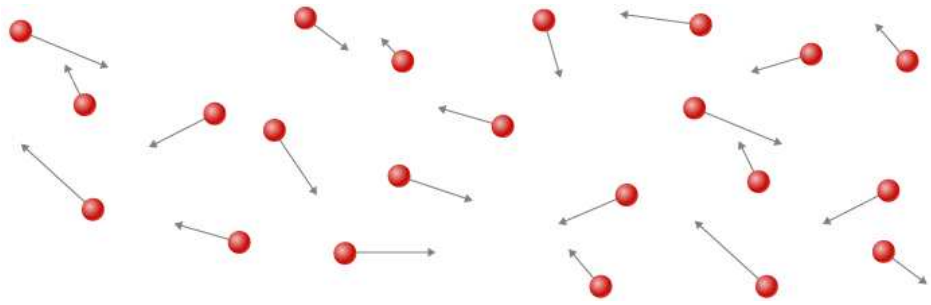
Statistical Mechanics & Kinetic Physics

We need to consider what's going on with a collection of a large number of particles. We'll be applying the math of probabilities to gases, in which particles are moving around randomly.

Outline: (A) Probabilistic motivations
(B) The 6D phase-space distribution function
(C) Time evolution: the Boltzmann/Vlasov equation
(D) Coulomb collisions

(A) Probabilistic Motivations

We will begin by thinking about the **ideal gas,** which is a gas of point-like particles that carry only kinetic energy, and which tend to stay widely separated from one another.



("Real gases" have complications like molecular vibrations & Coulomb forces that attract or repel when they get close.)

Thus, in an ideal gas, each particle has a mass m and a velocity $\mathbf{v}$.
The direction of the velocity vector is random.

The magnitude of the velocity vector (v) is describable by its kinetic energy

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) \qquad \text{(i.e., ignore relativity, so } \mathbf{p} = m\mathbf{v}\text{)}.$$

We will use E for kinetic energy here, because for now it's the **only** component of energy that we're considering for the ideal gas.

What's the **probability distribution of kinetic energy** for a room-full of particles of an ideal gas? In other words, if we pick out one random particle, what are the chances of finding its kinetic energy E in some range of values?

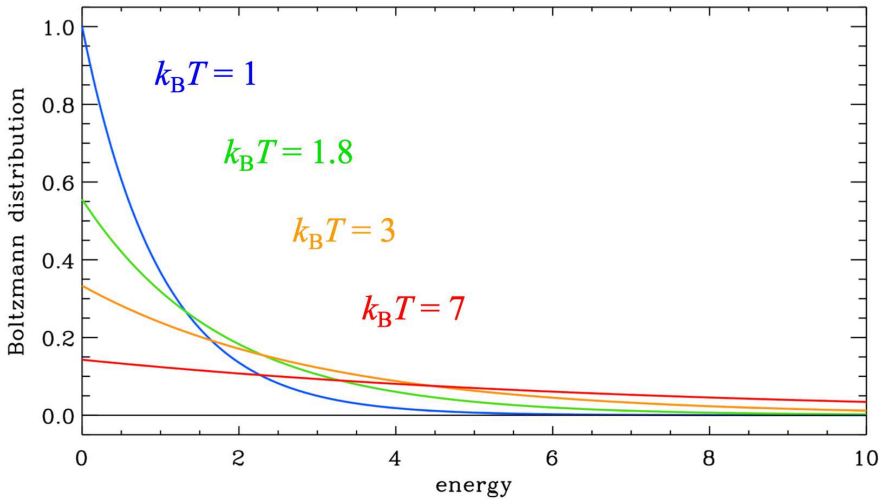
Although there are many forms the probability distribution can take, we'll start with the typical (non-relativistic, non-quantum) case of the equilibrium Boltzmann distribution. We'll discuss a few thought-experiments to help us understand why it takes the form it does.

In 1869, Ludwig Boltzmann first formulated the distribution as a proportionality, not an exact equality. The Boltzmann distribution is:

$$\boxed{\mathbb{P}(E) \propto e^{-E/k_{\mathrm{B}}T}} \qquad \text{where } k_{\mathrm{B}} = \text{Boltzmann's constant} \approx 1.38 \times 10^{-23}\ \mathrm{J/K}.$$

Here is where we introduce the concept of temperature T, and we'll always use units of Kelvins for it. The combination $k_{\mathrm{B}}T$ is an energy.

For finite T, it's more likely to find **low**-energy particles, and less likely to find **high**-energy particles. T tells us how *spread out* the distribution is:



Why does the Boltzmann distribution look the way it does?

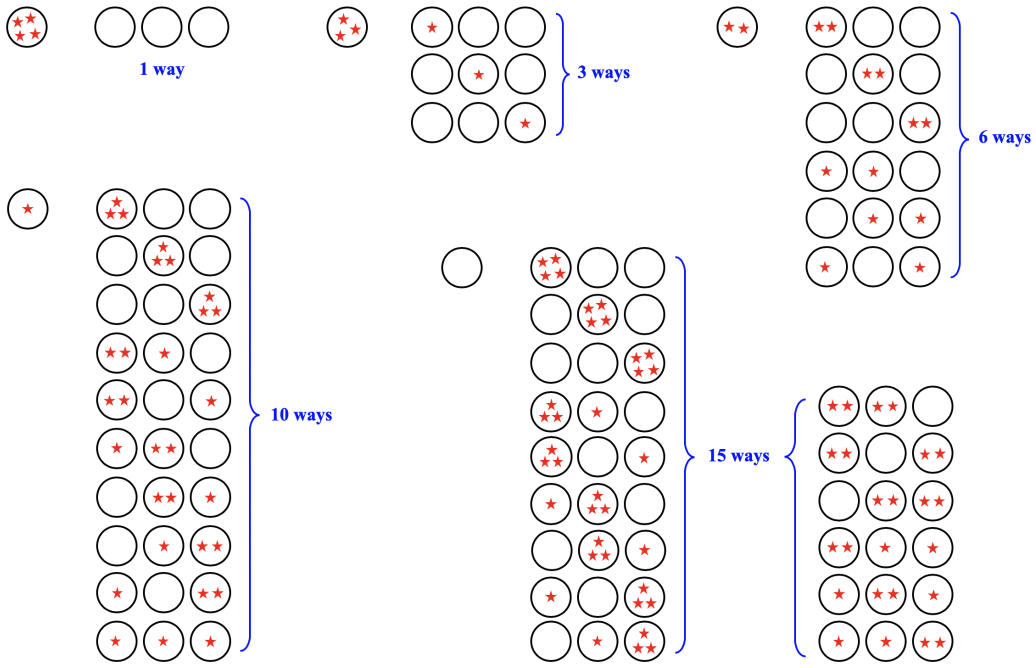
It may help to reduce it down to a smaller system: Let's say we have 4 particles, which are able to "pass around" 4 bits of energy.

Suppose that all possible "ways" of distributing the bits among the 4 particles are equally likely.

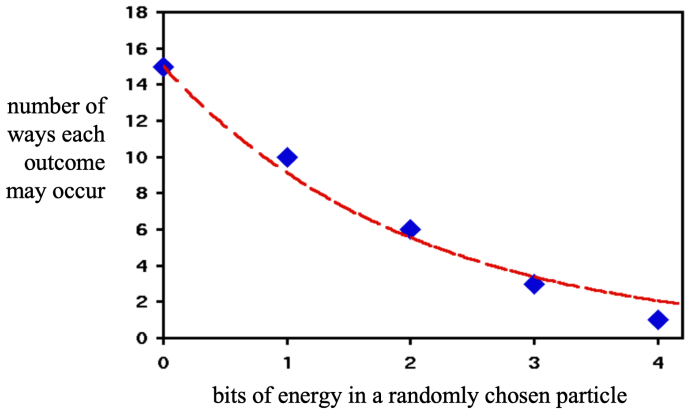
What's the probability that a randomly chosen particle will have 0, 1, 2, 3, or all 4 of those bits?

Let's go from high to low, and just count up the possible combinations:

- If the first particle has **all 4 bits,** there must be 0 in each of the others. There is only **1 way** to do this.
- If the first particle has **3 bits,** that 1 remaining bit of energy can be in any of the 3 other particles. Thus, **3 ways.**
- If the first particle has **2 bits,** there could be 2 in any one of the others (3 ways), or 1 with none and 1-each in the other two (3 ways), thus **6 ways** in all.
- We can count the rest by drawing them...



Thus, for 0/1/2/3/4 bits per particle, there are 15/10/6/3/1 ways:



More ways = higher probability. Same shape as the Boltzmann distribution.

If we started with 8 energy-bits for just 4 particles, the curve would look like a "higher-T" (more spread-out) version of the Boltzmann distribution.

In all cases, $E = 0$ is the most likely value for a particle's energy, because that allows the maximum number of arrangements for energy "bits" among the **other** particles.

There's another thought-experiment that may help one understand how the Boltzmann distribution arises from random **exchanges** of energy between particles (i.e., from collisions). I'll just point you to the paper that describes the **"coconuts & islanders"** story: https://arxiv.org/abs/1904.04669

(B) The 6D Phase-Space Distribution Function

In order to describe the physics of a gas, let's define another quantity that is related to the probability distribution:

The **phase-space density** (sometimes called the "single-particle distribution function") is written as:

$$f(\mathbf{r}, \mathbf{p}, t)$$

It's kind of a complete description of the system, which contains the most amount of information about a given species of particles that's possible to know (but still less detailed than if you were to follow the exact positions and velocities of **each** particle vs. time.

What are the units of f? Consider a number of particles in a 6-dimensional "box" that extends...

$$\left\{ \begin{array}{c} \text{from } x \text{ to } (x + dx) \\ \text{from } y \text{ to } (y + dy) \\ \text{from } z \text{ to } (z + dz) \\ \text{from } p_x \text{ to } (p_x + dp_x) \\ \text{from } p_y \text{ to } (p_y + dp_y) \\ \text{from } p_z \text{ to } (p_z + dp_z) \end{array} \right\}$$

The total number of particles in that box is defined as

$$N \;=\; f(\mathbf{r}, \mathbf{p}, t)\; dx\; dy\; dz\; dp_x\; dp_y\; dp_z \;=\; f(\mathbf{r}, \mathbf{p}, t)\; d^3\mathbf{r}\; d^3\mathbf{p}$$

$d^3\mathbf{r}$ is a 3D volume of physical space.
$d^3\mathbf{p}$ is a 3D volume of momentum space.
$d^3\mathbf{r}\ d^3\mathbf{p}$ is a 6D "volume" of phase space.

In other words, if we call $\Delta V = d^3\mathbf{r}\ d^3\mathbf{p}$, then

$$f(\mathbf{r},\mathbf{p},t) \;=\; \lim_{\Delta V \to 0} \left[\frac{\#\text{ of particles in } \Delta V}{\Delta V} \right] \quad .$$

Notes:

- We should really say f_s instead of just f; each particle "species" s has its own distribution function. For hydrogen plasma, $\boxed{s = p, e}$.
- Since much of this course's dynamics is non-relativistic ($\mathbf{p} = m\mathbf{v}$), it's often equivalent to express f as a function of velocity space...

$$d^3\mathbf{p} \;=\; m^3 d^3\mathbf{v}$$

Of course, we should never forget that the gas/plasma is a collection of discrete particles. Putting aside Heisenberg, an exact "grainy" expression for the distribution function is

$$f(\mathbf{r},\mathbf{p},t) \;=\; \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_i(t)]\ \delta[\mathbf{p} - \mathbf{p}_i(t)]$$

with maybe a $(2\pi)^3$ out front, depending on how the 3D delta functions are normalized. The sum over i is over **all** N particles in the system.

Since N is usually bigger than Avogadro's number, and there are typically still $>>>1$ particles even in the littlest boxes ($d^3\mathbf{r}$ and $d^3\mathbf{p}$), it's usually easier to treat f as a continuous probability density.

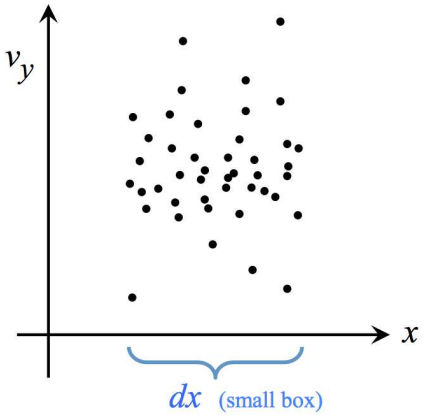
. . . . . .

This 6D phase space is kind of esoteric. Let's try to process what it means.

For just 1 particle, it's obvious that each of the 6 main parameters is *independent* of the others:

$$x(t)\ ,\quad y(t)\ ,\quad z(t)\ ,\quad p_x(t)\ ,\quad p_y(t)\ ,\quad p_z(t)\ .$$

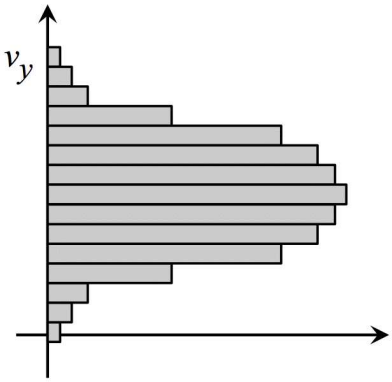
Now think about a small "parcel" of gas. Pick any pair of parameters:



No correlation between the 2 parameters.

Particles are randomly distributed in x.

But what about v_y? Collapse it into a histogram, and a pattern emerges…



We could do similar things with the other coordinates. If our brains were able to visualize higher dimensions, we could plot > 3 of them versus one another.

The (x, y, z) position coordinates help us specify properties "here," at a given location in space. Very similar to how classical fields (say, potential energy) depend on spatial position.

However, the (p_x, p_y, p_z) momentum coordinates are, in a sense, random and "microscopic," so we don't care about the details of which particle has which value of p_y (i.e., v_y), or whatever.

Typically, the things we **measure** come from integrating over the full "volume" of momentum space. What's left are quantities that depend on (x, y, z) and on *SOME* statistical aspects of the distribution in (p_x, p_y, p_z). Just not the gory details.

These integrated quantities are called **moments** of f . . . .

0th moment: number density $\qquad n(\mathbf{r}, t) = \int d^3\mathbf{p}\ f(\mathbf{r}, \mathbf{p}, t)$

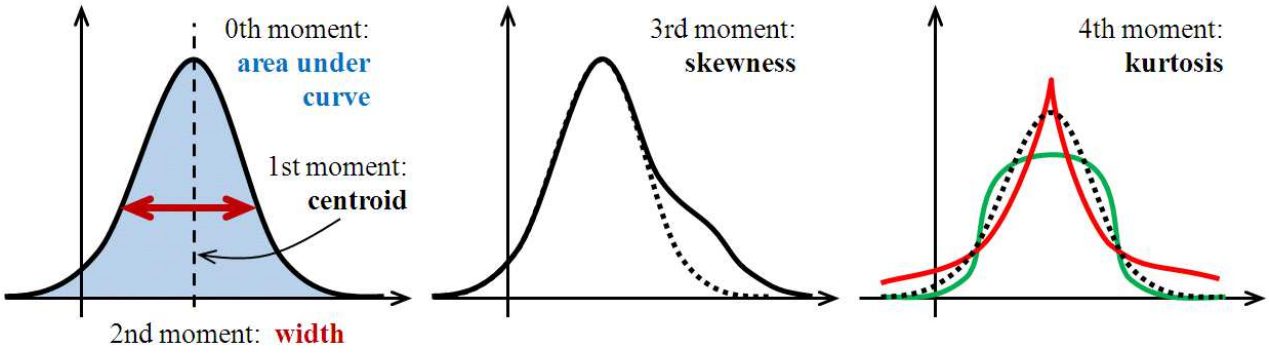
Here, the units of n are $(\#/\mathrm{m}^3)$. It's the number of particles per unit volume (of 3D physical space). The mass density ρ is just mn $(\mathrm{kg}/\mathrm{m}^3)$.

Note that n is still a function of $\mathbf{r}$ (position) & time.

Higher moments of f reveal more information. These are essentially weighted averages of the form

$$\langle \phi(\mathbf{p}) \rangle \;=\; \frac{\int d^3\mathbf{p}\;\; \phi(\mathbf{p})\; f(\mathbf{p})}{\int d^3\mathbf{p}\;\; f(\mathbf{p})} \;=\; \frac{1}{n}\int d^3\mathbf{p}\;\; \phi(\mathbf{p})\; f(\mathbf{p})$$

where ϕ is any function of the three momentum coordinates p_x, p_y, and p_z (or, often, the magnitude p).



Thus, if $\phi = \mathbf{p}$, we get the **1st moment,** or the mean momentum of the "bulk flow,"

$$\mathbf{p}_0 \;=\; \langle \mathbf{p} \rangle \qquad\qquad \text{more often, we use:} \qquad \mathbf{u} \;=\; \langle \mathbf{v} \rangle$$

When we deal with a gas as a continuous FLUID, we won't work with the "microscopic" velocity $\mathbf{v}$ at all anymore; we will only *squint our eyes* to see the bulk/centroid flow vector $\mathbf{u}$.

The **2nd moment** opens the door to thermodynamics... by measuring the standard deviation about the mean, or the "width" of the distribution in momentum or velocity space.

$$\left\{ \begin{array}{c} \text{internal energy} \\ \text{temperature} \\ \text{pressure} \end{array} \right\} \qquad \text{i.e.,} \quad \widetilde{U} \;=\; \langle E_{\mathrm{K}} \rangle \;\approx\; \left\langle \frac{1}{2} m v^2 \right\rangle \;\approx\; \left\langle \frac{p^2}{2m} \right\rangle$$

We'll define these more precisely later.

The **3rd moment** measures how f is skewed...

$$\text{heat flux} \;\sim\; \langle p^2 \mathbf{p} \rangle \qquad (\textit{net flux of kinetic energy in a particular direction})$$

...and this feature of $f(\mathbf{p})$ encodes **transport phenomena** such as heat conduction, friction, viscosity, and electrical resistivity.

The **4th moment** measures the so-called "kurtosis" of f... true departures from classical equilibrium thermodynamics.

For an equilibrium (e.g., Maxwell-Boltzmann) distribution, we'll have to deal only with the 0th, 1st, and 2nd moments.

. . . . . .

By this point, it's also now possible to define charge density & current density rigorously in terms of f.

Consider multiple charged species s (ignore neutrals):

$$\rho_c = \sum_s q_s n_s = \sum_s q_s \int d^3\mathbf{p}\ f_s \qquad \text{(charge density)}$$

$$\mathbf{J} = \sum_s q_s n_s \mathbf{u}_s = \sum_s q_s \int d^3\mathbf{p}\ \mathbf{v}\ f_s \qquad \text{(current density)}$$

Thus, Maxwell + Lorentz + Vlasov = a complete description of a plasma !?

Usually, in a plasma (on scales much bigger than λ_D), ρ_c is usually ≈ 0.

However, $\mathbf{J}$ can often be $\neq 0$.

(C) Time evolution: the Boltzmann/Vlasov equation

How does $f(t, \mathbf{r}(t), \mathbf{p}(t))$ evolve in time?

Let's derive a conservation equation for f. Essentially all other macro-scale conservation laws are derived from it.

There are several ways to derive it. One straightforward way is to think about how a small 6D box evolves from t_0 to $t_1 \equiv t_0 + dt$.

Consider a bunch of particles of the same species, that all start out in the nearby vicinity of position $\mathbf{r}_0$ and with velocity $\mathbf{v}_0$ (i.e., $\mathbf{p}_0 = m\mathbf{v}_0$).

Let's use subscript 0 for initial time; subscript 1 for final. If no particles are created or destroyed, then the total # of particles "in the box" should be the same at both times:

$$\# \;=\; f(\mathbf{r}_0, \mathbf{p}_0, t)\, d^3\mathbf{r}_0\, d^3\mathbf{p}_0 \;=\; f(\mathbf{r}_1, \mathbf{p}_1, t+dt)\, d^3\mathbf{r}_1\, d^3\mathbf{p}_1$$

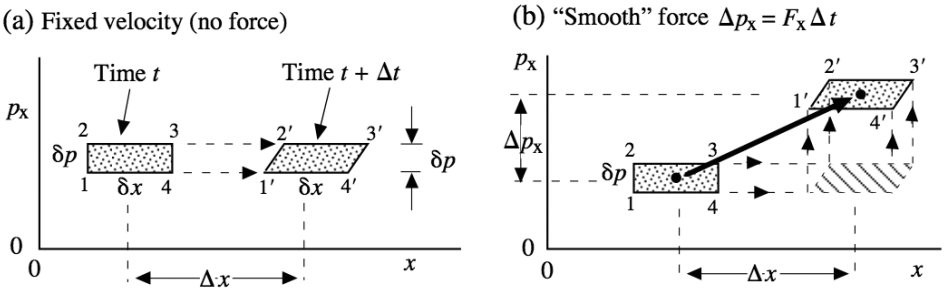
However, the box can move around in ($\mathbf{r}$, $\mathbf{p}$) space. In position space, the box is pushed with velocity $\mathbf{v}$, so that

$$\mathbf{r}_1(t+dt) \;=\; \mathbf{r}_0(t) + \mathbf{v}\, dt$$

and it's pushed in momentum space by a net force ($\mathbf{F} = m\mathbf{a} = d\mathbf{p}/dt$),

$$\mathbf{p}_1(t+dt) \;=\; \mathbf{p}_0(t) + \mathbf{F}\, dt$$

If the "forces" that push it around vary smoothly across the (tiny) box, then the box may be be stretched or distorted, but it's 6D volume won't change. Why? See 2D example:



This is an example of **Liouville's theorem** in classical mechanics; see Bradt & Olbert's notes (on web) for a deeper set of proofs of why this is true.

When this applies,

$$d^3\mathbf{r}_0\, d^3\mathbf{p}_0 \;=\; d^3\mathbf{r}_1\, d^3\mathbf{p}_1 \qquad \text{and thus} \qquad f(\mathbf{r}_0, \mathbf{p}_0, t) \;=\; f(\mathbf{r}_1, \mathbf{p}_1, t+dt)\;.$$

There's another way to estimate f at the new time, based on what we know about f at the old time. If the time step dt is tiny, then we can estimate f at the new time by **Taylor-expanding** around the old time:

$$f(\mathbf{r}_1, \mathbf{p}_1, t+dt) \;=\; f(\mathbf{r}_0, \mathbf{p}_0, t) \;+\; \left(\frac{df}{dt}\right)_0 dt \;+\; \cdots$$

and, if no particles are created or destroyed, then $\left(\dfrac{df}{dt}\right)_0 = 0\;.$

This isn't just a simple 1D derivative, though. As $t \to t + dt$, both $\mathbf{r}$ and $\mathbf{p}$ change, too. Thus, to compute it, we need to expand the total time derivative of

$$f\Big(t, x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)\Big)$$

via the chain rule...

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{dx}{dt}}_{v_x}\frac{\partial f}{\partial x} + \underbrace{\frac{dy}{dt}}_{v_y}\frac{\partial f}{\partial y} + \underbrace{\frac{dz}{dt}}_{v_z}\frac{\partial f}{\partial z} + \underbrace{\frac{dp_x}{dt}}_{F_x}\frac{\partial f}{\partial p_x} + \underbrace{\frac{dp_y}{dt}}_{F_y}\frac{\partial f}{\partial p_y} + \underbrace{\frac{dp_z}{dt}}_{F_z}\frac{\partial f}{\partial p_z}$$

(Newton's law: $\mathbf{F} = \mathrm{m}\mathbf{a} = \mathrm{d}\mathbf{p}/\mathrm{dt}$)

The $\mathbf{v}$ terms involve transport in & out of a "physical box."
The $\mathbf{F}$ terms involve acceleration/deceleration in & out of a "momentum box."

If $\dfrac{df}{dt} = 0$, **Vlasov equation:** particles are shuffled around in $(\mathbf{r}, \mathbf{v})$ phase space, but no particles are created or destroyed.

If $\dfrac{df}{dt} \neq 0$, There are sources & sinks (ionization, recombination, dust sublimation, Coulomb collisions); for just collisions, it's the **Boltzmann equation.**

Let's write the Vlasov equation in vector form...

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{F}\cdot\nabla_p f = 0} \qquad \text{i.e.,} \qquad \boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}\cdot\frac{\partial f}{\partial \mathbf{p}} = 0}$$

and keep in mind that $\mathbf{F} = m\mathbf{a}$ is just any "external" applied force that corresponds to an acceleration $\mathbf{a}$, like gravity.

. . . . . .

Later, we will take moments of the entire equation by multiplying each term by a given variable and integrating each term over $d^3\mathbf{p}$.

Lecture 5: fluid/MHD moments: $\left\{ \begin{array}{rl} \text{0th:} & \text{mass conservation} \\ \text{1st:} & \text{momentum conservation} \\ \text{2nd:} & \text{energy conservation} \end{array} \right\}$

(D) Coulomb Collisions

We now have to think more about how charged-particle collisions affect the distribution function.

Ideally, no particles are created or destroyed in small-enough boxes in 6D phase space (as long as we think of Lagrangian "moving boxes").

However, collisions occur on micro-scales that we often wish to ignore.
On average, the inter-particle E&M forces are too random to be considered as straightforward **F** terms in the Vlasov equation.

In practice, the $f(\mathbf{r}, \mathbf{v}, t)$ that we consider is a smooth **ensemble average** over a large number of trial samples (i.e., over lots of different realizations for the "grainy" delta-function version of f).

Performing that ensemble average is a topic for a stat–mech course, but I'll show you one way that collisions defy our intuition.

For charged particles in a plasma, think about the following term in the Vlasov equation:

$$\mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} \;=\; q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{p}}$$

and we'd like to work with "smooth" (ensemble-averaged) versions of the **E** and **B** fields, too.

Unfortunately, the above term contains products of fields with f itself.

Maxwell's equations say that **E** and **B** depend on both ρ_c and **J**; i.e., on positions and trajectories of charged particles $\implies$ **F** depends on f.

Thus, the above term is a product of two *partially correlated* quantities...

From our work on random walks, we thus know that

$$\left\langle \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} \right\rangle \;\neq\; \langle \mathbf{F} \rangle \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{p}}$$

(they would be = only if they were independent & uncorrelated!)

LHS: the "true" ensemble average term that we want to know.
RHS: $\langle \mathbf{F} \rangle$ represents smoothed E&M fields, and $\partial\langle f \rangle / \partial \mathbf{p}$ represents the evolution of a smoothed distrib. function. We'd prefer to work with these.

Thus, we write

$$\left\langle \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} \right\rangle \;=\; \langle \mathbf{F} \rangle \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{p}} \;-\; \mathcal{C}(f)$$

where $\mathcal{C}(f)$ is a **collision operator** that accounts for the subtle correlations that are strongest when charged particles fly by one another & interact electromagnetically.

Leaving off the brackets for simplicity (assuming ensemble averages on each term in the product), the collisional Boltzmann equation is written as

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \mathcal{C}(f)} \qquad \text{where we often use} \quad \mathcal{C}(f) \equiv \left(\frac{\partial f}{\partial t} \right)_{\rm coll}$$

The big problem: how do we evaluate $\mathcal{C}(f)$?

. . . . . .

We'll derive it "the right way" soon, but let me first prime your intuition by showing a useful approximation:

Max Krook (Bhatnagar, Gross, & Krook 1954, *Phys. Rev.,* 94, 511) worked out the so-called **BGK operator,**

$$\left(\frac{\partial f}{\partial t} \right)_{\rm coll} \approx \frac{f_0 - f}{\tau}$$

where τ is a collisional relaxation timescale, and f_0 is a "known" equilibrium distribution.

In the absence of spatial gradients & external forces, the Boltzmann equation has a simple analytic solution:

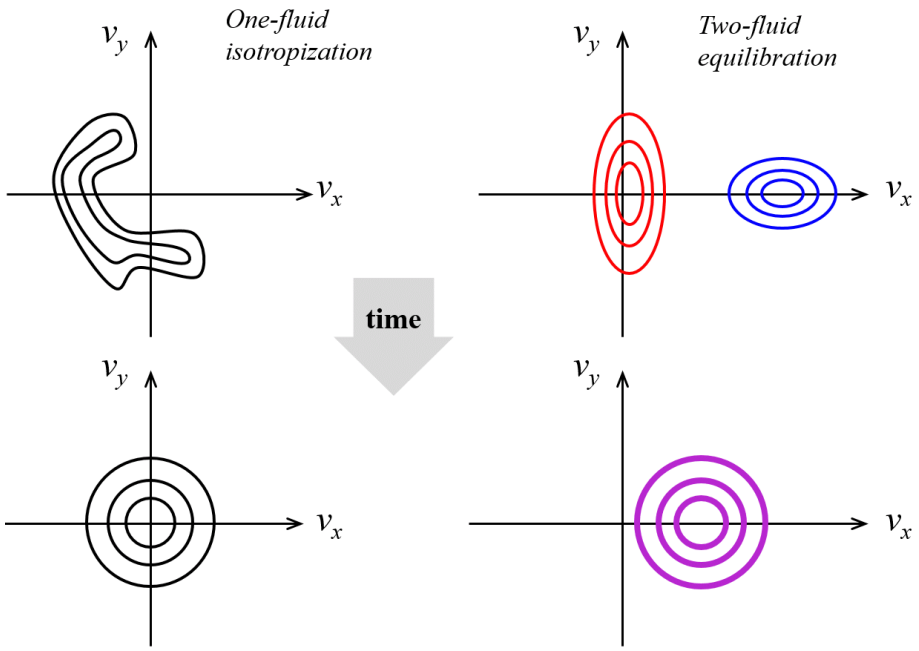
$$\text{For} \quad f = f_{\rm init} \quad \text{at} \quad t = 0, \qquad f(\mathbf{p}, t) \;=\; f_{\rm init}(\mathbf{p})\, e^{-t/\tau} \;+\; f_0(\mathbf{p}) \left(1 - e^{-t/\tau} \right)$$

i.e., f rapidly relaxes from $f_{\rm init}$ to f_0.

Downside: you need to know f_0 a-priori.

For many plasmas, f_0 will have a Maxwellian shape, but the values of its parameters (n, $\mathbf{u}$, T) may not be known.

Examples:



Later, we'll use the BGK operator to derive reasonable values for the plasma **transport coefficients.** For now, we should move on to the full Boltzmann collision operator.

. . . . . .

The collision operator is really **both** a source & a sink term.

When collisions occur, *some* particles inside the 6D "box" ($d^3\mathbf{r}\, d^3\mathbf{p}$) may be scattered out of the box, and some other particles may be scattered into it. For bookkeeping's sake, define:

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll}} = \mathcal{C}_{\text{in}}(f_1, f_2) - \mathcal{C}_{\text{out}}(f_1, f_2)$$

where we now must add subscripts. We follow particles of type 1, and our goal is to figure out how they interact with "target" particles of type 2.

$\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ are rates that describe how rapidly f gets altered (per second).

Formally, our Boltzmann equation looks like

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{r_1}} + \mathbf{F}_1 \cdot \frac{\partial f_1}{\partial \mathbf{p}_1} = \mathcal{C}_{\text{in}}(f_1, f_2) - \mathcal{C}_{\text{out}}(f_1, f_2) \ .$$

I often find it easier to do the "bookkeeping" when there are **finite numbers** of events. Thus, like before, let's deal with a specific time-span Δt.

Recalling how f is defined, we can state clearly that the absolute numbers of particles (of type 1) that are scattered in/out of the 6D box are

$$N_{\rm in} \;=\; \mathcal{C}_{\rm in}\, d^3\mathbf{r}_1\; d^3\mathbf{p}_1\; \Delta t$$

$$N_{\rm out} \;=\; \mathcal{C}_{\rm out}\, d^3\mathbf{r}_1\; d^3\mathbf{p}_1\; \Delta t$$

Notation: Collisions result in an ~abrupt change to the particle trajectories. Thus, for the INCOMING trajectories, we will use $\mathbf{v}_1$ and $\mathbf{v}_2$ (i.e., $\mathbf{p}_1 = m_1\mathbf{v}_1$, etc) in the inertial frame.

After the collision, for the OUTGOING trajectories, we will use $\mathbf{v}_1'$ and $\mathbf{v}_2'$.

Collisions occur when particles of type 1 & 2 share the same *physical* space, so

$$\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_1' = \mathbf{r}_2' \;\equiv\; \mathbf{r} \qquad\qquad d^3\mathbf{r}_1 = d^3\mathbf{r}_2 = d^3\mathbf{r}_1' = d^3\mathbf{r}_2' \;\equiv\; d^3\mathbf{r} \;\; .$$

How do we actually specify $N_{\rm in}$ and $N_{\rm out}$?

Start with $N_{\rm out}$. How many particles are scattered out of a tiny box of 6D phase space over time-span Δt?

Somewhere inside that box, *EACH* of the particle 1's that enters will undergo a collision with *EACH* of the particle 2's that are present in the box. Thus,

$$N_{\rm out} \;=\; N_{1,\rm enter}\, N_2 \;\; .$$

In other words, every pairwise collision *removes* a particle 1 from its little $d^3\mathbf{p}_1$ region of momentum space, and also *removes* a particle 2 from its little $d^3\mathbf{p}_2$ region of momentum space,

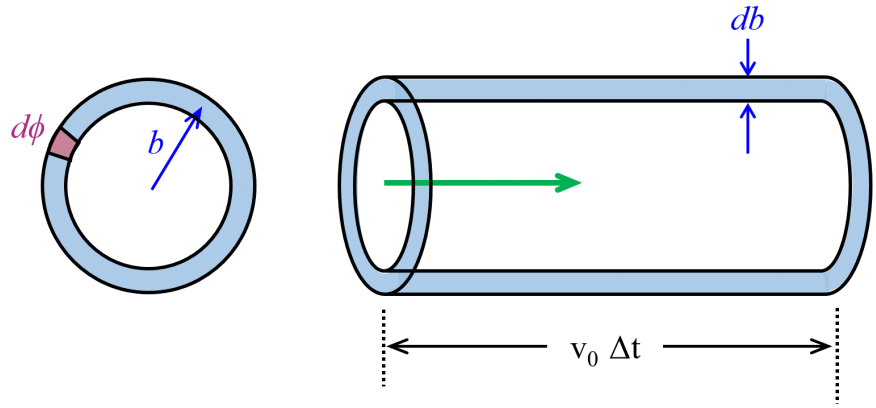
It's relatively easy to specify the number of "targets" in the volume:

$$N_2 \;=\; f_2(\mathbf{r}, \mathbf{v}_2, t)\, d^3\mathbf{r}\, d^3\mathbf{p}_2$$

What about particle 1's? We could compute N_1 the same way, but that just gives us the number of 1's that happen to be nearby at some *infinitesimal* instant of time. Not what we want!

Because the 1's and 2's are *streaming past one another,* we really want to know how many 1's ENTER the "front face" of the volume element over a (longer?) time span Δt.

For that, let's consider the same cylindrical area element we used before when discussing Coulomb collisions:



We draw the picture in a frame in which the type-2 targets are stationary, and the particle 1's are incoming. However, we still must keep in mind that particle 1's have $\mathbf{v}_1$ in the inertial frame, particle 2's have $\mathbf{v}_2$, etc.

We want to know how many collisions occur in a little volume element

$$d^3\mathbf{r} \;=\; dz\; dA \;=\; dz\; (b\; db\; d\phi)$$

But, to figure out $N_{1,\text{enter}}$, we note that the length of the cylinder is defined by the finite time-span Δt, and

$$dz \;=\; |\mathbf{v}|\,\Delta t \;=\; |\mathbf{v}_1 - \mathbf{v}_2|\,\Delta t$$

and thus,

$$N_{1,\text{enter}} \;=\; f_1(\mathbf{r},\mathbf{p}_1,t)\, d^3\mathbf{r}\, d^3\mathbf{p}_1 \;=\; f_1(\mathbf{r},\mathbf{p}_1,t)\, |\mathbf{v}_1 - \mathbf{v}_2|\; d^3\mathbf{p}_1\; (b\; db\; d\phi)\; \Delta t \;\; .$$

Thus,

$$\begin{aligned} N_{\text{out}} \;&=\; N_{1,\text{enter}}\, N_2 \\ &=\; f_1(\mathbf{r},\mathbf{p}_1,t)\, f_2(\mathbf{r},\mathbf{p}_2,t)\, |\mathbf{v}_1 - \mathbf{v}_2|\; d^3\mathbf{p}_1\; d^3\mathbf{p}_2\; d^3\mathbf{r}\; (b\; db\; d\phi)\; \Delta t \;\; . \end{aligned}$$

Note this number is symmetric: swap $1 \leftrightarrow 2$ and it doesn't change. Good! Also, as Δt increases, so does N_{out}. That also makes sense!

Thus, to obtain $\mathcal{C}_{\rm out}$, we divide $N_{\rm out}$ by $(d^3\mathbf{r}\; d^3\mathbf{p}_1\; \Delta t)$.

We also must acknowledge that, from species 1's standpoint, there's really a whole range of targets at this location... i.e., to account for them all we must integrate over all available values of $\mathbf{p}_2$, b, ϕ.

$$\mathcal{C}_{\rm out}(f_1, f_2) \;=\; \int d^3\mathbf{p}_2 \int db\; b \int d\phi\; f_1\, f_2\, |\mathbf{v}_1 - \mathbf{v}_2|$$

We simplify the notation by assuming f_1 is a function of $\mathbf{p}_1$, and f_2 is a function of $\mathbf{p}_2$.

Of course, in a mixture of many types of charged particles,

$$\mathcal{C}_{\rm out} \;=\; \sum_s \mathcal{C}_{\rm out}(f_1, f_s) \qquad\qquad \text{where } s = 1 \text{ is also in the sum!}$$

but let's just consider one species of 2's at a time.

. . . . . .

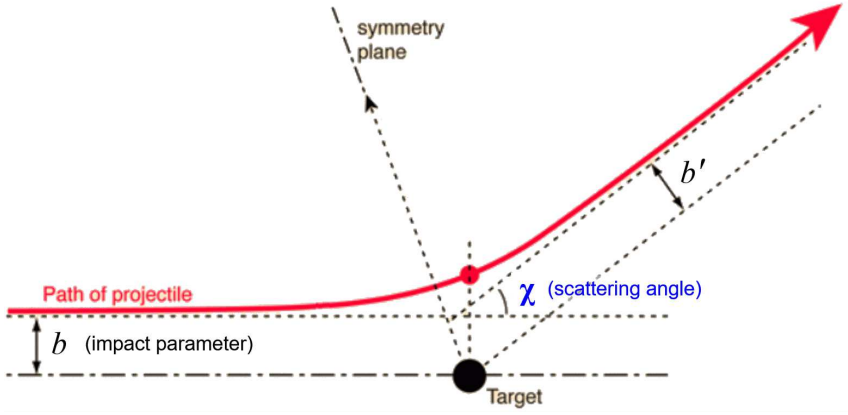
To obtain $N_{\rm in}$, the number of particles scattered INTO the volume element over time Δt, we reverse the roles of initial and final velocities.

Incoming: $\mathbf{v}_1'$ and $\mathbf{v}_2'$ Outgoing: $\mathbf{v}_1$ and $\mathbf{v}_2$

$$N_{\rm in} \;=\; f_1(\mathbf{r}, \mathbf{p}_1', t)\, f_2(\mathbf{r}, \mathbf{p}_2', t)\, |\mathbf{v}_1' - \mathbf{v}_2'|\; d^3\mathbf{p}_1'\; d^3\mathbf{p}_2'\; d^3\mathbf{r}\; (b'\; db'\; d\phi')\; \Delta t\;.$$

For elastic collisions, their paths are symmetric: $\left\{ \begin{array}{c} |\mathbf{v}_1 - \mathbf{v}_2| \;=\; |\mathbf{v}_1' - \mathbf{v}_2'| \\ b \;=\; b' \\ \phi \;=\; \phi' \end{array} \right\}$

See cartoon:



In homework, you may have to work out how the volume element *products* are equal (no matter the reference frame) before & after the collision:

$$d^3\mathbf{v}_1\; d^3\mathbf{v}_2 \;=\; d^3\mathbf{U}\; d^3\mathbf{v} \;=\; d^3\mathbf{U}'\; d^3\mathbf{v}' \;=\; d^3\mathbf{v}_1'\; d^3\mathbf{v}_2' \;\;.$$

Thus, we can drop **almost** all of the primes, and write

$$\mathcal{C}_{\rm in}(f_1, f_2) \;=\; \int d^3\mathbf{p}_2 \int db\; b \int d\phi\; f_1(\mathbf{r}, \mathbf{v}_1', t)\, f_2(\mathbf{r}, \mathbf{v}_2', t)\; |\mathbf{v}_1 - \mathbf{v}_2| \;\;.$$

and the full collision operator is

$$\boxed{\left(\frac{\partial f_1}{\partial t}\right)_{\rm coll,2} \;=\; \int d^3\mathbf{p}_2 \int db\; b \int d\phi\; |\mathbf{v}_1 - \mathbf{v}_2|\;\; (f_1'\, f_2' \;-\; f_1\, f_2)} \;\;.$$

Let's clarify:

- The left side of the Boltzmann equation follows f_1 through phase space, so the collision operator also specifies just one specific value of $\mathbf{p}_1$.
- The collision operator integrates over all possible values of the "target" momentum-space $\mathbf{p}_2$.
- For each collision (i.e., each pair of $\mathbf{p}_1$ and $\mathbf{p}_2$), the collision operator also integrates over the full range of possible geometries for the collision (i.e., b and ϕ), too.
- For each UNIQUE collision (i.e., each set of $\mathbf{p}_1$, $\mathbf{p}_2$, b, ϕ), everything is deterministic. There's a unique "outgoing" set of $\mathbf{p}_1'$ & $\mathbf{p}_2'$ values, and the quantity $f_1' f_2'$ is evaluated at those values.

Some books define a differential scattering cross section,

$$\sigma \;=\; \left|\frac{b\; db\; d\phi}{d\Omega}\right| \;=\; \left|\frac{b\; db\; d\phi}{\sin\chi\; d\chi\; d\phi}\right|$$

for which particles passing through the (b, ϕ) volume element are scattered into a solid angle element $d\Omega$, and the db integral can be turned into a $d\chi$ integral.

We haven't thought too much about this χ angle, but it's just the angular separation between $\mathbf{v}$ & $\mathbf{v}'$. For small-angle scattering, $\Delta v_\perp / v_0 \approx \tan\chi \approx \chi$.

Because $\chi \uparrow$ as $b \downarrow$, one would get $\sigma < 0$ unless one takes the absolute value, as above (for physical realism). It's possible to show that

$$\tan\left(\frac{\chi}{2}\right) = \frac{b_{\min}}{2b} \qquad \left(\text{where recall that} \quad b_{\min} = \frac{2|q_1 q_2|}{m_{12}|\mathbf{v}_1 - \mathbf{v}_2|^2} \right)$$

and thus,

$$\sigma = \frac{b_{\min}^2}{16\,\sin^4(\chi/2)} \ .$$

Note that for $b = b_{\min}$, $\chi \approx 53.1°$, and for $b = b_{\min}/2$, $\chi = 90°$.
We could have used $b_{\min}/2 = b_{90}$ in the denominator of Λ, and $\ln\Lambda$ would hardly be much different.

Also, note that $b\,db\,d\phi = b'\,db'\,d\phi'$, implies $\sigma(\Omega)\,d\Omega = \sigma(\Omega')\,d\Omega'$.

What will we *do* with the full Boltzmann collision term? The remainder of these notes provide the outlines of 4 derivations:

(a) Prove that collisions obey known conservation laws.

(b) In equilibrium, show that collisions drive us to a Maxwellian distribution.

(c) When dominated by many small-angle scatterings, show that the collision term turns into **advection & diffusion** in momentum space (i.e., the Fokker–Planck equation).

(d) Explore (qualitatively) how the system evolves when we start with an OUT-of-equilibrium situation.

(a) Conservation laws

One thing we'll do a lot later is take moments over each term in the Vlasov/Boltzmann equation, often *weighted* by specific functions of $\mathbf{p}_1$ or $\mathbf{v}_1$.

Consider this kind of weighted moment of the collision operator,

$$M_{12}(\Psi) \;=\; \int d^3\mathbf{p}_1 \; \Psi(\mathbf{p}_1) \left(\frac{\partial f_1}{\partial t}\right)_{\text{coll},2}$$

and we'll eventually use

$$\Psi(\mathbf{p}_i) \;=\; \left\{ \begin{array}{cl} m_i & \text{for mass conservation} \\ m_i \mathbf{v}_i & \text{for momentum conservation} \\ \frac{1}{2} m_i |\mathbf{v}_i|^2 & \text{for energy conservation} \end{array} \right\} \qquad i = 1 \text{ or } 2 \; .$$

Let's write out the standard version of the moment and call it

$$M_{12} \;=\; \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega \; \sigma(\Omega) \; |\mathbf{v}_1 - \mathbf{v}_2| \;\; (f_1' \, f_2' \;-\; f_1 \, f_2) \;\; \Psi(\mathbf{p}_1)$$

Alternately, we saw that if we swapped the un-primed (initial) and primed (final) variables, we can make use of properties of elastic collisions to simplify. Thus,

$$\begin{aligned} M_{12}' \;&=\; \int d^3\mathbf{p}_1' \int d^3\mathbf{p}_2' \int d\Omega' \; \sigma(\Omega') \; |\mathbf{v}_1' - \mathbf{v}_2'| \;\; (f_1 \, f_2 \;-\; f_1' \, f_2') \;\; \Psi(\mathbf{p}_1') \\ &=\; \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega \; \sigma(\Omega) \; |\mathbf{v}_1 - \mathbf{v}_2| \;\; (f_1 \, f_2 \;-\; f_1' \, f_2') \;\; \Psi(\mathbf{p}_1') \end{aligned}$$

The trick is to notice that

$$M_{12} \;\;=\;\; M_{12}' \;\;=\;\; \frac{M_{12} + M_{12}'}{2}$$

and thus,

$$M_{12}(\Psi) = \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega \; \sigma(\Omega) \, |\mathbf{v}_1 - \mathbf{v}_2| \; (f_1' \, f_2' - f_1 \, f_2) \left[\frac{\Psi(\mathbf{p}_1) - \Psi(\mathbf{p}_1')}{2} \right]$$

This may seem a bit convoluted, but note that, for $\Psi(\mathbf{p}_1) = m_1$, we know that the particle's mass is unchanged before and after the collision, so we know

$$\Psi(\mathbf{p}_1) - \Psi(\mathbf{p}_1') = 0 \qquad \text{and thus} \qquad \int d^3\mathbf{p}_1 \left(\frac{\partial f_1}{\partial t}\right)_{\text{coll},2} = 0$$

i.e., the zeroth moment of the Boltzmann collision operator vanishes, which means that *collisions do not create or destroy mass* ($M_{12} = 0$ for mass).

Consider $\Psi(\mathbf{p}_1) = m_1\mathbf{v}_1$ (momentum of particle 1). We know that $\mathbf{v}_1 \neq \mathbf{v}_1'$, so the weighted moment M_{12} does not vanish in this case.

This means that collisions *CAN* provide a net, macro-scale source (if $M_{12} > 0$) or sink (if $M_{12} < 0$) of momentum to a population of particles. Think back to Brownian motion. The big pollen grain can be slowed down by the smaller molecules.

We can understand what's going on by swapping the 1 and 2 indices. This gives us a weighted moment over the Boltzmann equation for species 2, when those particles are colliding with "target" particles of type 1:

$$M_{21}(\Psi) \;=\; \int d^3\mathbf{p}_2 \;\Psi(\mathbf{p}_2) \left(\frac{\partial f_2}{\partial t}\right)_{\text{coll},1}$$

Thus,

$$M_{21} \;=\; \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega \;\sigma(\Omega)\;|\mathbf{v}_1 - \mathbf{v}_2|\;\;(f_1'\,f_2' \;-\; f_1\,f_2)\;\;\Psi(\mathbf{p}_2)$$

As above, we can also swap the primes and unprimes to obtain M_{21}'.

I won't show the details, but if we construct the sum

$$M_{12} \;+\; M_{12}' \;+\; M_{21} \;+\; M_{21}'$$

we can write it as the standard integral above, with integrand

$$[\Psi(\mathbf{p}_1) \;-\; \Psi(\mathbf{p}_1') \;+\; \Psi(\mathbf{p}_2) \;-\; \Psi(\mathbf{p}_2')]$$

However, we do know that the **total** momentum is conserved in a binary elastic collision. Remembering that $\Psi(\mathbf{p}_i) = m_i\mathbf{v}_i$, we know that

$$\Psi(\mathbf{p}_1) + \Psi(\mathbf{p}_2) \;=\; \Psi(\mathbf{p}_1') + \Psi(\mathbf{p}_2')$$

so

$$M_{12} \;+\; M_{12}' \;+\; M_{21} \;+\; M_{21}' \;=\; 0 \;\;.$$

We already know that $M_{12} = M'_{12}$, and presumably it's clear also that $M_{21} = M'_{21}$. Thus,

$$M_{12} = -M_{21} \qquad \text{i.e.,} \qquad \int d^3\mathbf{p}_1 \; m_1\mathbf{v}_1 \left(\frac{\partial f_1}{\partial t}\right)_{\text{coll},2} = -\int d^3\mathbf{p}_2 \; m_2\mathbf{v}_2 \left(\frac{\partial f_2}{\partial t}\right)_{\text{coll},1}$$

i.e., the rate at which particles of type 1 gain momentum due to collisions with particles of type 2 is equal to the rate at which particles of type 2 lose momentum due to collisions with particles of type 1. *Collisions conserve momentum.*

You can show that the above is true for **kinetic energy,** too.

. . . . . .

(b) Equilibrium

There are a number of different statistical theorems that can be proved (e.g., Boltzmann's H theorem, which is related to entropy and the 2nd law of thermodynamics), but there's a more straightforward way to see where this is all going.

In a homogeneous system ($\nabla f = 0$) with no forces acting on it ($\mathbf{F} = 0$), the Boltzmann equation reduces to

$$\frac{\partial f_1}{\partial t} = \left(\frac{\partial f_1}{\partial t}\right)_{\text{coll},2} = \int d^3\mathbf{p}_2 \int db \; b \int d\phi \; |\mathbf{v}_1 - \mathbf{v}_2| \; (f'_1 \, f'_2 - f_1 \, f_2)$$

and if we ultimately reach a time-steady equilibrium, then both sides = 0.

One way to make the collision term vanish is to find a distribution that obeys a "detailed balance" condition,

$$f'_1 \, f'_2 - f_1 \, f_2 = 0 \qquad \text{i.e.,} \qquad f_1(\mathbf{p}_1) \, f_2(\mathbf{p}_2) = f_1(\mathbf{p}'_1) \, f_2(\mathbf{p}'_2) \; .$$

- LHS: depends only on conditions before the collision.
- RHS: depends only on conditions after the collision.

It's equivalent to write this as

$$\ln f_1 + \ln f_2 = \ln f'_1 + \ln f'_2 \; .$$

Note that, in equilibrium, the sum over each particle's $\ln f$ must be the same before and after each collision.

Thus, the total $(\ln f)$ behaves like a **collisional invariant** (e.g., like total mass, total momentum, or total kinetic energy)!

Let's remember what those were. For elastic collisions:

$$\begin{aligned} m_1 + m_2 &= m_1' + m_2' && \text{(duh)} \\ \mathbf{p}_1 + \mathbf{p}_2 &= \mathbf{p}_1' + \mathbf{p}_2' && \text{(less trivial)} \\ (p_1)^2/2m_1 + (p_2)^2/2m_2 &= (p_1')^2/2m_1 + (p_2')^2/2m_2 && \text{(see earlier lectures)} \end{aligned}$$

Collisional invariants aren't easy to find. Is the above an exhaustive list?

Maybe $(\ln f)$ isn't a brand new invariant. Let's investigate what it implies if $(\ln f)$ is, in fact, merely a **linear combination** of the existing invariants.

That *ansatz* actually works. Assume one can write

$$\ln f_i \;=\; c_1 m_i \;+\; \mathbf{c}_2 \cdot \mathbf{p}_i \;+\; c_3 \, \frac{p_i^2}{2m_i}$$

where we must note that $\mathbf{c}_2$ is a vector because we need to fold in momentum but still have the final result be a scalar.

This is just a concrete application of the "method of Lagrange multipliers."

The linear and quadratic terms can be combined into a single term (with some algebraic "completing the square"), with

$$\ln f \;=\; \widetilde{c}_1 \;+\; \widetilde{c}_2 \, (\mathbf{p} - \mathbf{p}_0)^2$$

and the constants can be redefined to obtain the equilibrium (i.e., Maxwell-Boltzmann) form of the distribution,

$$f(\mathbf{p}) \;=\; e^{\widetilde{c}_1} \, e^{\widetilde{c}_2 (\mathbf{p} - \mathbf{p}_0)^2} \;\propto\; \exp\left[-\frac{1}{2mk_{\mathrm{B}}T} \, (\mathbf{p} - \mathbf{p}_0)^2 \right]$$

. . . . . .

(c) Fokker–Planck Equation

Recall the weighted moment over the collision operator. It's the sum of two terms: $M_{12} = M_{12,\text{in}} + M_{12,\text{out}}$

$$M_{12,\text{in}} \;=\; + \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega\; \sigma(\Omega)\; |\mathbf{v}_1 - \mathbf{v}_2|\; f_1' f_2'\; \Psi(\mathbf{p}_1)$$

$$M_{12,\text{out}} \;=\; - \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega\; \sigma(\Omega)\; |\mathbf{v}_1 - \mathbf{v}_2|\; f_1 f_2\; \Psi(\mathbf{p}_1)$$

no matter what the Ψ function represents.

Also remember that the first term can also be written as

$$M_{12,\text{in}}' \;=\; + \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega\; \sigma(\Omega)\; |\mathbf{v}_1 - \mathbf{v}_2|\; f_1 f_2\; \Psi(\mathbf{p}_1')$$

Thus, another way to write the full moment is

$$\begin{aligned}\widetilde{M}_{12} \;&=\; M_{12,\text{in}}' + M_{12,\text{out}} \\ &=\; \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega\; \sigma(\Omega)\; |\mathbf{v}_1 - \mathbf{v}_2|\; f_1 f_2 \left[\Psi(\mathbf{p}_1') - \Psi(\mathbf{p}_1)\right] .\end{aligned}$$

Now we're (finally!) going to express the primed quantities *in terms of* the unprimed quantities.

Let's limit ourselves to just weak, small-angle scattering: $\mathbf{v}_1' \;=\; \mathbf{v}_1 + \Delta\mathbf{v}$,

and because $\Delta\mathbf{v}$ is small in comparison to $\mathbf{v}_1$, we can expand $\Psi(\mathbf{v}_1') = \Psi(\mathbf{v}_1 + \Delta\mathbf{v})$ in a Taylor series about the initial velocity $\mathbf{v}_1$

$$\Psi(\mathbf{v}_1 + \Delta\mathbf{v}) \;=\; \Psi(\mathbf{v}_1) \;+\; \left(\frac{\partial\Psi}{\partial\mathbf{v}}\right)_{\mathbf{v}_1} \cdot \Delta\mathbf{v} \;+\; \frac{1}{2}\left(\frac{\partial^2\Psi}{\partial\mathbf{v}\partial\mathbf{v}}\right)_{\mathbf{v}_1} : \Delta\mathbf{v}\Delta\mathbf{v} \;+\; \cdots$$

and to be clear,

$$\left(\frac{\partial\Psi}{\partial\mathbf{v}}\right) \cdot \Delta\mathbf{v} \;=\; \sum_i \frac{\partial\Psi}{\partial v_i}\,\Delta v_i \qquad\qquad \text{(3 terms in sum)}$$

$$\left(\frac{\partial^2\Psi}{\partial\mathbf{v}\partial\mathbf{v}}\right) : \Delta\mathbf{v}\Delta\mathbf{v} \;=\; \sum_{i,j} \frac{\partial^2\Psi}{\partial v_i \partial v_j}\,\Delta v_i \Delta v_j \qquad\qquad \text{(9 terms in sum)}$$

Traditionally we neglect all higher-order terms, mainly because we focus on *weak* collisions, for which $\Delta\mathbf{v}$ is a small parameter, and we see that

$$\widetilde{M}_{12} \;=\; \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \int d\Omega\; \sigma(\Omega)\; |\mathbf{v}_1 - \mathbf{v}_2|\; f_1\, f_2 \;\;\times$$

$$\times \;\left[\left(\frac{\partial \Psi}{\partial \mathbf{v}} \right)_{\mathbf{v}_1} \cdot \Delta\mathbf{v} \;+\; \frac{1}{2} \left(\frac{\partial^2 \Psi}{\partial \mathbf{v} \partial \mathbf{v}} \right)_{\mathbf{v}_1} : \Delta\mathbf{v}\Delta\mathbf{v} \right]$$

If we can manipulate this integrand to get it into a form like...

$$\widetilde{M}_{12} \;=\; \int d^3\mathbf{p}_1\; \Psi(\mathbf{p}_1)\; \{\text{something}\}$$

then we would have a usable expression for

$$\left(\frac{\partial f_1}{\partial t} \right)_{\text{coll},2} \;=\; \{\text{something}\} \qquad\qquad (\text{with no primes!})$$

This can be done with integration by parts. (Once for 1st order terms, twice for 2nd order terms.)

Let's not work out the math for all 12 terms in class, but the upshot is to realize that

$$\int dv_i\; f_1\; \frac{\partial \Psi}{\partial v_i} \;=\; [f_1\, \Psi] \;-\; \int dv_i\; \Psi\; \frac{\partial f_1}{\partial v_i}$$

and if we integrate over all velocity space, the $[f_1\, \Psi]$ term vanishes because $f_1 \to 0$ in the limit of $v_i \to \pm\infty$.

The end result is the **Fokker–Planck equation:**

$$\boxed{ \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll},2} \;=\; -\sum_i \frac{\partial}{\partial v_i} \Big(f_1 \langle\langle \Delta v_i \rangle\rangle \Big) \;+\; \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} \Big(f_1 \langle\langle \Delta v_i \Delta v_j \rangle\rangle \Big) }$$

where

$$\left\{ \begin{array}{c} \langle\langle \Delta v_i \rangle\rangle \\ \langle\langle \Delta v_i \Delta v_j \rangle\rangle \end{array} \right\} \;=\; \int d^3\mathbf{p}_2 \int d\Omega\; \sigma(\Omega)\; |\mathbf{v}_1 - \mathbf{v}_2|\; f_2(\mathbf{p}_2) \left\{ \begin{array}{c} \Delta v_i \\ \Delta v_i \Delta v_j \end{array} \right\}$$

Units of these coefficients:

- $\langle\langle \Delta v_i \rangle\rangle = v/t$: i.e., acceleration (the minus sign on the term means deceleration), or "advection" in velocity space. This term provides **dynamic friction.**
- $\langle\langle \Delta v_i \Delta v_j \rangle\rangle = v^2/t$: analogous to ℓ^2/t spatial diffusion coefficients. This term provides **diffusion in velocity space.**

When the advection & diffusion coefficients are known, the Boltzmann equation becomes "just" a differential equation, which is easier to deal with than an integro-differential equation.

(Exact solutions of F–P equation for cosmic rays: arXiv:1703.02554)

In equilibrium, the collision term $(\partial f_1/\partial t)_{\rm coll,2} \to 0$, so the advection/friction term must **balance** the diffusion term. Here's the *fluctuation–dissipation* theorem again!

Heuristically, we should know that "pure" diffusion would never let us reach a time-steady equilibrium: it just keeps diffusing out and out, to ever higher temperatures! The dynamical friction keeps that from happening.

. . . . . .

Note that the Fokker–Planck coefficients $\langle\langle \Delta v_i \rangle\rangle$ and $\langle\langle \Delta v_i \Delta v_j \rangle\rangle$ are valid for ANY kind of binary collisions. We've spent a lot of time thinking about one specific kind: Coulomb collisions (i.e., mediated by electrostatic attraction/repulsion).

In that case, we know $\sigma(\Omega)$ and can do that integral, while still not specifying $f(\mathbf{p})$. The result is

$$\langle\langle \Delta v_i \rangle\rangle \;=\; \left(\frac{4\pi q_1^2 q_2^2 \, \ln\Lambda}{m_1 m_{12}} \right) \frac{\partial H}{\partial v_i} \quad , \qquad \langle\langle \Delta v_i \Delta v_j \rangle\rangle \;=\; \left(\frac{4\pi q_1^2 q_2^2 \, \ln\Lambda}{m_1^2} \right) \frac{\partial^2 G}{\partial v_i \partial v_j}$$

where G and H are the **Rosenbluth potentials:**

$$H(\mathbf{v}_1) \;=\; \int d^3\mathbf{p}_2 \, \frac{f_2(\mathbf{p}_2)}{|\mathbf{v}_1 - \mathbf{v}_2|} \quad , \qquad G(\mathbf{v}_1) \;=\; \int d^3\mathbf{p}_2 \, f_2(\mathbf{p}_2) \, |\mathbf{v}_1 - \mathbf{v}_2| \quad .$$

and the $\partial/\partial v_i$ derivatives are taken with respect to $\mathbf{v}_1$.

(d) Non-Equilibrium Evolution

We've seen that if all species are Maxwellians, and they're in thermal equilibrium with one another (i.e., $T_1 = T_2$, $\mathbf{u}_1 = \mathbf{u}_2$), then

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll},2} = 0 \ .$$

Thus, whenever we think about non-zero collision terms, we must be dealing with an OUT-of-equilibrium situation.

Let us think briefly about 3 sub-cases where we "initialize" the system with some non-equilibrium-ness...

d1. **Energetic particles:** species 1 is a single "test particle" zooming in to interact with a Maxwellian distribution of species 2's.

d2. **Transport coefficients:** these are self-collisions (i.e., species 1 is the same as species 2) with initially non-Maxwellian distribution.

d3. **Multi-fluid equilibration:** start with Maxwellians (for species 1 $\neq$ species 2), each with different temperatures or bulk flow speeds.

. . . . . .

(d1) Test particles:

This situation is close to the idealized Coulomb scattering problem we examined earlier. Just replace the single "particle 2" by a distribution.

Recall that thinking about particle–particle scattering in the frame of the target led us to a "cylinder model" for the total slowing-down rate,

$$\frac{\langle \Delta v_\parallel \rangle}{\Delta t} = n_2 \, v_0 \int d\phi \int db \; b \; \Delta v_\parallel$$

where $\Delta v_\parallel$ is a function of b, v_0, and charges & masses of the particles.

For a distribution of targets, we simply need to

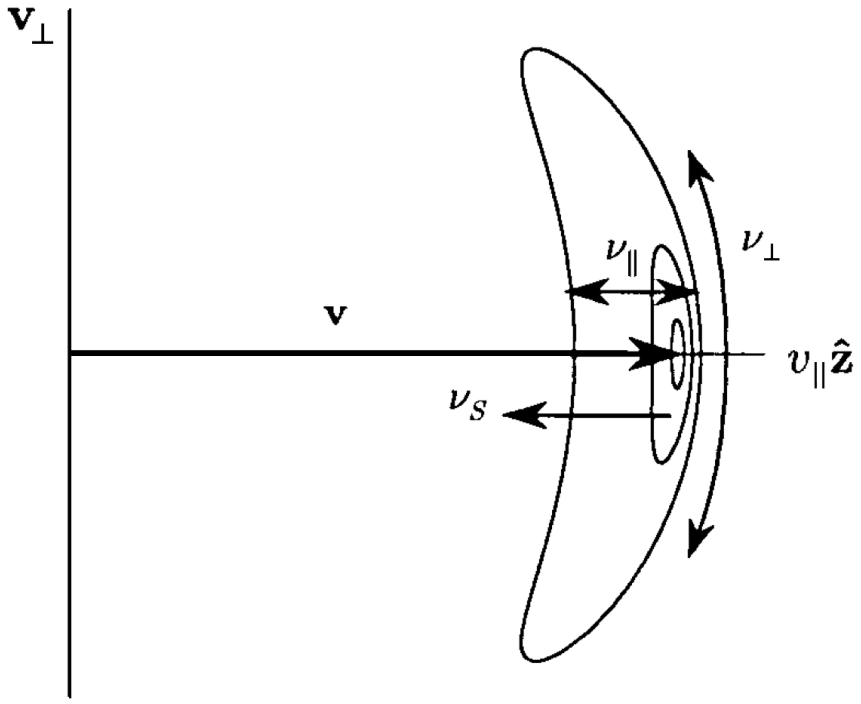
$$\text{replace} \qquad n_2 \, v_0 \qquad \text{by} \qquad \int d^3\mathbf{p}_2 \; f(\mathbf{p}_2) \; |\mathbf{v}_1 - \mathbf{v}_2|$$

and you can verify that if $f(\mathbf{p}_2)$ was a delta function, we'd recover our earlier result.

Heuristically, now we can reinterpret:

- "Slowing down" means that $\langle \mathbf{v}_1 \rangle$ will relax to $\mathbf{u}_2$, the mean/centroid of $f(\mathbf{p}_2)$ (assuming $\mathbf{v}_1$ is larger).
- "Perpendicular diffusion" means that $\langle |\mathbf{v}_1|^2 \rangle^{1/2}$ will come into equilbrium with $v_{\mathrm{th},2}$, the r.m.s. thermal spread of $f(\mathbf{p}_2)$.
- There is also now a (slower) "parallel diffusion" (i.e., $\langle |\Delta \mathbf{v}_\parallel|^2 \rangle \neq 0$), even though this didn't happen in the particle–particle scattering problem.

Probability distribution of $\mathbf{v}_1$ values evolves from an initial δ function to...



If $f(\mathbf{p}_2)$ is Maxwellian, one can straightforwardly turn the crank on the $d^3\mathbf{p}_2$ integral. I won't go through it; just compare the new result with the old result:

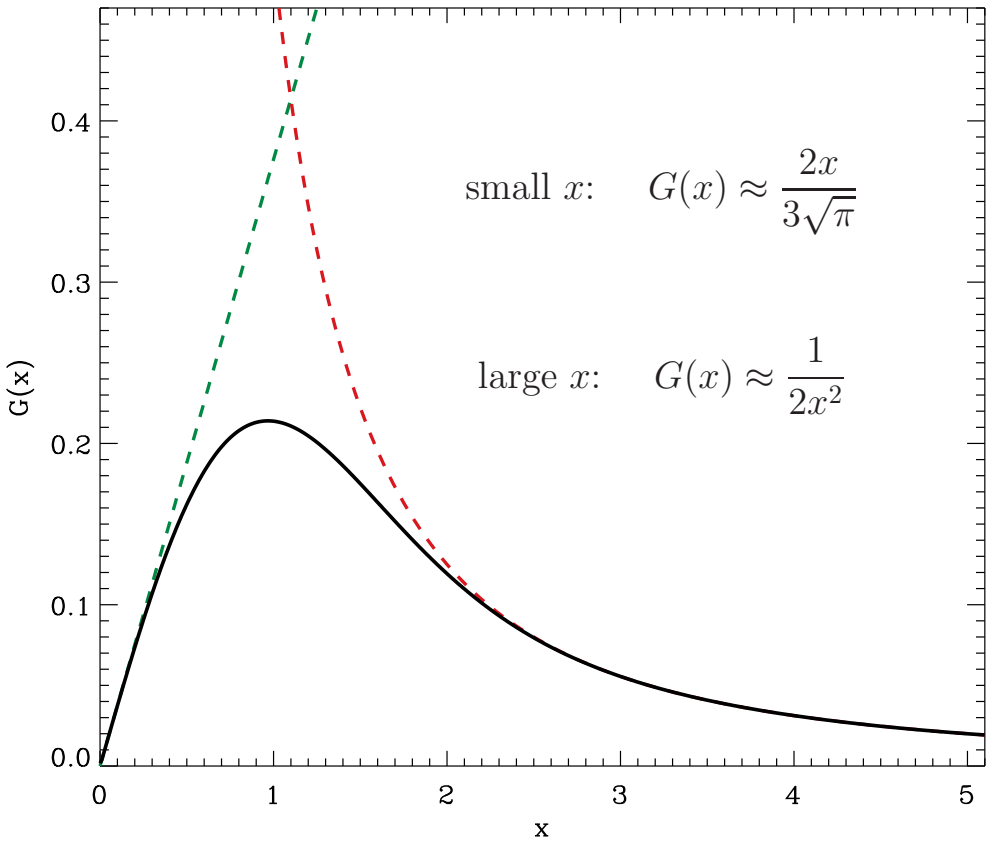
$$\text{OLD:} \qquad \frac{dv_{\parallel 1}}{dt} = -\nu_{\mathrm{S1}} \left(v_{\parallel 1} - v_{\parallel 2} \right) \qquad \text{where} \qquad \nu_{\mathrm{S1}} = \frac{4\pi\, q_1^2\, q_2^2\, n_2\ \ln\Lambda}{m_1\, m_{12}\, |\mathbf{v}_1 - \mathbf{v}_2|^3} \ .$$

$$\text{NEW:} \qquad \frac{dv_{\parallel 1}}{dt} = -2\overline{\nu}_{\mathrm{S1}}\ v_{\mathrm{th},2}\, G(x) \qquad \text{where} \qquad \overline{\nu}_{\mathrm{S1}} = \frac{4\pi\, q_1^2\, q_2^2\, n_2\ \ln\Lambda}{m_1\, m_{12}\, v_{\mathrm{th},2}^3}$$

where also,

$$v_{\rm th,2} = \sqrt{\frac{2k_{\rm B}T_2}{m_2}} \ , \qquad x = \frac{|\mathbf{v}_1 - \mathbf{u}_2|}{v_{\rm th,2}} \ , \qquad G(x) = \frac{1}{2x^2}\left[{\rm erf}(x) \ - \ \frac{2x}{\sqrt{\pi}}\, e^{-x^2}\right]$$

$G(x)$ is the Chandrasekhar (1943, *ApJ,* 97, 255) function:



Thus, for small relative motions (i.e., $|\mathbf{v}_1 - \mathbf{u}_2| \ll v_{\rm th,2}$), the evolution of parallel speed is **friction-like.**

However, if $|\mathbf{v}_1 - \mathbf{u}_2| \gg v_{\rm th,2}$, the test particle zips through so fast that the frictional force rapidly declines to zero. This is called **Coulomb runaway.**

The $\parallel$ and $\perp$ diffusion equations contain comparable RHS functions that behave differently in the two above limits.

. . . . . .

(d2) Transport coefficients:

Can large-scale gradients maintain (i.e., feed) a non-Maxwellian distribution? Yes, but only if there exist "self-collisions" (i.e., particles of a given species colliding with one another).

For this course, these effects are under the umbrella of "resistive MHD." Thus, we'll come back to it after talking about "ideal MHD."

. . . . . .

(d3) Multi-fluid equilibration:

How does the collision operator show up in multi-species fluid equations?

We'll have to come back to this one later, too, once we've introduced the multi-species fluid equations.

. . . . . .