You're getting three (or four?) courses in one:

- **Astrophysical Plasmas:** focusing on two aspects...
  - **Collisions & transport phenomena:** understanding how charged particles interact with one another on micro-scales to produce macro-effects like conductivity & viscosity.
  - **Magnetohydrodynamics (MHD):** an approximate fluid description of a magnetized plasma (ubiquitous in space) and some liquid conductors (e.g., molten planetary interiors).
- **Gravitational Dynamics:** the study of how mutually gravitating bodies interact... from 2-body Keplerian motion to ($N \gg 1$)-body galaxies.
- **Radiative Processes:** treating a collection of photons as something as continuous and "fluid" as a collection of particles, and solving for how the radiation field interacts with matter.

Our (my?) challenge is to show how these fields are unified by some fundamental physical processes.

Lots of our time will be spent separating these processes into:

- **Macroscopic** physics (continuous, classical "forces and fields"), vs.
- **Microscopic** physics (statistics of many individual random events).

The bridge between the worlds is typically described by **transport phenomena,** in which we also must distinguish between

- **Collisionless** systems (where particles don't interact with one another; only with large-scale forces/fields), vs.
- **Collisional** systems (where particles do interact with one another).

Examples of 'particles:" ions, electrons, neutral atoms, photons, ice/dust grains, planetesimals, stars, dark matter particles?

We'll start with some introductory background/review.

**Syllabus review...** These lecture notes are the main "textbook." Please read everything, even if we skip some parts in class.

## Background/Review Topic 1 of 2: Differential Equations

I'll assume you're relatively comfortable with ordinary differential equations (ODEs). Knowing the *integrating factor method* for first-order ODEs should certainly be in your toolbox (see the useful–formula handout).

There are a number of classical partial differential equations (PDEs) that show up frequently enough for us to review them at the beginning of this course. Let's review three of them:

### (1a) The Advection/Transport Equation

Baked into many conservation laws are expressions that look like

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f) = 0$$

for some continuous function $f(\mathbf{r}, t)$.

In some cases (i.e., an incompressible fluid), the flow velocity $\mathbf{v}$ obeys

$$\nabla \cdot \mathbf{v} = 0 \ ,$$

and in that case,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \equiv \frac{Df}{Dt} = 0$$

i.e., $f =$ constant along the flow path.
$Df/Dt$ is the **advective derivative** (sometimes called the convective, material, substantial, total, Lagrangian, Stokes, etc. ... derivative)

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Before we move on, let's just pause to think about what the advective derivative really means. Consider an **example** in which we're sitting in a meadow, looking at the trees. Define $f$ as the density of trees at any point in the meadow.

If we're just sitting still, the trees are (slooooowly) growing where they stand. Nothing is moving around in space. Thus, $f = f(t)$ only, and the total derivative

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t}$$

is just the partial derivative with respect to time.

However, what if we're in a boat on a river, drifting along the $x$ direction, and the trees are getting denser/thicker as we go down river. In our reference frame, $f = f(t, x)$, and $x$ itself is a function of time.

$$f = f(t, x(t)) \ , \qquad \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{dx}{dt}}_{v_x} \frac{\partial f}{\partial x} \qquad \text{(chain rule!)}$$

If $f$ increases as $x$ increases, then $\partial f/\partial x > 0$. We see the summed effect of 2 kinds of "increase" in tree density: in time, and in space.

In general, if we're looking at a little "fluid parcel" (that's evolving in time AND moving around in space with velocity $\mathbf{v}$), the total change in some quantity $f$ associated with the parcel (density, temperature, etc.) is given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$$

Remember that $\partial f/\partial t$ is just the change in the *local* value of $f$ at a fixed location, as multiple parcels move through it. But often we want to track what's going on with **one** parcel as it moves, and for this we want $Df/Dt$.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Back to PDEs. The 1D Cartesian version of the advection equation is

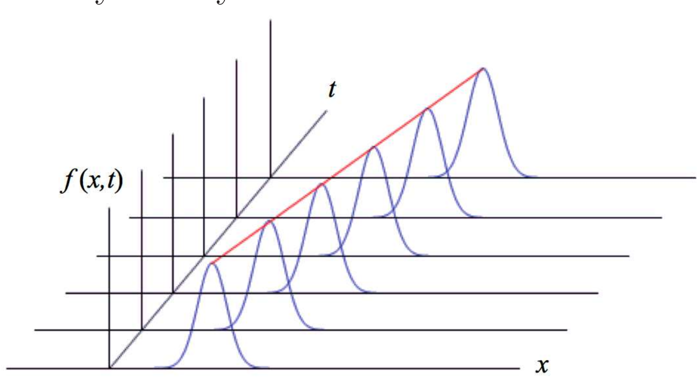
$$\boxed{\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} = 0} \qquad \text{with initial condition } f(x,t) = f_0(x) \text{ at } t = 0.$$

and for now let's assume $v_x = \text{constant}$.

D'Alembert showed that the solution to the advection equation is

$$f(x,t) = f_0(x - v_x t) \qquad \text{(i.e., a pulse moving with speed } v_x\text{)}$$

where $f_0$ can essentially be any well-behaved function of one variable.



It's easy to verify that this satisfies the equation, with the chain rule.

However, there's also a rigorous way to obtain this solution, using Fourier transforms. It's overkill for this particular equation, but it generalizes to much more complicated PDEs.

**Joseph Fourier** himself worked this out [for the heat equation] around 1801, while deployed in Egypt in Napoleon's army.

For a continuous function $f(x)$, define the Fourier transform and inverse Fourier transform:

$$g(k) = \int_{-\infty}^{+\infty} dx \; f(x) \; e^{ikx}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; g(k) \; e^{-ikx}$$

Take the advection equation PDE, multiply each term by $e^{ikx}$, and integrate over all $x$,

$$\int_{-\infty}^{+\infty} dx \; e^{ikx} \left( \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} \right) = 0$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, e^{ikx} f \; + \; v_x \int_{-\infty}^{+\infty} dx \, e^{ikx} \frac{\partial f}{\partial x} = 0$$

<u>1st term on LHS:</u> This is just $\partial g / \partial t$.

<u>2nd term on LHS:</u> As long as $f(x) \to 0$ as $x \to \pm\infty$, it can be evaluated by *integration by parts.*

The transformed equation becomes:

$$\frac{\partial g}{\partial t} - ikv_x g = 0$$

which shows why Fourier transforms are useful for "converting" derivatives into products. This is just a 1st order ODE in time,

$$g(k,t) = g_0(k) \, e^{ikv_x t} \; .$$

To get the desired solution $f(x,t)$, we perform the inverse transform,

$$f(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\; g_0(k)\; e^{-ik(x-v_x t)} \qquad (\text{“full solution”})\;.$$

It's clear that $g_0(k)$ is just the Fourier transform of the **initial condition** $f_0(x) = f(x,0)$, i.e.,

$$f_0(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\; g_0(k)\; e^{-ikx}$$

Thus, if we take the full solution above, and substitute $\xi = x - v_x t$, we see that the full solution is just the inverse transform

$$f(x,t) \;=\; f_0(\xi) \;=\; f_0(x - v_x t)$$

just as Fourier and D'Alembert found.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

If the above was hand-wavy, I should note that we can do it more rigorously. The full solution contains $g_0(k)$, which is the transform of the initial condition. Plugging that in directly,

$$\begin{aligned} f(x,t) \;&=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\; e^{-ik(x-v_x t)} \left[\int_{-\infty}^{+\infty} dx'\; f_0(x')\; e^{ikx'}\right] \\ &=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx'\; f_0(x') \left[\int_{-\infty}^{+\infty} dk\; e^{-ik(x-x'-v_x t)}\right] \end{aligned}$$

This is helpful only if you've seen that the Fourier transform of a **constant function** is the Dirac delta function (and vice versa); i.e.,

$$\delta(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\; e^{-ikx} \;\;.$$

Thus, our full solution is

$$\begin{aligned} f(x,t) \;&=\; \int_{-\infty}^{+\infty} dx'\; f_0(x')\; \delta(x - x' - v_x t) \\ &=\; f_0(x - v_x t)\;\;. \end{aligned}$$

---

## (1b) The Wave Equation

I'm sure you've seen it before. It's kind of similar to the advection equation, since solutions "propagate" in space. In 1D,

$$\boxed{\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}} \qquad \text{again assuming} \quad c = \text{constant} \ .$$

D'Alembert also showed there are pulse-propagating solutions. If we define

$$\alpha = x + ct \qquad\qquad \beta = x - ct$$

$$\text{then one can transform the wave equation into} \qquad \frac{\partial^2 f}{\partial \alpha \, \partial \beta} = 0$$

which can be integrated twice to obtain

$$f(x,t) = A(\alpha) + B(\beta) = A(x+ct) + B(x-ct)$$

meaning that waves can propagate to the left or the right, depending on the initial condition.

We could've also used the Fourier transform method to obtain this solution.

Alternately, one can look for *separable* solutions of the form $f(x,t) = X(x)T(t)$. Those are sinusoids: $\exp\left[\, i(kx \pm \omega t) \,\right]$, with $c = |\omega/k|$.

---

## (1c) The Diffusion Equation

Also called the heat equation, or Fick's law(s) of diffusion. In general,

$$\frac{\partial f}{\partial t} = \nabla \cdot [D(\mathbf{r}) \, \nabla f]$$

but if $D$ = constant and we limit ourselves to 1D Cartesian geometry,

$$\boxed{\frac{\partial f}{\partial t} = D \, \frac{\partial^2 f}{\partial x^2}}$$

where $D$ is the **diffusion coefficient** (units of length$^2$/time). It shows up in a lot of physical contexts!

The Fourier transform method gives us an elegant solution; equivalent in many ways to a **Green's function** solution. Let's derive it.

Taking the transform of each term like before,

$$\frac{\partial g}{\partial t} - D \int_{-\infty}^{+\infty} dx \; e^{ikx} \; \frac{\partial^2 f}{\partial x^2} \; = \; 0 \; .$$

To evaluate the 2nd term, one needs to perform integration by parts **twice,** and assume both $f$ and $\partial f / \partial x$ both $\to 0$ as $|x| \to \infty$. Thus,

$$\frac{\partial g}{\partial t} + k^2 D g \; = \; 0$$

which is essentially the same ODE (in time!) as before, with the solution

$$g(k,t) \; = \; g_0(k) \, e^{-k^2 D t} \; .$$

Remember that $g_0(k)$ is the Fourier transform of the initial condition $f_0(x)$, so take the inverse transform...

$$\begin{aligned} f(x,t) \; &= \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{-ikx} \; g_0(k) \; e^{-k^2 D t} \\ &= \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{-ikx - k^2 D t} \; \left[ \int_{-\infty}^{+\infty} dx' \, f_0(x') \, e^{ikx'} \right] \\ &= \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx' \; f_0(x') \; \left[ \int_{-\infty}^{+\infty} dk \; e^{-k^2 D t + ik(x' - x)} \right] \; . \end{aligned}$$

As long as $D > 0$, the final integral in square brackets is well-behaved. It can be looked up, or one can "complete the square" of the quadratic function of $k$ in the exponent...

$$-k^2 D t + ik(x' - x) \; = \; -Dt(k - k_0)^2 - \frac{(x - x')^2}{4Dt}$$

where $k_0$ is just a function of other "constants" (i.e., anything except $k$). Thus,

$$f(x,t) \; = \; \int_{-\infty}^{+\infty} dx' \; f_0(x') \; \left\{ \frac{1}{\sqrt{4\pi D t}} \exp\left[ -\frac{(x - x')^2}{4Dt} \right] \right\} \; .$$

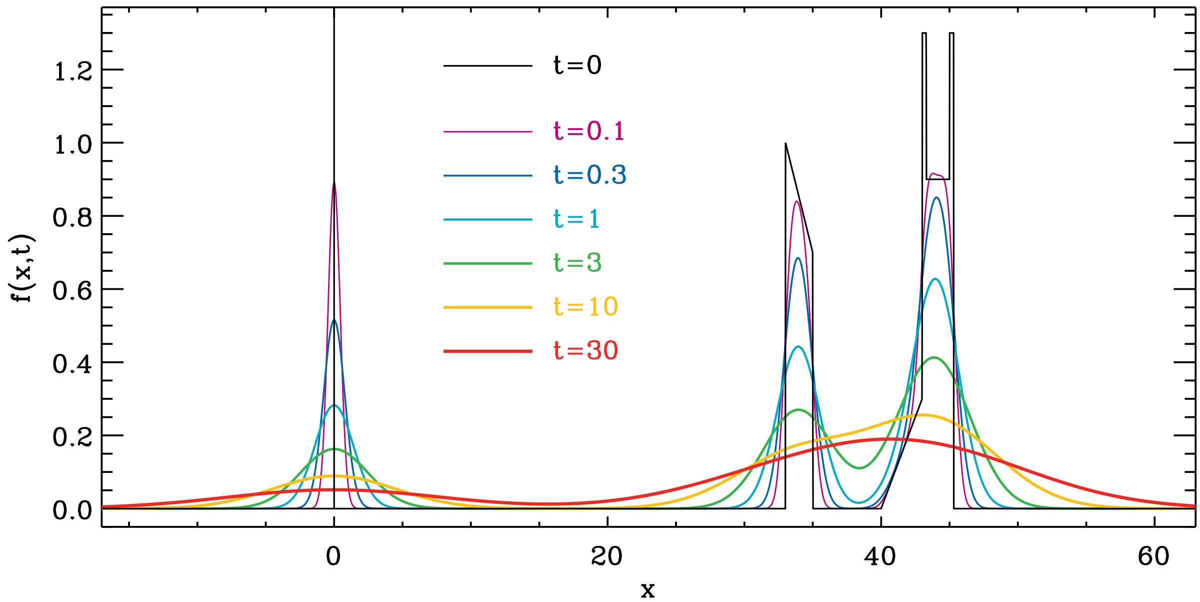
The quantity in {} is a "kernel" that plays the same role as the Green's function.

If $f_0(x') = \delta(x')$, then

$$f(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{x^2}{4Dt}\right] \quad ,$$

which is a normal distribution with $\sigma = \sqrt{2Dt}$.

Example solutions ($D = 1$) for a delta function & an arbitrary $f_0(x)$:




As $t \to \infty$, $f(x,t)$ "wants" to diffuse to a constant value.

If $x$ subtends all space, then $\boxed{f \to 0}$ .

Despite looking like a hybrid of the advection & wave equations, the diffusion equation has solutions that exhibit some very **different** properties:

1. There's a clear **directionality** of time to the solution; initial structure is irreversibly smeared out as $t$ marches on.

2. Formally, diffusion happens at **infinite speed.** If one starts with a delta function at $t = 0$, then at any $t > 0$ there is a finite value of $f(x,t)$ at all values of $x$. This is unrealistic; resolving it requires a closer look at the underlying physics.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

It's worthwhile to look at **moments** (or **expectation values**) of this solution... i.e., averages taken over all $x$, weighted over both $f(x,t)$ (which acts like a probability distribution) and over some other functions of $x$.

The mean value of $x$ at a given time $t$ is given by

$$\langle x \rangle \;\equiv\; \frac{\int dx \; x \; f(x,t)}{\int dx \; f(x,t)} \;.$$

For the version of $f$ obtained from a Dirac $\delta$ function initial condition, the denominator is normalized to 1.

The numerator is an odd function... so $\langle x \rangle = 0$. Thus, pure diffusion always maintains the initial "center of gravity" of a distribution.

What about the mean value of $x^2$ ?

$$\langle x^2 \rangle \;=\; \int dx \; x^2 \; f(x,t) \;=\; \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{+\infty} dx \; x^2 \, e^{-x^2/(4Dt)} \;=\; \rightsquigarrow \;=\; 2Dt$$

and we see that $2Dt$ is essentially $\sigma^2$ (i.e., the variance, or the square of the standard deviation) of this **normal distribution.**

Thus, for classical diffusion the r.m.s. "width" of the distribution increases as $\boxed{\sigma \propto t^{1/2}}$.

FYI, there are non-classical "disordered" systems for which the diffusion is "anomalous:"

$$\sigma \propto t^{\alpha} \qquad \left\{ \begin{array}{ll} \alpha > 1/2 & \text{superdiffusion} \\ \alpha < 1/2 & \text{subdiffusion} \end{array} \right\}$$

In some cases (e.g., fractal media), $\alpha$ corresponds to something real. Other times, it's just an empirical *fitting parameter,* and the actual physics is more complicated than "just" a modified type of diffusion.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

In 3D, the diffusion equation is $$\frac{\partial f}{\partial t} \;=\; D \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

and it's straightforward to show that

$$\langle r^2 \rangle \;=\; \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle \;=\; \boxed{6Dt} \;.$$

## Background/Review Topic 2 of 2: Classical Forces/Fields

We'll concentrate only on two of the four fundamental forces of the Standard Model: electromagnetism & gravity.

### E&M:

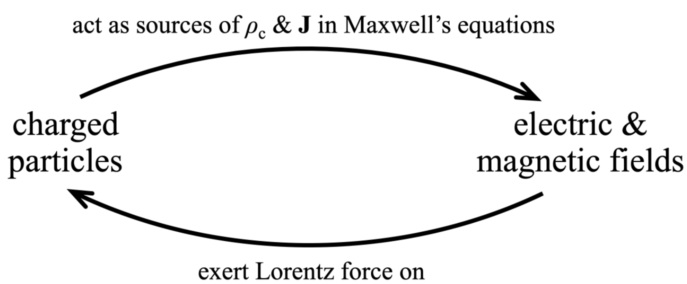
I'll be using Gaussian/c.g.s. units ("astronomer's units") as a default. Why?

- We need to stay connected to our history... I'll try to cite relevant milestone papers (all of which likely used cgs), and you're encouraged to *always* follow the bread crumbs back into the literature.
- **E** and **B** essentially have the same units.
- Electric charges are "natural" (Coulomb force: $F = q_1 q_2 / r^2$).

Whenever we deal with E&M, we'll also be thinking about **plasmas.**

Plasmas are not really a "vacuum" because they're filled with particles. However, we'll use the vacuum E&M equations and treat the particles as discrete, point-like "add-on" sources of charge.

Thus, forget **D** and **H**. Just electric field **E** and magnetic field **B**.



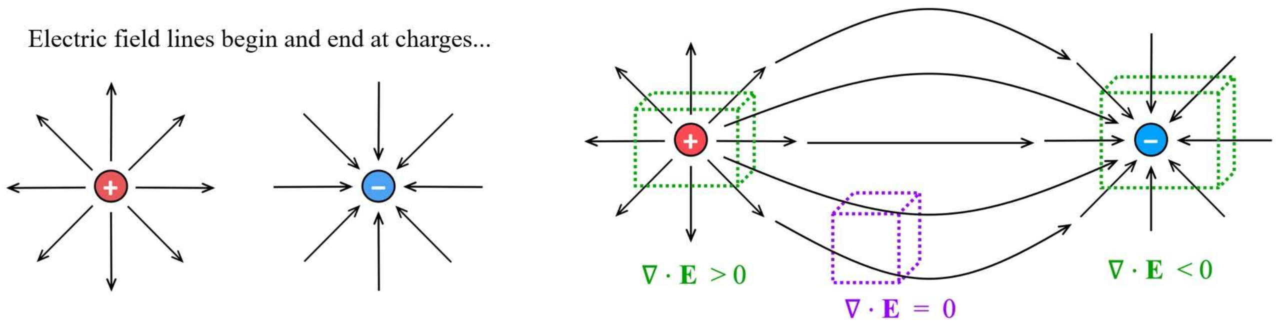

If you know **E** and **B**, you know the non-relativistic **Lorentz force** exerted by them on a particle with charge $q$,

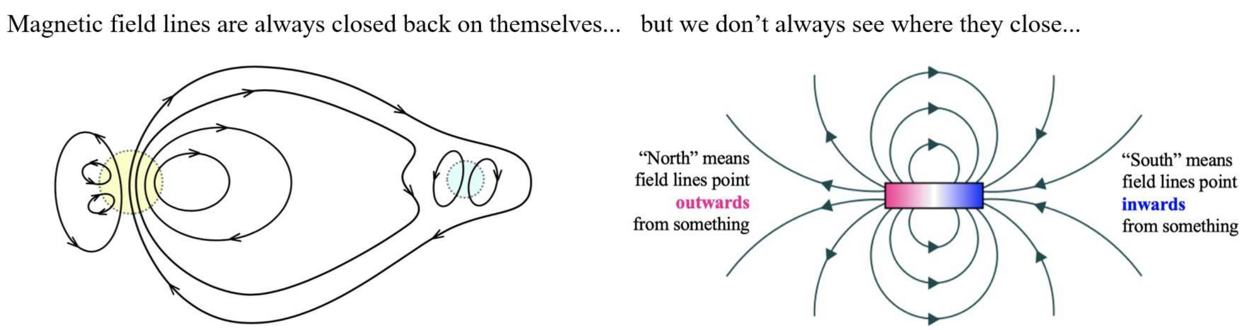
$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

- Note that a stationary charge only feels the effects of an electric field, and that force is parallel to **E**.
- *Motion* is required to feel the effects of magnetic field, and the Lorentz force is perpendicular to both **B** and the particle's current velocity vector.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are continuous vector fields. At each point in space, you can draw a little vector, and then later "connect" them with **field lines.**




For $\mathbf{E}$, field lines = *lines of force.* In absence of other forces, particles will be accelerated parallel to field lines. Electric field lines begin & end with electric charges.

Magnetic field lines are all CLOSED... there are no magnetic charges.

The magnetic Lorentz force *isn't* parallel to the field lines, but they're still useful:

- Some pieces of physics make the most sense when thinking about "how many" field lines cross through a surface.
- Sometimes one can *see them!* Plasma often organizes itself ALONG field lines – i.e., rates of conductivity, particle transport, etc., are quick/easy in the $\parallel$ direction; not in the $\perp$ direction. **(coronal loops)**
- In some ways, we'll see that field lines have some elasticity/tension to them; i.e., they behave like taut wires.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

The combined system of E&M fields and charged particles obeys all four **Maxwell's equations,** which relate $\mathbf{E}$ & $\mathbf{B}$ to:

- $\rho_c$ (charge density): just what it says: how much electric charge is concentrated into a given volume (statcoulombs/cm$^3$).
- $\mathbf{J}$ (current density): how much charge is <u>in motion</u> in a given volume (units: $\rho_c \mathbf{v}$... also: statamperes/cm$^2$). To get total current $I$ (in statamperes) passing through a given surface area $d\mathbf{A}$, integrate over $\mathbf{J} \cdot d\mathbf{A}$.

We'll soon write rigorous definitions for $\rho_c$ and $\mathbf{J}$ as sums over the properties of charged particles.

In Maxwell's equations, $\rho_c$ & $\mathbf{J}$ mainly are "sources" for $\mathbf{E}$ & $\mathbf{B}$. However, $\mathbf{E}$ & $\mathbf{B}$ also feed back on $\rho_c$ & $\mathbf{J}$, so the whole system is complex & coupled.

The first of Maxwell's equations...

$$\nabla \cdot \mathbf{E} = 4\pi\rho_c \qquad \text{(Gauss's law)}$$

which tells us how electric fields are *generated* by charge imbalances.

We already know this from Coulomb's law: a point-charge in a vacuum exerts an electric field. The field due to a point charge $q_1$ (at the origin) is

$$\mathbf{E}(\mathbf{r}) = q_1 \frac{\mathbf{r}}{|\mathbf{r}|^3} \qquad \left[ \text{ with } \quad \rho_c(\mathbf{r}) = q_1\,\delta(\mathbf{r}) \right] .$$

The associated Lorentz force on a test particle $q_2$, due to the point charge $q_1$ at the origin, has a magnitude

$$|\mathbf{F}| = \frac{|q_1 q_2|}{r^2}$$

and the **electrostatic potential energy** due to these two charges is the work done bringing in the test charge from infinity,

$$U_{\mathrm{E}} = -\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \frac{q_1 q_2}{r}$$

which we'll use later to compare with other kinds of energy.

The 2nd Maxwell's equation is:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)}$$

Another way to generate an electric field is to have a time-varying magnetic field. Faraday invented the *dynamo.*

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

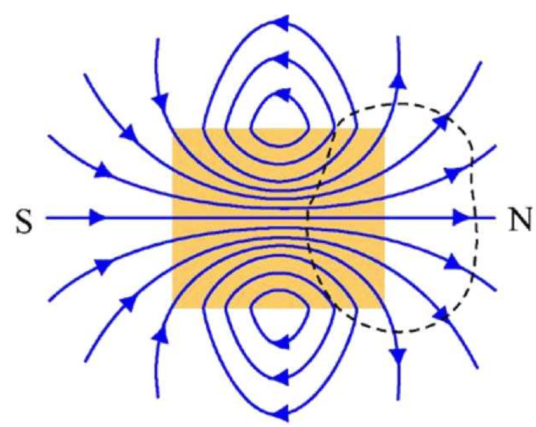
The last two Maxwell's equations involve the magnetic field.

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(law of no magnetic monopoles)}$$

Also known as the conservation of magnetic flux. Any closed 3D volume must have equal amount of **field lines** poking into it as poking out of it.

Integral version of $\nabla \cdot \mathbf{B} = 0$ given by the divergence theorem:

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$



Magnetic Flux

The last of Maxwell's equations:

$$\nabla \times \mathbf{B} = \frac{4\pi \mathbf{J}}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère's law)}$$

Magnetic fields can be generated by either currents (moving charges relative to one another), or by time-variability of the electric field.

In plasma physics, we will often note that the **displacement current** term ($\partial \mathbf{E}/\partial t$) has a tiny magnitude compared to the other terms, and it will be ignored.

However, for E&M waves propagating in a pure vacuum ($\mathbf{J} = 0$), the displacement current term is important.

With what we've given so far, we can demonstrate **charge conservation:**

Take $\nabla\cdot$ Ampère's law, and then use Gauss's law, to obtain

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J} \;=\; 0$$

which says that any local creation or destruction of charge in a box is balanced exactly by the 'flux' of charge in or out of the box.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

One can manipulate Maxwell's equations into a **wave equation** for electromagnetic fluctuations.

Take the curl of both sides in the vacuum version of Ampère's law:

$$\nabla \times (\nabla \times \mathbf{B}) \;=\; \frac{1}{c}\,\nabla \times \frac{\partial \mathbf{E}}{\partial t} \;=\; \frac{1}{c}\,\frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

and we can use Faraday's law to rewrite the right-hand side:

$$\nabla \times (\nabla \times \mathbf{B}) \;=\; -\frac{1}{c^2}\,\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}}{\partial t}\right) \;.$$

Lastly, use a vector identity to rework the left-hand side:

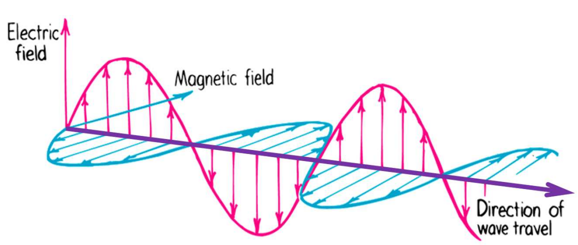
$$\nabla \times (\nabla \times \mathbf{B}) \;=\; \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$$

and, since $\nabla \cdot \mathbf{B} = 0$, it leaves us with a wave equation,

$$\nabla^2 \mathbf{B} \;=\; \frac{1}{c^2}\,\frac{\partial^2 \mathbf{B}}{\partial t^2} \qquad \text{with phase speed} \qquad \boxed{V_{\rm ph} \;=\; c\;.}$$

We could have started by taking the curl of Faraday, then plugging in Ampère, and we'd get the same wave equation for **E**.

Then, either Faraday or Ampère tell us that wavelike solutions for **E** & **B** are mutually perpendicular, and both $\perp$ to the direction of propagation **k**:



Throughout the course, we'll deal a lot with electromagnetic radiation that passes through gas & plasma. Atoms & ions absorb and/or scatter radiation.

(e.g., if $\mathbf{J} \neq 0$ when charged particles are present, there would be a first-order $\partial/\partial t$ term in the wave equation: waves would be **damped**)

Last piece of E&M review: One can show how Maxwell's equation lead to an expression for electromagnetic energy conservation, with a net "loss" when $\mathbf{J} \neq 0$:

$$\frac{\partial}{\partial t}\left(U_{\mathrm{E}} + U_{\mathrm{B}}\right) + \nabla \cdot \mathbf{S} \;=\; -\mathbf{J} \cdot \mathbf{E}$$

where

$$U_{\mathrm{E}} = \frac{|\mathbf{E}|^2}{8\pi} \;\; , \quad U_{\mathrm{B}} = \frac{|\mathbf{B}|^2}{8\pi} \qquad \text{(electric \& magnetic energy densities)}$$

$$\mathbf{S} = \frac{c}{4\pi}\left(\mathbf{E} \times \mathbf{B}\right) \qquad \text{(Poynting flux)} \;\; .$$

Note that the RHS of energy equation $< 0$ if $\mathbf{E}$ and $\mathbf{J}$ point in the same direction. Where does the energy go when currents drain it from E&M fields?

Think about resistors in circuits, or the tungsten elements of old incandescent light bulbs: this RHS term is often called **Ohmic or Joule heating.**

## Gravity:

Newtonian gravity is purely attractive, so it behaves similarly to the electrostatic force between 2 oppositely charged particles.

$$\nabla \cdot \mathbf{E} \;=\; 4\pi \rho_c$$

$$\mathbf{F} = q_{\text{test}}\mathbf{E} \qquad\qquad \mathbf{E} = -\nabla\phi$$

$$\nabla^2 \phi \;=\; -4\pi\rho_c$$

$$\nabla \cdot \mathbf{g} \;=\; -4\pi G \rho$$

$$\mathbf{F} = m_{\text{test}}\mathbf{g} \qquad\qquad \mathbf{g} = -\nabla\Phi$$

$$\nabla^2 \Phi \;=\; 4\pi G\rho$$

For point particles, both gravity & the Coulomb force drop off as $1/r^2$. (If both particles are electrons, gravity is weaker by a factor of $\sim 10^{42}$.)

However, while $q$ comes with both $+$ and $-$ signs, mass is only positive. As we add together enough particles to make human-sized (or astro-sized) objects, mass keeps accumulating, but charges (mostly) cancel out!

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

For a test particle of mass $m$, a radial distance $r$ away from a source particle of mass $M$, the force is

$$\mathbf{F} \;=\; -\left(\frac{GmM}{r^2}\right)\hat{\mathbf{e}}_r \;=\; -GmM\,\frac{\mathbf{r}}{|\mathbf{r}|^3} \quad .$$

Generalizing to a continuous distribution of mass with density $\rho(\mathbf{r})$,

$$\mathbf{F}(\mathbf{r}) \;=\; Gm \int d^3\mathbf{r}'\;\rho(\mathbf{r}')\,\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} \quad .$$

Related quantities:

$$\mathbf{F} = -\nabla U_g \;\left\{ \begin{array}{c} \text{force } \mathbf{F} \\ \text{potential energy } U_g \end{array} \right\} \quad \boxed{\begin{array}{c} \mathbf{F} = m\mathbf{g} \\ U_g = m\Phi \end{array}} \quad \left\{ \begin{array}{c} \text{acceleration } \mathbf{g} \\ \text{potential } \Phi \end{array} \right\} \; \mathbf{g} = -\nabla\Phi$$

It's often easier to work with the **scalar potential** $\Phi$. Once it's known for a given source, it's straightforward to compute $U$ (for the source) and $\mathbf{g}$ or $\mathbf{F}$ (for a test particle).

The main constituitive equation for the potential is

$$\Phi(\mathbf{r}) \;=\; -G \int d^3\mathbf{r}' \; \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}$$

which can be derived after knowing that

$$\nabla \left( \frac{1}{|\mathbf{r}' - \mathbf{r}|} \right) \;=\; \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}$$

where the gradient derivatives are taken with respect to $\mathbf{r}$ (and $\mathbf{r}'$ is held fixed).

The differential form of the potential equation is

$$\nabla^2 \Phi \;=\; 4\pi G \rho \qquad \text{(Poisson's equation)},$$

which is equivalent to Gauss's law in E&M.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Later, we'll want to know how particles move in **bound orbits** around gravitational sources. For the simple case of spherically symmetric potentials, one can define two useful velocity-like quantities:

1. The **circular speed** $v_{\rm circ}(r)$, which is given by equating the magnitudes of the gravitational and centripetal accelerations,

   $$|\mathbf{g}| \;=\; \frac{v_{\rm circ}^2}{r} \qquad \rightsquigarrow \qquad \boxed{v_{\rm circ}(r) \;=\; \sqrt{r\,|\nabla\Phi|}}$$

   and the associated angular frequency is $\Omega = v_{\rm circ}/r$.

2. The **escape speed** $v_{\rm esc}(r)$, which is given by equating the particle's (positive) kinetic energy with the (negative) potential energy of the central source,

   $$\frac{1}{2} m v_{\rm esc}^2 \;=\; |U_g| \qquad \rightsquigarrow \qquad \boxed{v_{\rm esc}(r) \;=\; \sqrt{2\,|\Phi|}} \;.$$

For a point-mass source...

$$\Phi(r) \;=\; -\frac{GM}{r} \;, \qquad v_{\rm circ}(r) \;=\; \sqrt{\frac{GM}{r}} \;, \qquad v_{\rm esc}(r) \;=\; \sqrt{\frac{2GM}{r}}$$

...and the motions are ideally **Keplerian.**