General Philosophy

The format will be a "take-home" exam with problems having a similar scope as the homework problem sets. Completed exams will be due 48 hours after they are given to you, and the text that will be at the top of the exam is as follows:

> **GUIDELINES:** Open-book, open-notes, no collaboration. Feel free to use the useful-formula document provided earlier. The four problems are weighted slightly differently, in proportion to their relative complexity: #1 (20%), #2 (20%), #3 (30%), #4 (30%).

Some of the advice given below that involves *memorizing* is a bit of a relic from the closed-book past, but it's probably still good advice for what's best to retain in the long term.

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1. Intro/Review Material

- Know the definition of the advective derivative D/Dt and what its two components mean.
- For the three basic types of PDEs (advection/diffusion/wave equations) know them when you see them, and know the meanings of their coefficients. For the 1D diffusion equation, know that, for a delta-function initial condition, the solution is a gaussian with an r.m.s. width (i.e., standard deviation) of $\sigma = \sqrt{2Dt}$.
- E&M: In your favorite set of units, know how the Lorentz force depends on $\mathbf{E}$ and $\mathbf{B}$, and know at least the two Maxwell's equations with divergences. (If I can't memorize Ampere & Faraday, I won't ask you to, either.) Still, it's important to understand the physical meanings of all four Maxwell's equations.

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2. Random Walks & the Langevin Equation

- The basic nature of random walks should be understood by realizing that the following three statements are saying essentially the same thing, for $N \gg 1$:
  1. If it would have taken N steps, in a straight line, to reach a given distance from your starting location, then it really takes about N^2 steps to get there while random-walking.
  2. If the time is fixed and your step-size is ℓ, then you can reach a distance of ℓN by walking in straight line, but only a shorter distance $\ell\sqrt{N}$ by random-walking.
  3. Random walks behave in a similar way as solutions to the diffusion equation, with the diffusion coefficient given by $D \sim \ell^2/\tau$ (where ℓ and τ are the individual step-sizes in length and time).
- When N is not $\gg 1$, the above ideas about diffusion aren't quite so true any more. At short times, random-walking particles move "ballistically" ($r \sim t$), and only at longer times do they move diffusively ($r \sim t^{1/2}$).

- The Langevin equation: Don't worry about the actual equation, but know what kind of a system is being described by it. A key feature of the system is that it can only come into thermal equilibrium when the source of the random motions is "balanced" by a non-zero source of dissipation/drag.

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3. Plasmas & Coulomb Collisions

- Know that plasmas like to maintain quasi-neutrality. If charges become separated, an $\mathbf{E}$-field is created that forces them back together quickly and "shorts itself out."

- Know how a charged particle's Larmor gyrofrequency depends on the magnetic field.

- For random collisions in a gas of number density n with mean inter-particle speed v, you should understand how the mean-free-path $\ell_{\rm mfp}$ and mean time between collisions $\tau_{\rm coll}$ can be written as
$$\ell_{\rm mfp} = \frac{1}{n\sigma} \ , \qquad \tau_{\rm coll} = \frac{1}{n\sigma v}$$
where σ is the collisional cross section.

- Understand how Coulomb collisions give rise to typical cross sections much larger than if the particles were impacting one another directly. You don't need to memorize the expressions for the impact parameters $b_{\rm min}$ and $b_{\rm max}$ (or the plasma parameter Λ), but understand what they mean in general.

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4. Kinetic Theory & Vlasov/Boltzmann Equation

- Make sure you understand the basics of what the phase-space distribution function $f(\mathbf{r}, \mathbf{p}, t)$ describes, and that its most important moments are:
$$n = \int d^3\mathbf{p} \ f \ , \qquad n\mathbf{u} = \int d^3\mathbf{p} \ f \, \mathbf{v} \ , \qquad U = \frac{3}{2} n k_{\rm B} T = \int d^3\mathbf{p} \ f \ \left(\frac{1}{2} m v^2 \right)$$
i.e., number density n, bulk flow velocity $\mathbf{u}$, thermal energy U, and temperature T.

- Know that the idea of "no particles being created or destroyed" is equivalent to the Vlasov/ Boltzmann equation:
$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$
and also know that Coulomb collisions can make that right-hand side $\neq 0$, as long as we're working with an appropriately large-scale or ensemble-averaged f.

- Know that, in a system undergoing frequent collisions, the most likely end-state is thermal equilibrium with all species having the same flow speed $\mathbf{u}$, same temperature T, and a Maxwell-Boltzmann distribution of individual random velocities/momenta...
$$f_{\rm MB}(\mathbf{p}) \propto \exp \left[-\frac{(\mathbf{p} - \mathbf{p_0})^2}{2 m k_{\rm B} T} \right]$$
where $\mathbf{p}_0 = m\mathbf{u}$ for sub-relativistic speeds.

5. Fluid Moments & Magnetohydrodynamics (MHD)

- For a gas in Maxwell-Boltzmann equilibrium, the pressure is a scalar, with $P_{\mathrm{gas}} = nk_{\mathrm{B}}T$.
- For systems that vary on space/time scales that are big compared to the particle microphysics (i.e., Larmor gyrations, Debye screening, and collisional mean-free paths), the MHD conservation equations are important to know:

$$\text{Mass conservation:} \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \;=\; 0$$

$$\text{Momentum conservation:} \qquad \rho \frac{D\mathbf{u}}{Dt} + \nabla P_{\mathrm{gas}} - \rho \mathbf{g} - \frac{1}{c}\mathbf{J} \times \mathbf{B} \;=\; 0$$

$$\text{Thermal energy conservation:} \qquad \frac{D}{Dt}\left(P_{\mathrm{gas}}/n^{5/3}\right) \;=\; 0 \qquad \text{(adiabatic limit)}$$

$$\text{Magnetic induction equation:} \qquad \frac{\partial \mathbf{B}}{\partial t} \;=\; \nabla \times (\mathbf{u} \times \mathbf{B}) \,+\, D_{\mathrm{B}}\, \nabla^2 \mathbf{B}$$

  It's also useful to understand the physical meaning of each term in these equations.

- In MHD, one can also write the current density $\mathbf{J}$ as being proportional to $\nabla \times \mathbf{B}$, so that the $\mathbf{J} \times \mathbf{B}$ force can be broken up into two terms interpretable as magnetic tension and magnetic pressure. The exact terms don't need to be memorized, except perhaps for the magnetic pressure, which is a key component of the plasma β ratio:

$$P_{\mathrm{mag}} = \frac{B^2}{8\pi} \quad \text{(in Gaussian/cgs units)} \;\;, \qquad \beta \;=\; \frac{P_{\mathrm{gas}}}{P_{\mathrm{mag}}}$$

  $\beta \ll 1$: magnetic field is important (i.e., it tells the fluid how to flow).
  $\beta \gg 1$: magnetic field is unimportant (i.e., it gets pushed around).

6. Applications of Ideal MHD

- A so-called "force-free" magnetostatic equilibrium is one with $\mathbf{J} \times \mathbf{B} = 0$. There are two ways to make that happen:

  1. $\mathbf{J} = 0$ describes a potential field (i.e., $\mathbf{B} = -\nabla\psi$) which tends to be a minimum magnetic-energy state. One obtains the field by solving Laplace's equation ($\nabla^2\psi = 0$).
  2. If $\mathbf{J}$ is parallel to $\mathbf{B}$, then the field is twisted and can be described by $\nabla \times \mathbf{B} = \alpha\mathbf{B}$, where the torsion parameter α must be constant along field lines.

- Ideal MHD waves—with small (linear) amplitudes, adiabatic pressure fluctuations, and sinusoidal oscillations in space/time—break up into two categories. Define a plane that contains the wavevector $\mathbf{k}$ and the background magnetic field $\mathbf{B}_0$, and:

1. Fluctuations OUT of the plane are **Alfvén waves,** which are incompressible (i.e., no density perturbations) and driven by magnetic tension, like a plucked guitar string. The only non-zero amplitudes are perpendicular to the background magnetic field:

$$\frac{u_{1\perp}}{V_{\rm A}} = \frac{B_{1\perp}}{B_0} \qquad \text{and the Alfvén speed defined as} \qquad V_{\rm A} = \frac{B_0}{\sqrt{4\pi\rho_0}} \ .$$

   For Alfvén waves propagating parallel to $\mathbf{B}_0$, their phase speed is $V_{\rm A}$.

2. Fluctuations IN of the plane are **magnetosonic waves,** which are compressible and driven by a combination of the gas-pressure gradient and the magnetic $\mathbf{J} \times \mathbf{B}$ force. In the limit of $B_0 \to 0$ they become acoustic waves, with longitudinal oscillation amplitudes given by

$$\frac{u_1}{c_{s0}} = \frac{\rho_1}{\rho_0} \qquad \text{and the sound speed defined as} \qquad c_{s0} = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

   and $\gamma = 5/3$ for a monatomic ideal gas. Sound waves propagate parallel to $\mathbf{k}$ with a phase speed c_{s0}. For nonzero magnetic fields, the properties of magnetosonic waves depend on $V_{\rm A}$ and c_{s0} and the angle between $\mathbf{k}$ and $\mathbf{B}_0$.

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7. Resistive MHD & Plasma Physics Beyond MHD

- Not in exam.
