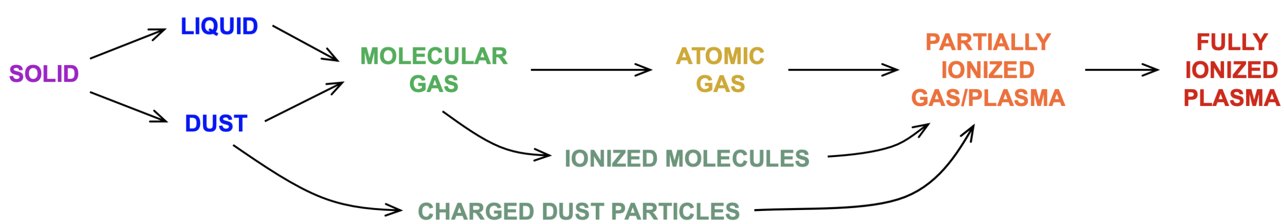
PLASMA PHYSICS

What is a plasma? It's an ionized gas, in which the component **charged particles** interact with (and/or generate) **E** and **B** fields to the extent that the system behaves "collectively."

The term "plasma" was coined by Irving Langmuir in 1928. His immediate inspiration was **blood plasma.** Blood contains particle-like cells surrounded by a liquid medium, and the analogy is that charged particles don't act in isolation; they require the continuous "medium" of the E&M fields to make the system behave as a well-coupled system.

(Following the chain of etymology, blood plasma was named after *protoplasm,* a 19th century term for liquids inside living cells. *proto* + *plasm* comes from the Greek for "first thing formed," and there were many weird ideas about this stuff being the basis of all life...)

One can often think of a "breakup" sequence with increasing T (or stronger radiation field) as you go from left to right:



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If you step back and squint your eyes, most plasmas are **quasi-neutral,** i.e., since it most likely came from a neutral source of matter, the free positive charges are *balanced* by free negative charges... i.e.,

$$\rho_c = \sum_s q_s n_s \approx 0 \ .$$

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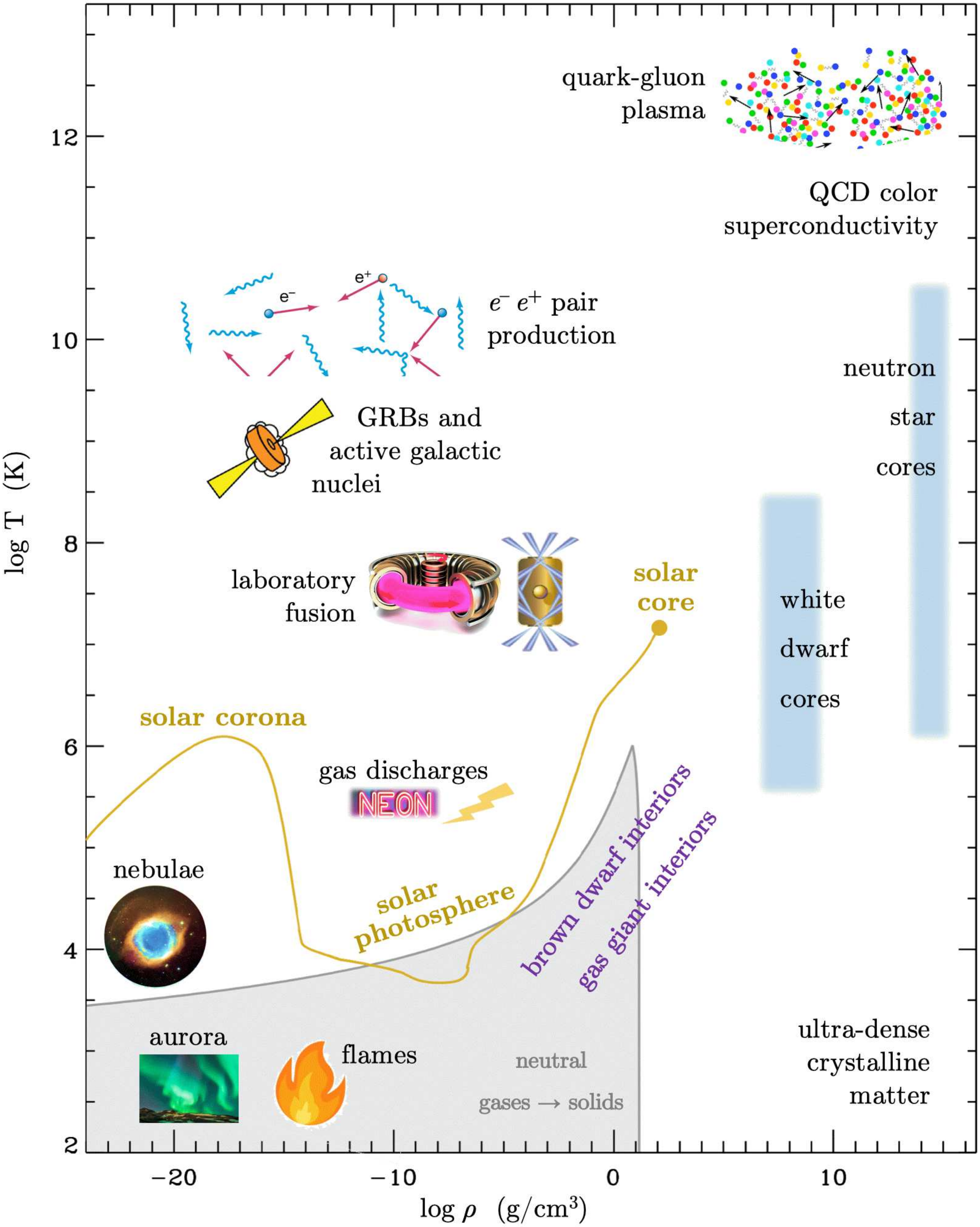
We'll examine 3 aspects of charged particle motion in a plasma:
(1) why quasi-neutrality, (2) Larmor gyromotions, (3) Coulomb collisions.

But first... some perspective...

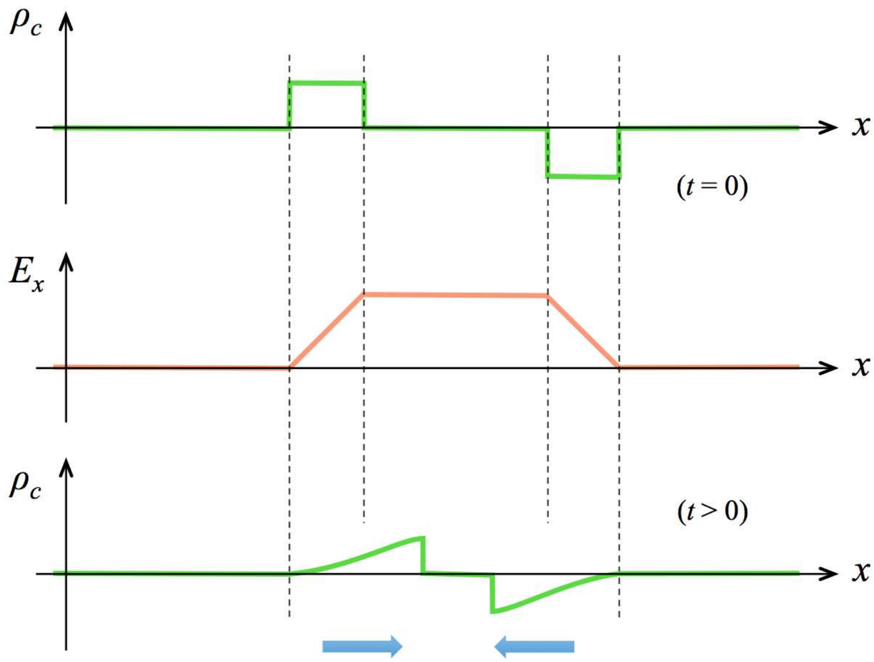
Phase diagram for astrophysical/planetary/heliophysical plasmas:



(1) What would happen if someone imposed some "charge separation?"

Extreme case: put all + charges over here, and all − charges over there?

What about something more subtle? Assume an overall balance of + and − charges, but a slight imbalance in charge density ρ_c,



$$\text{Gauss' law:} \quad \nabla \cdot \mathbf{E} = 4\pi\rho_c \qquad \text{which in 1D Cartesian is:} \quad \frac{\partial E_x}{\partial x} \propto \rho_c$$

and that results in the E_x distribution above.

How then do charged particles respond to this "new" electric field?

$$\text{Lorentz force:} \quad \mathbf{F} = q\mathbf{E} \ , \qquad \text{or in 1D,} \quad F_x = qE_x$$

so for $E_x > 0$, + charges will move $\longrightarrow$, and − charges will move $\longleftarrow$.

The excess charges will converge (driving a net current), and $\rho_c \to 0$ everywhere. The electric field **shorts itself out.**

$\Longrightarrow$ Thus, plasmas "want" to be quasi-neutral, overall.

(How fast does this happen? You'll explore that in homework...)

We'll soon see an important exception that acts on small scales (*Debye shielding*) but quasi-neutrality is usually pretty safe to assume.

(2) Larmor Gyro-Motions

Prior to discussing how particles move around "in bulk" in a magnetized plasma, we should note that there's one important feature of *individual* particle motion (when $\mathbf{B} \neq 0$) that never really goes away when thinking statistically:

You've probably seen it derived before, but I'll include it here in the notes, just in case.

Consider a uniform $\mathbf{B}$, and recall the Lorentz force:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) \qquad (\text{for } \mathbf{E} = 0).$$

Define $\mathbf{B} = B_z\hat{\mathbf{e}}_z$ as constant along z-axis.

The 3 components of the Lorentz equation of motion are:

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{q}{mc}(v_y B_z) = \Omega\, v_y \\ \frac{dv_y}{dt} &= \frac{q}{mc}(-v_x B_z) = -\Omega\, v_x \\ \frac{dv_z}{dt} &= 0 \qquad \text{i.e., } v_z = \text{constant (depends on init. cond.)} \end{aligned}$$

and we define

$$\Omega = \frac{q|\mathbf{B}|}{mc} = \left\{ \begin{array}{c} \text{Larmor frequency} \\ \text{cyclotron frequency} \\ \text{“gyrofrequency”} \end{array} \right\} = \text{constant}, \quad \text{if } \mathbf{B} = \text{constant}.$$

Take note that there is **no** Lorentz magnetic force parallel to $\mathbf{B}$. Particles moving *along the field* act like there's no $\mathbf{B}$-field at all.

But what about the equations for v_x and v_y? We're faced with two **coupled** differential equations...

One trick for solving them is to take d/dt of both sides, and then substitute from the other...

$$\frac{d^2 v_x}{dt^2} = \Omega \frac{dv_y}{dt} = -\Omega^2 v_x \quad \longrightarrow \quad \boxed{\ddot{v}_x + \Omega^2 v_x = 0}$$

$$\frac{d^2 v_y}{dt^2} = -\Omega \frac{dv_x}{dt} = -\Omega^2 v_y \quad \longrightarrow \quad \boxed{\ddot{v}_y + \Omega^2 v_y = 0}$$

Simple harmonic oscillators. The solutions are sinusoids, but the original 2 equations show that v_x must be 90° out of phase with v_y.

$$\begin{aligned} \text{Thus, if} \quad & v_x(t) = v_\perp \sin(\Omega t) \\ \text{then} \quad & v_y(t) = v_\perp \cos(\Omega t) \\ \text{and...} \quad & v_z(t) = v_\parallel \end{aligned}$$

where both $v_\parallel$ and $v_\perp = \sqrt{v_x^2 + v_y^2}$ are constants.

Thus, the kinetic energy of the particle,

$$E_{\rm K} = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m \left(v_\parallel^2 + v_\perp^2 \right) = \text{constant, too!}$$

i.e., a magnetic field "accelerates" particles by changing their direction, but it doesn't speed them up or slow them down.

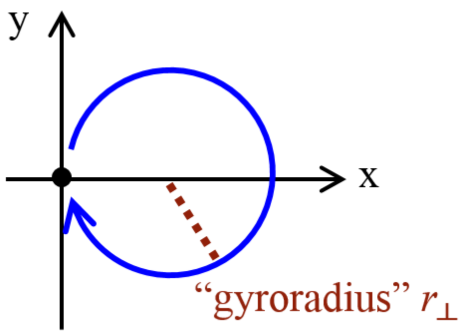
Why? $\mathbf{F}$ is always perpendicular to $\mathbf{v}$, so $\mathbf{F}$ does no net **work** on the particle (work $\propto \mathbf{F} \cdot \mathbf{v}$).

To get particle position (x, y, z) versus time, we must integrate...

$$\begin{aligned} x(t) &= \int dt \; v_x(t) = x_0 + \frac{v_\perp}{\Omega} (1 - \cos \Omega t) \\ y(t) &= \int dt \; v_y(t) = y_0 + \frac{v_\perp}{\Omega} \sin \Omega t \\ z(t) &= \int dt \; v_z(t) = z_0 + v_\parallel t \end{aligned}$$

where these have been normalized so that the positions are (x_0, y_0, z_0) at $t = 0$.

Define the **gyroradius** $r_\perp = v_\perp / \Omega$.



Positively charged particles have $\Omega > 0$, and thus their gyro-motion is **left-hand polarized** (i.e., use "left-hand rule" with thumb pointing along **B**).

Negatively charged particles have $\Omega < 0$, and thus their gyro-motion is right-hand polarized.

In many plasmas that we'll be dealing with $r_\perp$ is *tiny* in comparison to all other length-scales of the system.

Thus, if $v_\parallel \neq 0$, charged particles travel **helical** paths aligned with the field.... but if you "blur your eyes," particles mainly just flow along **B**.

Later, we'll prove that in some limiting cases (i.e., ideal MHD), particles are essentially "tied" to field lines.

And since + and − charged particles go in opposite senses, there's sometimes a net current **J** associated with gyro-motion (when particle-particle collisions are weak).

<u>Note units:</u> Ω = frequency in radians/sec... we sometimes also see

$$\nu_{\rm L} = \frac{\Omega}{2\pi} \quad \text{as the Larmor frequency in cycles/sec}$$

(e.g., in the Zeeman effect in spectral lines).

I'll skip over the laundry list of other kinds of drift-like motions that can *perturb* Larmor orbits in a collisionless plasma.

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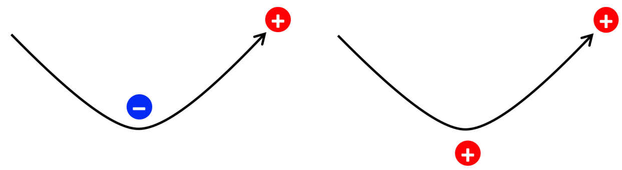
(3) Coulomb Collisions

Now we can start applying what we know about the diffusive effects of micro–scale random walk motions to the real-world example of charged particles in a plasma.

- 1 collision = a hyperbolic “encounter”
- >>1 collisions (following 1 particle) = random walk
- Electromagnetic force acts over long ranges (compared to how neutral atoms/molecules collide)



We often consider collisions to be “micro-physics,” but we want to know how they end up defining the diffusive **transport properties** of a plasma on MACRO-scales.

Individually, “collisions” (really: electromagnetic scattering events) share energy & momentum between the particles.

Collectively, when there are large-scale gradients in a fluid, collisions can act as catalysts to facilitate transport of macroscopic energy & momentum.

If there’s complete homogeneity & equilibrium, with an exact Maxwellian distribution everywhere, there’s no possibility for **net** transport of anything from point A to point B.

. . . .

In an inhomogeneous gas of charged particles, Coulomb collisions (between species i and species j) can:

1. transport momentum	• viscosity $(i = j)$ • friction $(i \neq j,\ \mathbf{u}_i \neq \mathbf{u}_j)$
2. transport thermal energy	• heat conduction $(i = j)$ • temperature isotropization $(i = j,\ T_{\parallel,i} \neq T_{\perp,i})$ • temperature equilibration $(i \neq j,\ T_i \neq T_j)$
3. transport magnetic energy	• electrical resistivity/conductivity $(\mathbf{u}_{\rm ion} \neq \mathbf{u}_e)$ (i.e., when there’s a net current)

Our goal will be to derive how these effects depend on the micro-properties of individual Coulomb collisions.

We'll soon be working with a single-fluid approach to plasma physics: MHD.

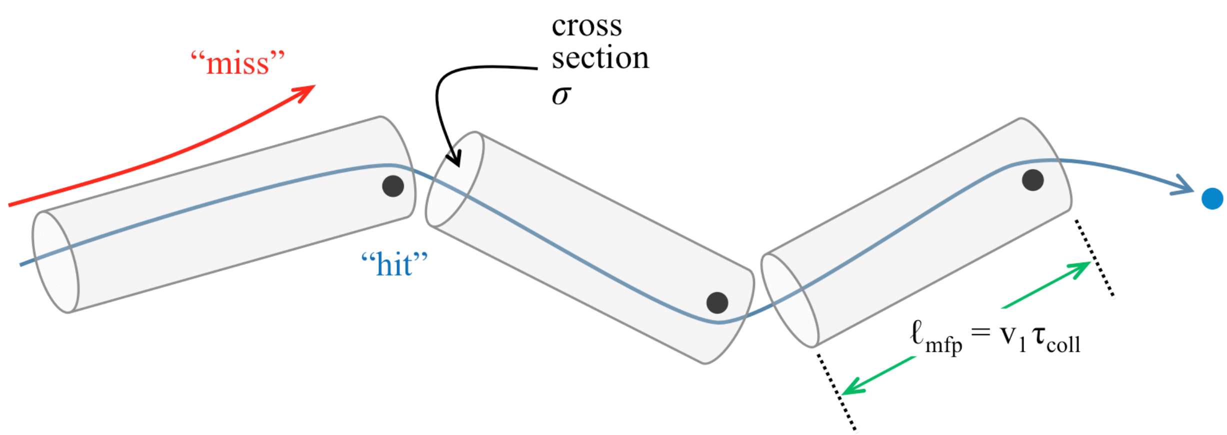
MHD is only really valid when particle-particle interactions (not necessarily 'collisions') are *frequent enough* to maintain:

- Maxwell-Boltzmann velocity distributions, and
- Common temperatures & flow speeds for all species (e^-, p^+, ions).

This happens when the **mean free path** between collisions is $\ll$ the important macroscopic length scales of the system.

But what is the mean free path ℓ_{mfp} ?

Consider a test particle (labeled 1), careening through a field of stationary "targets" (labeled 2):



The way it's drawn here, particles 1 & 2 are oppositely charged.

Look at just one "leg" of this journey as a cylinder. The volume of this cylinder, inside of which is only <u>one</u> particle of type 2, is

$$V_2 = \sigma \, \ell_{\text{mfp}} \ .$$

However, if we know the number density of particles of type 2,

$$n_2 = \frac{\#\text{ of particles}}{\text{unit volume}} = \frac{1 \text{ particle}}{V_2} \qquad \text{Thus,} \qquad \boxed{\ell_{\text{mfp}} = \frac{1}{n_2 \, \sigma}}$$

It makes sense. If either n_2 or σ were increased, it would require us to make the cylinder-length *shorter* (i.e., to make ℓ_{mfp} smaller) in order to keep just one particle inside each cylinder.

For neutral atoms, the "billiard-ball" collisional cross section is of order

$$\begin{aligned} \sigma &\approx \pi r^2 \qquad \text{where } r \approx \text{a few Å} \\ &\approx 10^{-15}\ \text{cm}^2 \qquad \text{(tiny!)} \end{aligned}$$

However, evaluating σ for ions & electrons isn't easy; it should be *larger* because their **E**-fields extend their influence beyond their physical sizes.

Note: Another key quantity is the mean time between collisions. If particle 1 is incoming with speed v_1, it will traverse the cylinder in time

$$\tau_{\text{coll}} = \frac{\ell_{\text{mfp}}}{v_1}\ , \qquad \text{so that} \qquad \tau_{\text{coll}} = \frac{1}{n_2\,\sigma\,v_1}$$

which is sometimes given as a frequency, $\nu_{\text{coll}} = n_2\,\sigma\,v_1$.

Our next goal will be to compute τ_{coll} for charged-particle collisions.

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Let's write out the **electrostatic forces** involved. Test particle 1 feels the Lorentz force from target ("field") particle 2, with

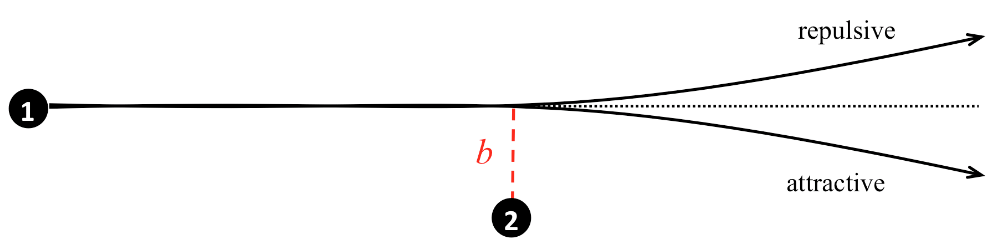
$$\mathbf{F}_{12} = \{\text{ force on 1 due to 2 }\} = q_1\mathbf{E}_2$$

$$\text{and we know} \qquad \Phi_2 = \frac{q_2}{|\mathbf{r}|}\ , \qquad \text{so that} \quad \mathbf{E}_2 = -\nabla\Phi_2 = -\frac{q_2\,\mathbf{r}}{|\mathbf{r}|^3}$$

where $\mathbf{r}$ is the relative displacement between 1 & 2.

$$\text{Thus, Coulomb's law:} \qquad \boxed{\mathbf{F}_{12} = -\frac{q_1\,q_2\,\mathbf{r}}{|\mathbf{r}|^3}}\ .$$

It's **repulsive** for like charges ... **attractive** for opposite. We'll find that, in a statistical sense, it doesn't matter! (It's the *spread* that matters.)



Note the impact parameter b (minimum value of $|\mathbf{r}|$, or distance of closest approach). Smaller $b \longrightarrow$ bigger deflection!

Let's be a bit more rigorous and write down the full equations of motion.

Start with position vectors ($\mathbf{r}_1$, $\mathbf{r}_2$) and velocity vectors ($\mathbf{v}_1$, $\mathbf{v}_2$) defined with respect to a *stationary* reference frame. There are 4 equations:

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1 \qquad \frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2 \qquad m_1\frac{d\mathbf{v}_1}{dt} = \frac{q_1\, q_2\,(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \qquad m_2\frac{d\mathbf{v}_2}{dt} = \frac{q_1\, q_2\,(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

The forces are equal and opposite ($\mathbf{F}_{12} = -\mathbf{F}_{21}$).

You can verify that the equations make sense by putting one particle at the origin, and looking at the direction in which the RHS vector points (in or out).

Everything will be easier to deal with if we transform to the **center-of-mass (CM)** reference frame. Taking linear combinations of what we already know...

Define CM positions & velocities...

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{U} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

and relative positions & velocities...

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \ .$$

If we add the two equations of motion, we find that $\dfrac{d\mathbf{U}}{dt} = 0$

i.e., the CM velocity is unchanged by this kind of "flyby."

If we take $m_1 \times \{$the $\mathbf{v}_2$ equation$\}$, and subtract it from $m_2 \times \{$the $\mathbf{v}_1$ equation$\}$, we get...

$$m_{12}\frac{d\mathbf{v}}{dt} = \frac{q_1\, q_2\, \mathbf{r}}{|\mathbf{r}|^3}$$ where the **reduced mass** is $$m_{12} \equiv \frac{m_1 m_2}{m_1 + m_2} \ .$$

It's often easier to remember it as $$\frac{1}{m_{12}} = \frac{1}{m_1} + \frac{1}{m_2} \ .$$

Reduced mass measures the "inertia" of what's in motion in the CM frame.

Note that if the target particle is so massive that it doesn't move (i.e., $m_2 \gg m_1$), then $m_{12} \approx m_1$.

If the two particles have the same mass ($m_1 = m_2 \equiv m$), then $m_{12} = m/2$.

Anyway, let's work with $\mathbf{v}$ and the reduced equation of motion, and we can always transform back to the inertial frame (i.e., $\mathbf{v}_1$ and $\mathbf{v}_2$) later.

. . . .

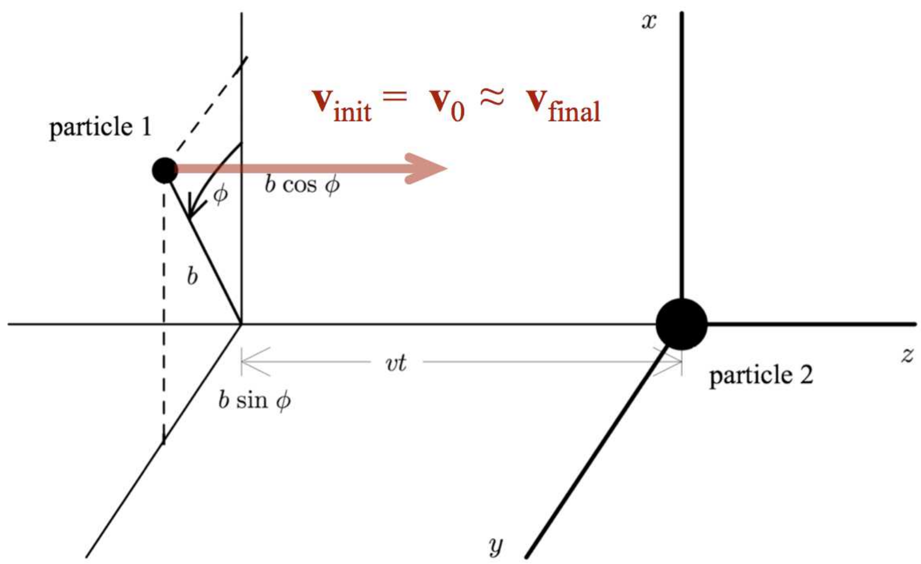
What happens during a "collision" encounter?

First let's think about relatively large impact parameters b, in which it's a glancing blow (i.e., a weak deflection). In this case,

$$|\mathbf{v}_{\rm init}| \;\gg\; |\Delta\mathbf{v}|$$

where $\Delta\mathbf{v} = \mathbf{v}_{\rm final} - \mathbf{v}_{\rm init}$, like in my sketch above.

It seems strange, but we can then **assume** (to zero-order) that particle 1's trajectory remains a straight line! We'll verify this assumption later.



Put particle 2 at the origin, so that $\mathbf{r} = \mathbf{r}_1$ & $\mathbf{v} = \mathbf{v}_1$ (assume $b =$ constant)

So, what is the time-dependent relative position $\mathbf{r}(t)$ in this system?

$$\mathbf{r} \;=\; x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z \qquad \left\{ \begin{array}{l} x = b\cos\phi \\ y = b\sin\phi \\ z = v_0 t \end{array} \right\}$$

where we define the ∼unchanged relative velocity as $\mathbf{v} = v_0\,\hat{\mathbf{e}}_z$.

We also define $t = 0$ as the time of minimum distance between the two particles, so that

$$r \;=\; |\mathbf{r}| \;=\; \sqrt{b^2 + v_0^2 t^2}\ .$$

Of course, we know there really is a (small) deflection to the trajectory. Let's integrate the equation of motion to evaluate that deflection.

Take the equation of motion and multiply both sides by dt,

$$m_{12}\, d\mathbf{v} \;=\; \frac{q_1\, q_2\, \mathbf{r}}{r^3}\, dt$$

and then integrate both sides over time...

from $\quad t = -\infty$ (at which $\mathbf{v} = \mathbf{v}_{\rm init}$) $\quad$ to $\quad t = +\infty$ (at which $\mathbf{v} = \mathbf{v}_{\rm final}$)

then the result is the total "momentum impulse"

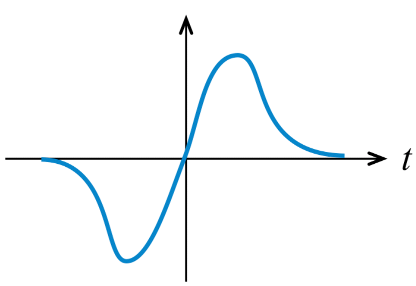
$$m_{12}\,(\mathbf{v}_{\rm final} - \mathbf{v}_{\rm init}) \;=\; m_{12}\Delta\mathbf{v} \;=\; \int_{-\infty}^{+\infty} dt\; \frac{q_1\, q_2\, \mathbf{r}(t)}{|\mathbf{r}(t)|^3} \;\; .$$

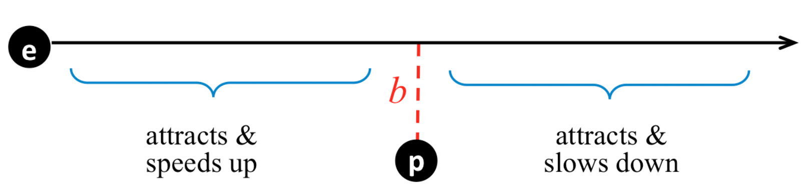
Here's where the (zero-order) assumption of a straight trajectory comes in handy...

$$m_{12} \left\{ \begin{array}{c} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{array} \right\} \;=\; q_1\, q_2 \int_{-\infty}^{+\infty} \frac{dt}{(b^2 + v_0^2 t^2)^{3/2}} \left\{ \begin{array}{c} b\cos\phi \\ b\sin\phi \\ v_0 t \end{array} \right\} \;\; .$$

Note: $\Delta v_z = 0$, since the integrand is an **odd function** of t

To understand this physically, assume the particles have opposite charges:





The changes in the z direction cancel out, to zero order (i.e., at the level of our simple linear trajectory assumption).

Thus, the only major deflection is in a direction **perpendicular** to the initial velocity vector.

Let's evaluate $\quad \Delta\mathbf{v}_\perp \;=\; (\Delta v_x)\hat{\mathbf{e}}_x + (\Delta v_y)\hat{\mathbf{e}}_y$

From now on, in this derivation, let us use subscripts $\|$ and $\perp$ to refer to motions parallel and perpendicular to $\mathbf{v}_{\rm init}$.

$$\Delta\mathbf{v}_\perp = \frac{q_1\, q_2\, b}{m_{12}} \left(\hat{\mathbf{e}}_x \cos\phi \,+\, \hat{\mathbf{e}}_y \sin\phi\right) \int_{-\infty}^{+\infty} \frac{dt}{(b^2 + v_0^2 t^2)^{3/2}}$$

where the quantity in parentheses is a $\perp$ unit vector.

$$\text{The integral is:} \qquad \left[\frac{t}{b^2\sqrt{b^2+v_0^2t^2}}\right]_{-\infty}^{+\infty} = \left(\frac{1}{b^2\,v_0}\right) - \left(-\frac{1}{b^2\,v_0}\right)$$

$$\text{Thus} \qquad \Delta\mathbf{v}_\perp = \left(\hat{\mathbf{e}}_x \cos\phi \,+\, \hat{\mathbf{e}}_y \sin\phi\right) \frac{2\, q_1\, q_2}{m_{12}\, v_0\, b}$$

and it makes intuitive sense that a smaller b means MORE deflection.

Let's simplify by defining the **Landau length,**

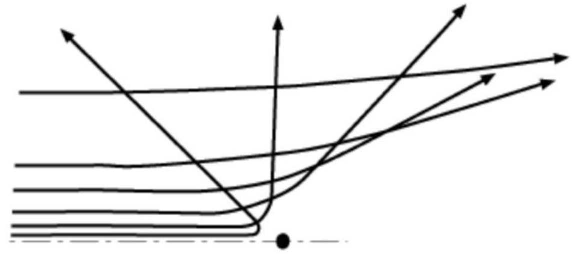
$$b_{\rm min} \equiv \left|\frac{2\, q_1\, q_2}{m_{12}\, v_0^2}\right|$$

which we also call a "minimum impact parameter" (you'll see why soon).

$$\text{Thus, the magnitude} \qquad \boxed{|\Delta\mathbf{v}_\perp| = v_0\, \frac{b_{\rm min}}{b}} \ .$$

We assumed $|\Delta\mathbf{v}| \ll v_0$, so this means that this result is valid for $b \gg b_{\rm min}$.

Note that our straight-trajectory assumption breaks down when $|\Delta\mathbf{v}| \approx v_0$, which is equivalent to $b \approx b_{\rm min}$. The velocity vector gets deflected by about $\gtrsim$1 radian, and that's about as big a deflection as we'll need to deal with. This justifies calling it $b_{\rm min}$.



For typical astrophysical plasmas, $b_{\rm min}$ is of order *a few* to *tens* of Å (atom to molecule size). This is pretty much as small as you can get without worrying about quantum effects.

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Okay, this is going to be useful, but it's still based on the constant-trajectory assumption.

That can't be completely correct, because it neglects something else we know about perfectly "elastic" collisions like these (i.e., in which $\mathbf{F}_{12} = -\mathbf{F}_{21}$):

Total energy must be conserved, too!

I'd like to show that this implies that the initial **kinetic energy** must be equal to the final kinetic energy.

At $t = -\infty$ and $t = +\infty$, there is "no potential energy" (i.e., particles are **far away** from one another and not feeling any electrostatic potential). So at these particular times, total = kinetic. Thus, $K_{\text{init}} = K_{\text{final}}$.

In inertial frame, the total kinetic energy of the 2-particle system is

$$K = \frac{1}{2} m_1 |\mathbf{v}_1|^2 + \frac{1}{2} m_2 |\mathbf{v}_2|^2$$

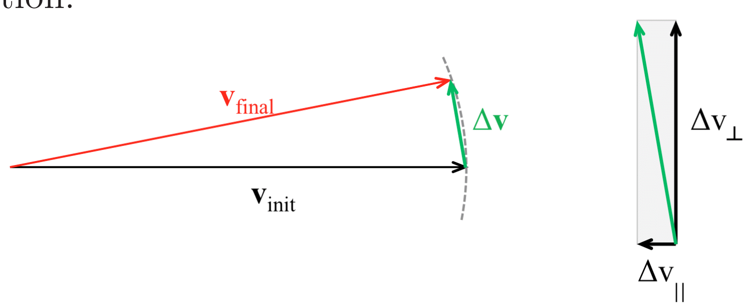
and one can use some algebra to show that

$$K = \frac{1}{2} M |\mathbf{U}|^2 + \frac{1}{2} m_{12} |\mathbf{v}|^2 \qquad \text{(where } M = m_1 + m_2\text{)}$$

so if $d\mathbf{U}/dt = 0$, as we saw above, then $K_{\text{init}} = K_{\text{final}}$ requires that $|\mathbf{v}_{\text{init}}|^2 = |\mathbf{v}_{\text{final}}|^2$. Thus, in an elastic collision, the direction of $\mathbf{v}$ can change, but its magnitude doesn't (at least when comparing initial & final states).

Recall that we're talking about the relative $\mathbf{v}$ here, not $\mathbf{v}_1$ or $\mathbf{v}_2$. In the *inertial frame,* a low-mass particle 1 can get a big "gravity assist" from a high-mass particle 2.

So, look more closely at the vector deflection $\Delta\mathbf{v}$, which comes only from a change in direction:



Yes, $\Delta\mathbf{v}$ is dominated by what we found above for $\Delta v_\perp$, but there's **also** got to be a small $\Delta v_\parallel$ component.

We'll use a sneaky trick to evaluate $\Delta v_\parallel$.

$$\text{If we say} \quad \begin{cases} \mathbf{v}_{\rm init} = \mathbf{v}_0 \\ \mathbf{v}_{\rm final} = \mathbf{v}_0 + \Delta\mathbf{v} \end{cases}$$

then the equality of magnitudes implies

$$\begin{aligned} |\mathbf{v}_0|^2 &= |\mathbf{v}_0 + \Delta\mathbf{v}|^2 \\ &= |\mathbf{v}_0|^2 + 2\mathbf{v}_0 \cdot \Delta\mathbf{v} + |\Delta\mathbf{v}|^2 \end{aligned}$$

$$\text{Due to cancellation, we get} \qquad 2\mathbf{v}_0 \cdot \Delta\mathbf{v} = -|\Delta\mathbf{v}|^2 \ .$$

- The LHS gives us the projected component of $\Delta\mathbf{v}$ *parallel* to $\mathbf{v}_0$.
- In the limit of small deflections, the RHS is dominated by the *perpendicular* component.

$$\text{Thus, we can approximate} \qquad 2v_0 \Delta v_\parallel \approx -|\Delta v_\perp|^2 \ .$$

$$\text{And, recalling} \ \frac{\Delta v_\perp}{v_0} = \frac{b_{\rm min}}{b} \ , \quad \text{then} \quad \boxed{\Delta v_\parallel \approx -\frac{1}{2v_0}|\Delta v_\perp|^2 \approx -\frac{v_0}{2}\left(\frac{b_{\rm min}}{b}\right)^2 \ .}$$

The parallel velocity change is always **negative** (when compared with initial direction of motion). It results in a relative slowdown: **friction/drag!**

As expected, if $(b_{\rm min}/b) \ll 1$, the magnitude of parallel slowdown is even tinier than $|\Delta v_\perp|$.

We now have estimates for individual changes in $\mathbf{v}$ in the two directions. How do they lead to **collective effects?**

We need to consider the cumulative effect of multiple random collisions... i.e., what's the average $\nu_{\rm coll}$ for a particle 1 traversing through a field of particle 2's ?

First, let's think separately about two different regimes:

- Large-angle collisions ($0 < b \lesssim b_{\min}$)
- Small-angle collisions ($b > b_{\min}$)

$$\left.\right\} \quad \nu_{\text{coll}} = \nu_{\text{LA}} + \nu_{\text{SA}} \ .$$

Note how $b_{\min}$ (which we already have an expression for) serves as a convenient divider between these two regimes.

For all b values smaller than $b_{\min}$, let's go back to the classical cylinder cartoon & assume cross section $\sigma = \pi b_{\min}^2$. This accounts for *all* large-angle collisions!

Thus, the collision **rate** (# of events per second, at some position in space) for large-angle (LA) collisions is

$$\nu_{\text{LA}} = n_2 \, v_0 \, (\pi b_{\min}^2) = \boxed{\frac{4\pi \, q_1^2 \, q_2^2 \, n_2}{m_{12}^2 \, v_0^3}} \ .$$

. . . .

We cannot follow this same procedure for computing ν_{SA} for small-angle deflections. In this case, $\sigma \to \infty$. However, these large-b collisions are very weak, so it's possible that the SUM over all of them still converges to a finite collision rate.

To compute ν_{SA}, we have to sum over a large range of b values, which isn't easy, since the Δv deflections depend on b itself.

> ⋆⋆⋆ When keeping track of how $\Delta v_{\parallel}$ and $\Delta v_{\perp}$ "accumulate" over multiple collisions, we *really* ought to transform back into the reference frame of particle 1. After all, these cumulative effects are happening to one given particle. However, I'll follow Callen, who expresses everything in the CM frame, and only transforms back into the inertial frame at the end. For weak, small-angle collisions, the leading-order result is the same as a more rigorous approach.

Let's start with the **slowing-down effect** of $\Delta v_{\parallel}$.
(This is "easy" because each interaction has the same sign; they add.)

As a test particle flies through a cloud of particle 2's, the total $\langle \Delta v_\parallel \rangle$ will keep increasing in magnitude as a function of time.

$$\longrightarrow \quad \{\underline{\text{more}} \text{ particle 2's mean } \underline{\text{more}} \text{ slowing down.}\} \quad \longleftarrow$$

Consider a cylindrically symmetric model of collisions again, but just for a single choice of impact parameter b & ϕ (for now).

Over a time span Δt, particle 1 will encounter N targets of species 2 at this value of b, and their effects will add linearly...

$$\langle \Delta v_\parallel \rangle \; = \; N \, \Delta v_\parallel \qquad \text{where} \qquad N \; = \; n_2 \, dV_b$$

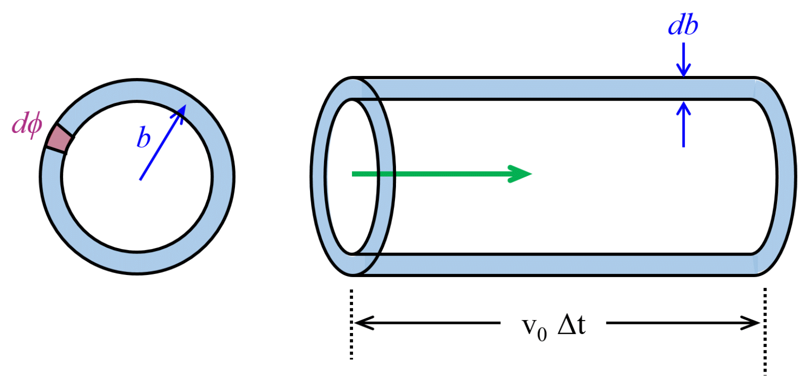
and dV_b is the volume of a piece of a shell-like "ring," centered on impact parameter b.

To let you know where we're going with this, note that it's going to be possible to write down the ratio

$$\frac{\langle \Delta v_\parallel \rangle}{\Delta t} \; \approx \; \frac{v_\parallel}{\tau_{\text{coll,SA}}} \; = \; \nu_{\text{SA}} \, v_\parallel$$

and we'll be able to pull out a value for ν_{SA}.

Anyway, each collision has a unique pair of values for (b, ϕ). Thus, each "target" species 2 corresponds to any one of the little red boxes viewed along the axis...



When we look at multiple collisions, each box gets filled up.

So, instead of summing up the deflections "in order" (i.e., in real time), let's **bin them** into their boxes. To account for them all, we'll integrate over cross-sectional area dA.

Thus, the volume element is given by

$$dV_b = (v_0\,\Delta t)dA$$

and the cross-sectional area element is $\quad dA = b\,db\,d\phi$

and we'll integrate, $\quad \langle \Delta v_\parallel \rangle = n_2\, v_0\, \Delta t \int d\phi \int db\; b\; \Delta v_\parallel$

To account for *all* weak, small-angle collisions, we'll try taking

$$b:\; b_{\rm min} \longrightarrow \infty \qquad\qquad \phi:\; 0 \longrightarrow 2\pi \quad .$$

The ϕ integral is easy, since $\Delta v_\parallel$ doesn't depend on ϕ.

Remember that $\quad \Delta v_\parallel = -\frac{v_0}{2}\left(\frac{b_{\rm min}}{b}\right)^2 \quad ,$

so...

$$\langle \Delta v_\parallel \rangle = -\pi\, n_2\; v_0^2\; \Delta t\;\; b_{\rm min}^2 \int \frac{db}{b} = -\pi\, n_2\; v_0^2\; \Delta t\; \left(\frac{4\, q_1^2\, q_2^2}{m_{12}^2\, v_0^4}\right) \int \frac{db}{b}$$

The integral gives $(\ln b)$, which **diverges** for both $b \to 0$ and $b \to \infty$.

- This explains why we use $b_{\rm min}$ as a LOWER cutoff, and treated the large-angle collisions differently. Our $b_{\rm min}/b$ model didn't apply to large-angle collisions, anyway.
- Electromagnetic interactions formally go on forever, but in practice we need an UPPER cutoff. I'll just give it, then defend it...

$$\boxed{b_{\rm max} = \lambda_{\rm D} = \sqrt{\frac{k_{\rm B} T_e}{4\pi e^2\, n_e}}} \qquad \text{(for electrons)}$$

The **Debye length** $\lambda_{\rm D}$ is the fundamental (microscopic) length scale that defines **charge separation** in a plasma.

To understand this concept better, consider 2 extreme cases...

(A) Two isolated charges, separated by distance r. For particle 1 = e^- and particle 2 = p^+, the electrostatic potential energy felt by 1 (due to 2) is

$$|U_{12}| = \frac{e^2}{r} \ .$$

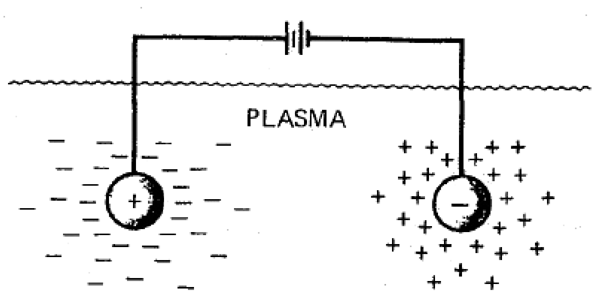
(B) Now consider another pair of charges, embedded in a hydrogen plasma. There are N other nearby p's, balanced by N other nearby e's. Since $N \gg 1$, the effect of particle 2 is drowned out. Effectively, $N \approx N + 1$, so

$$|U_{12}| \approx 0 \ .$$

However, Debye & Hückel (1923) realized that if one deposited a 'new' proton into an already neutral plasma,

$$\text{the} \ \left\{ \begin{array}{c} \text{other protons} \\ \text{electrons} \end{array} \right\} \ \text{would be slightly} \ \left\{ \begin{array}{c} \text{repelled away from} \\ \text{attracted toward} \end{array} \right\} \ \text{it!}$$

Thus, on small scales ($r \approx \lambda_D$), each charge builds up a "screening cloud" of oppositely charged particles around it that act to cancel out its net charge.



$$\text{A test particle at} \ \left\{ \begin{array}{c} r \ll \lambda_D \\ r \gg \lambda_D \end{array} \right\} \ \text{"sees" particle 2 with} \ |U_{12}| \approx \left\{ \begin{array}{c} e^2/r \\ 0 \end{array} \right\}$$

and Debye & Hückel showed that a general solution for particles in thermal equilibrium is

$$|U_{12}| = \frac{e^2}{r} \exp\left(-\frac{r}{\lambda_D}\right)$$

Thus, when $b \gg \lambda_D$, the above collision theory **breaks down** because screening effects block out the single-particle electrostatic forces.

$$\boxed{\text{i.e.,} \quad b_{max} \approx \lambda_D} \ .$$

Anyway,

$$\int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b} = \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \qquad \text{and...}$$

$$\frac{b_{\rm max}}{b_{\rm min}} = \frac{\lambda_{\rm D}}{2e^2/(m_{12}v_0^2)} \qquad \text{for electrons \& protons}$$

$$\approx \frac{\lambda_{\rm D}\, k_{\rm B}T}{e^2} \qquad \text{for } \textit{thermal} \text{ motions } \left(\frac{1}{2}mv_0^2 = k_{\rm B}T\right)$$

$$= 4\pi n\, \lambda_{\rm D}^3 \equiv \Lambda \qquad \text{the } \textbf{plasma parameter.}$$

Λ is $\approx$ the number of particles inside a "Debye sphere."

- Plasma physics is valid only for $\Lambda \gg 1$, which is good because this is the situation for most astrophysical systems.
- $\Lambda \ll 1$ applies for crystals, degenerate WDs, metallic gas-giant cores.

In collision theory, we call $\ln\Lambda$ the **Coulomb logarithm.**

Most astrophysical plasmas have $\Lambda \approx 10^5$ to 10^{15}, so: $\boxed{\ln\Lambda \approx 10 \rightarrow 30\ .}$

Quantum aside: There are some cases where $b_{\rm min}$ *isn't* the right quantity to use in the definition of Λ. For high enough temperatures, the particle's **de Broglie wavelength** $\lambda_{\rm dB}$ exceeds $b_{\rm min}$, and we need to use

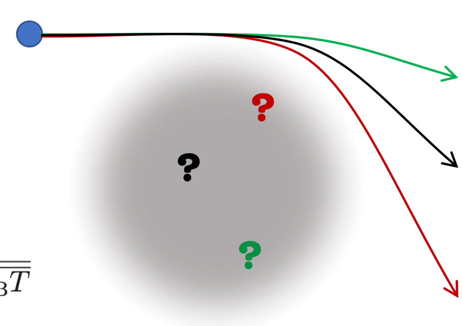
$$\Lambda = \frac{b_{\rm max}}{\max\{b_{\rm min}, \lambda_{\rm dB}\}}\ .$$

If we know exactly where the particle is, we can state clearly when large-angle collisions occur, and when they don't...

But if there's **quantum uncertainty,** there may be a chance of large or small angle collisions. To be safe, we've got to use the size of the "cloud."

$$\lambda = \frac{h}{p}\ , \quad \text{and} \quad \frac{3}{2}k_{\rm B}T = \frac{p^2}{2m_{12}}\ , \quad \text{so} \rightsquigarrow \quad \lambda = \frac{h}{\sqrt{3m_{12}k_{\rm B}T}}$$

However, $b_{\rm min}$ goes as $1/v^2 = 1/T$, so de Broglie wins at high enough T.



Thus, the collective effect of all small-angle (large b) collisions gives

$$\frac{\langle \Delta v_{\parallel} \rangle}{\Delta t} = -\frac{4\pi\, q_1^2\, q_2^2\, n_2}{m_{12}^2\, v_0^2} \ln \Lambda$$

and this is a statistically averaged equation of motion for how much a test particle **slows down** (vs. time) in the CM frame.

It's not too bad an approximation to rewrite the LHS as $\displaystyle \frac{\langle \Delta v_{\parallel} \rangle}{\Delta t} \approx \frac{dv_{\parallel}}{dt}$,

and if we write the RHS as a rate times $v_0 = v_{\parallel}$, then we get an equation of **collisional friction,**

$$\boxed{\frac{dv_{\parallel}}{dt} = -\nu_{\mathrm{SA}}\, v_{\parallel}} \quad \text{where} \quad \nu_{\mathrm{SA}} = \frac{4\pi\, q_1^2\, q_2^2\, n_2}{m_{12}^2\, v_0^3} \ln \Lambda = \nu_{\mathrm{LA}} \ln \Lambda \quad !$$

Because $\ln \Lambda \approx 10$–30, the cumulative effect of small-angle collisions **dominates** the rate due to large-angle collisions, by about an order of magnitude. Traditionally, we ignore ν_{LA}.

We've seen how test particles slow down. Recall that $\Delta v_{\perp}$ was a *larger* effect. How does that kind of deflection affect a test particle after $N \gg 1$ collisions?

We computed $\langle \Delta v_{\parallel} \rangle = \langle \Delta v_z \rangle$. How about summing up $\langle \Delta v_x \rangle$ or $\langle \Delta v_y \rangle$?

$$\text{Recall} \qquad \Delta v_x = \Delta v_{\perp} \cos\phi = v_0 \cos\phi \frac{b_{\min}}{b}$$

So, as before,

$$\langle \Delta v_x \rangle = N\, \Delta v_x = n_2\, v_0\, \Delta t \int_0^{2\pi} d\phi \int db\; b\; v_0 \cos\phi \frac{b_{\min}}{b}$$

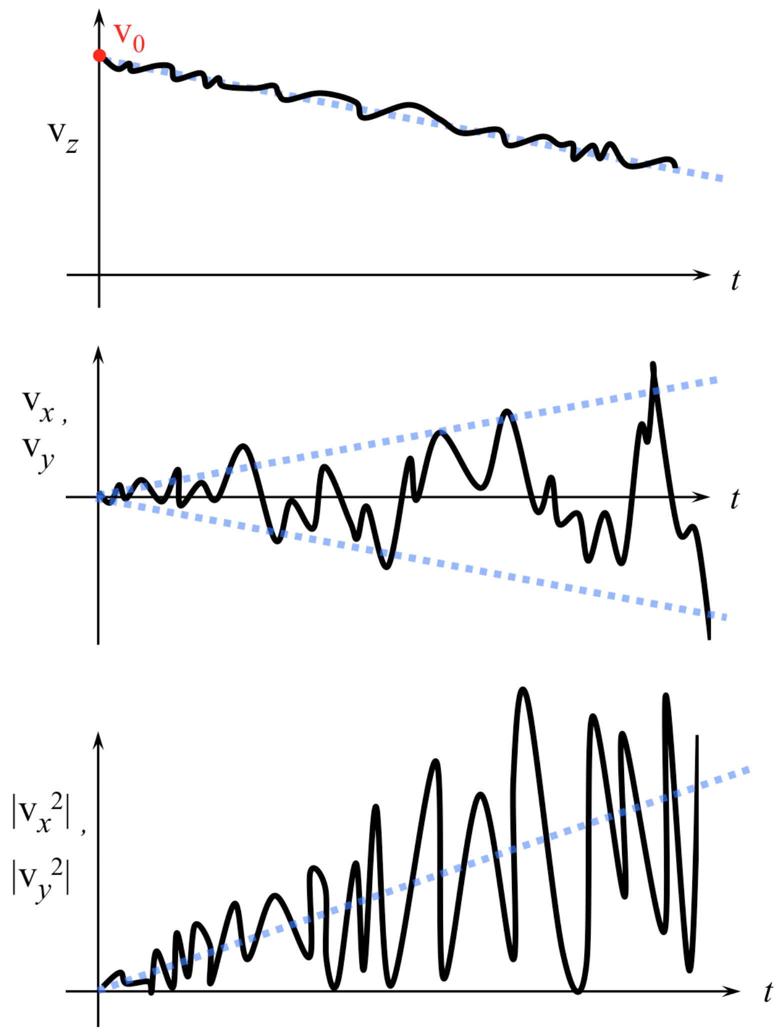
and the ϕ integral is zero!

The same goes for $\langle \Delta v_y \rangle$, so $\boxed{\langle \Delta \mathbf{v}_{\perp} \rangle = 0}$ too.

Sound familiar? This looks like a random walk, in which there is no preferred direction for the test particle to drift into.

Continuing the analogy with random walk, we know there <u>is</u> an overall $\perp$ drift.

The ensemble-mean 'displacement' (in velocity) doesn't change with time, but the **r.m.s.** grows:



Thus, we can sum the squared 'displacements' like we did in random-walk theory...

$$\begin{aligned} \langle \Delta v_\perp^2 \rangle &= N \, |\Delta \mathbf{v}_\perp|^2 \\ &= n_2 \, v_0 \, \Delta t \int d\phi \int db \;\; b \; |\Delta \mathbf{v}_\perp|^2 \end{aligned}$$

...and one can turn the crank on the algebra, just like before.

OR we could do it the easy way. Remember that, for elastic collisions,

$$\Delta v_\parallel \;=\; -\frac{1}{2v_0}|\Delta \mathbf{v}_\perp|^2 \;\; .$$

Thus, if we make the (reasonable) leap of faith that

$$\langle \Delta v_\parallel \rangle \;=\; -\frac{1}{2v_0}\langle \Delta v_\perp^2 \rangle$$

then we can use the slowing-down equation to show that

$$\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t} \;=\; \frac{8\pi\, q_1^2\, q_2^2\, n_2}{m_{12}^2\, v_0}\; \ln \Lambda$$

which is the same answer we would have gotten from turning the crank on the algebra above.

We have two coupled equations:

$$\boxed{\frac{dv_\parallel}{dt} \;=\; -\nu_{\rm SA}\, v_\parallel} \qquad \text{and} \qquad \boxed{\frac{d\langle v_\perp^2 \rangle}{dt} \;=\; \frac{dv_\perp^2}{dt} \;=\; +2\,\nu_{\rm SA}\, v_\parallel^2}$$

Both friction (parallel) and diffusion (perpendicular) are present, which is reminiscent of the *fluctuation–dissipation theorem.*

Eventually any initial $v_\parallel$ slows down via “friction,” and maintains random fluctuations around zero. Simultaneously, the $v_\perp$ motions *ramp up* from zero and saturate with random fluctuations (around zero). Eventually, all three (v_x, v_y, v_z) fluctuations are in thermal equilibrium with one another, and the initial state is forgotten.

. . . .

Note that a key parameter in both equations is $v_\parallel = v_0$.

These are still equations for the evolution of a single **test particle** that was shot in with an initial velocity.

What we really want are equations that tell us how **two distributions** of particles interact with each other via collisions.

. . . .

We'll get there, but first we should think about getting out of the CM frame and back to equations for species 1 (or 2) in the inertial frame.

$$\text{Recall:} \qquad \mathbf{v} \;=\; \mathbf{v}_1 - \mathbf{v}_2 \qquad\qquad \mathbf{U} \;=\; \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{M}$$

where $M = m_1 + m_2$. Thus,

$$\mathbf{v}_1 \;=\; \mathbf{U} + \frac{m_2\mathbf{v}}{M} \qquad\qquad \mathbf{v}_2 \;=\; \mathbf{U} - \frac{m_1\mathbf{v}}{M}$$

Look at small differential changes in velocity, keeping in mind that
$\mathbf{U}$ = constant in any elastic collision:

$$d\mathbf{v}_1 \;=\; +\frac{m_2}{M}d\mathbf{v} \qquad\qquad d\mathbf{v}_2 \;=\; -\frac{m_1}{M}d\mathbf{v}$$

$$\rightsquigarrow \quad \boxed{m_{12}\Delta\mathbf{v} \;=\; m_1\Delta\mathbf{v}_1 \;=\; -m_2\Delta\mathbf{v}_2} \quad \text{(essentially momentum conservation!)}$$

Thus, the coupled equations for a single particle of species 1 can be written as

$$\frac{dv_{\parallel 1}}{dt} \;=\; -\nu_{\mathrm{S1}}\,\left(v_{\parallel 1} - v_{\parallel 2}\right) \qquad \text{and} \qquad \frac{dv_{\perp 1}^2}{dt} \;=\; +2\,\nu_{\mathrm{S1}}\,|\mathbf{v}_1 - \mathbf{v}_2|^2$$

$$\text{where} \qquad \nu_{\mathrm{S1}} \;=\; \frac{4\pi\, q_1^2\, q_2^2\, n_2\ \ln\Lambda}{m_1\, m_{12}\, |\mathbf{v}_1 - \mathbf{v}_2|^3}\ .$$

. . . .

Eventually, we'll get to the point where these "particle-by-particle" equations are transformed into equations for how a distribution of test particles $f(\mathbf{v}_1)$ is changed when encountering a distribution of field particles $f(\mathbf{v}_2)$.