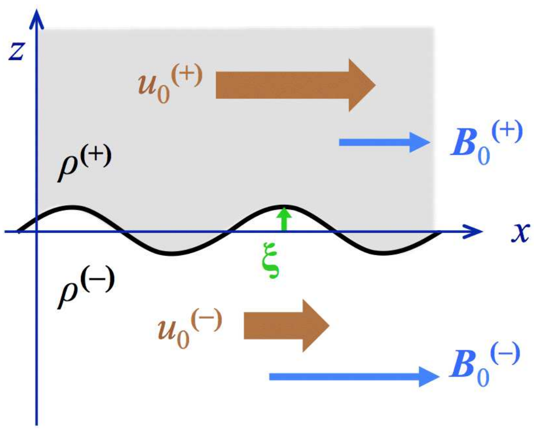

The Instability Menagerie. Consider a gravitationally stratified medium in the z direction (with constant gravity $\mathbf{g} = -g\hat{\mathbf{e}}_z$) separated into two background states at the $z = 0$ plane:



In this problem you will study the linear instability of this **ideal MHD** medium to small perturbations in the horizontal interface.

Properties above the interface have superscript (+) symbols, and properties below the interface have superscript (−) symbols.

The zero-order background quantities are specified at the $z = 0$ plane. Vector zero-order quantities (i.e., $\mathbf{u}_0$ and $\mathbf{B}_0$) are pointing only in the x direction, and can be considered constants.

Big assumptions: The linear velocity perturbations $\mathbf{u}_1$ are incompressible and irrotational; i.e.,

$$\nabla \cdot \mathbf{u}_1 = 0 \;, \qquad \nabla \times \mathbf{u}_1 = 0$$

and the displacement vector $\boldsymbol{\xi}$ is defined by

$$\mathbf{u}_1 \;=\; \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u}_0 \cdot \nabla \boldsymbol{\xi} \qquad \text{(i.e., crests \& troughs drift along with } \mathbf{u}_0) \;.$$

Assume all 1st order quantities vary as $\exp(-i\omega t + ik_x x + ik_y y + ik_z z)$, and consider k_x and k_y to be known and freely choosable parameters.

Note: Parts (a)–(e) below involve assembling together different pieces of this problem. They will be brought together in part (f), and applied in parts (g)–(i).

(a) For the above geometry, solve the linearized induction equation for $\mathbf{B}_1$ as a function of $\mathbf{u}_1$.

(b) Similarly, solve the x component of the linearized momentum equation for the gas pressure perturbation P_1 as a function of u_{1x}.

(c) In this problem, we can write $\mathbf{u}_1 = -\nabla\psi$. Combine this with the assumptions given above to show how ψ obeys Laplace's equation. Write a linearized version of that equation to show how it puts specific constraints on k_z (if k_x and k_y are arbitrary). Use that to write out the explicit height dependence of $\psi(z)$ in both domains.

Hint: If faced with 2 possible solutions (±), choose the one that is physically realistic (i.e., finite) in its respective z domain. This may be a different choice in the upper and lower domains!

(d) Combine the above expression with the definition of $\mathbf{u}_1$ to specifically write u_{1x} as a function of the vertical displacement ξ_z.

(e) If we assume total pressure balance between the two domains, show that the 1st order component of that total pressure balance can be written as

$$P_1^{(+)} + \frac{B_0^{(+)} B_{1x}^{(+)}}{4\pi} - \rho_0^{(+)} g \xi_z \;=\; P_1^{(-)} + \frac{B_0^{(-)} B_{1x}^{(-)}}{4\pi} - \rho_0^{(-)} g \xi_z \;.$$

Hint: The interface is not just $z = 0$. It is perturbed by the first-order perturbation.

(f) Substitute in the results from parts (a)–(d) into both sides of the total pressure balance equation from part (e), such that each term is proportional to ξ_z. After canceling out the ξ_z terms, show that the result agrees with the following:

$$\rho_0^{(+)}\left[-\omega_+^2 - gk_\perp\right] + \frac{k_x^2[B_0^{(+)}]^2}{4\pi} \;=\; \rho_0^{(-)}\left[\omega_-^2 - gk_\perp\right] - \frac{k_x^2[B_0^{(-)}]^2}{4\pi}$$

where $\omega_\pm = \omega - k_x u_0^{(\pm)}$ and $k_\perp^2 = k_x^2 + k_y^2$.

(g) For a static, field-free medium (i.e., all $\mathbf{u}_0 = \mathbf{B}_0 = 0$), show that the result from part (f) reduces to the traditional Rayleigh–Taylor instability criterion given in class.

(h) Modify part (g) by adding a magnetic field in the lower region ($B_{0x}^{(-)} \neq 0$). Show that both kinds of magnetic R–T instabilities (interchange and undular) behave in *qualitatively* the same ways as was described in the lecture notes.

(i) **Extra credit:** You've done enough, but if you're really curious, you can derive the Kelvin–Helmholtz instability criterion, too. Neglect background magnetic fields (i.e., assume $\mathbf{B}_0 = 0$), but impose nonzero shear flows ($\mathbf{u}_0 \neq 0$ in both regions), and derive the instability criterion given in the lecture notes. Assume $k_y = 0$, and thus $k_x = k_\perp$.

. . . .

(a) In class, we showed that the right-hand side of the ideal MHD induction equation can be broken up into 4 (easier to use) terms:

$$\frac{\partial \mathbf{B}}{\partial t} \;=\; \nabla \times (\mathbf{u} \times \mathbf{B}) \;=\; \mathbf{u}(\nabla \cdot \mathbf{B}) \;-\; \mathbf{B}(\nabla \cdot \mathbf{u}) \;+\; (\mathbf{B} \cdot \nabla)\mathbf{u} \;-\; (\mathbf{u} \cdot \nabla)\mathbf{B}$$

The 1st term is always zero. The 2nd term in our problem is also zero, due to the 0th order part ($\mathbf{u}_0$ = constant) and the 1st order part (incompressible perturbations) both being zero. Thus, the 3rd and 4th terms can be linearized as follows,

$$\frac{\partial \mathbf{B}_1}{\partial t} \;=\; (\mathbf{B}_0 \cdot \nabla)\mathbf{u}_1 \;-\; (\mathbf{u}_0 \cdot \nabla)\mathbf{B}_1$$

Thus, the sinusoidal dependence can be inserted,

$$-i\omega \mathbf{B}_1 \;=\; ik_x B_0 \mathbf{u}_1 \;-\; ik_x u_0 \mathbf{B}_1$$

and the equation is rearranged to obtain

$$\boxed{\mathbf{B}_1 \;=\; \left(\frac{k_x B_0}{k_x u_0 - \omega}\right) \mathbf{u}_1} \;.$$

(b) The ideal momentum equation,

$$\rho \frac{\partial \mathbf{u}}{\partial t} \;+\; \rho \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla P + \rho \mathbf{g} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}$$

can be linearized and simplified for our problem. For incompressible fluctuations, we know $\rho_1 = 0$. Using that, in combination with the knowledge that $\mathbf{u}_0$ and $\mathbf{B}_0$ are constants, we get

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} \;+\; \rho_0 \mathbf{u}_0 \cdot \nabla \mathbf{u}_1 \;=\; -\nabla P_1 + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi}$$

(also making use of the fact that the constant $\mathbf{g}$ is cancelled out by the zero-order total pressure gradient). We want to examine the x component of this equation, and we can cancel out the Lorentz force because it is perpendicular to $\mathbf{B}_0$, which lies along the x direction. Thus, the surviving terms are

$$-i\omega\rho_0 u_{1x} + ik_x\rho_0 u_0 u_{1x} = -ik_x P_1 \quad \Longrightarrow \quad \boxed{P_1 = \frac{(\omega - k_x u_0)\rho_0 u_{1x}}{k_x}} \ .$$

(c) The potential ψ in an incompressible & irrotational flow obeys Laplace's equation,

$$\nabla^2\psi = 0 \quad \Longrightarrow \quad k^2\psi = 0$$

and because $\psi \neq 0$, we require that

$$k_x^2 + k_y^2 + k_z^2 = 0 \quad \Longrightarrow \quad k_z = \pm i k_\perp \quad \left(\text{where } k_\perp = \sqrt{k_x^2 + k_y^2}\,\right) \ .$$

This means that the height dependence of the potential is $\psi(z) \propto \exp[\mp k_\perp z]$.

In physically realistic environments, exponentials shouldn't be allowed to blow up to infinity. Thus,

$$\text{In the } \left\{ \begin{array}{l} \text{upper } (+) \\ \text{lower } (-) \end{array} \right\} \text{ domain, we need to specify } \left\{ \begin{array}{ll} \psi(z) \propto e^{-k_\perp z} & (k_z = +ik_\perp) \\ \psi(z) \propto e^{+k_\perp z} & (k_z = -ik_\perp) \ . \end{array} \right.$$

(d) Lastly, we combine together the definition of the displacement ($\mathbf{u}_1 = -i[\omega - k_x u_0]\boldsymbol{\xi}$) with the definition of the potential ($\mathbf{u}_1 = -i\mathbf{k}\psi$), to write u_{1x} in terms of ξ_z. Specifically, we write $u_{1z} = -ik_z\psi$, and plug it into

$$u_{1x} = -ik_x\psi = \left(\frac{k_x}{k_z}\right) u_{1z} = \frac{-i(\omega - k_x u_0)k_x}{k_z}\,\xi_z \ .$$

Thus, using the two above solutions for k_z, we have

$$\boxed{u_{1x} = \frac{\sigma(\omega - k_x u_0)k_x\xi_z}{k_\perp} \qquad \text{where } \sigma = \left\{ \begin{array}{ll} -1 & \text{in the upper } (+) \text{ domain,} \\ +1 & \text{in the lower } (-) \text{ domain.} \end{array} \right.}$$

(e) Since there are no large zero-order vertical motions, the total pressure (the sum of the gas and magnetic pressures) must obey hydrostatic equilibrium,

$$\frac{\partial P_{\rm tot}}{\partial z} = -\rho g \ .$$

We want to specify total pressure balance at the interface ($P_{\rm tot}^{(+)} = P_{\rm tot}^{(-)}$), but we need to realize that even though the subscript-0 quantities are specified at $z = 0$, the interface *itself* is not at $z = 0$. Upward or downward displacements (ξ_z) result in a weak stratification of the zero-order quantities. Integrating the hydrostatic equilibium equation gives

$$P_{\rm tot} = (P_{\rm tot,0} + P_{\rm tot,1}) - \rho_0 g \xi_z$$

where the last term is a new 1st order term that supplements the "locally perturbed" $P_{\rm tot,1}$ term. The latter must then be written in terms of the perturbed gas pressure P_1 and the perturbed magnetic pressure. Note that we can write

$$\begin{aligned} B^2 = |\mathbf{B}_0 + \mathbf{B}_1|^2 &= |\mathbf{B}_0|^2 + 2\mathbf{B}_0 \cdot \mathbf{B}_1 + \{\text{2nd order terms}\} \\ &\approx B_0^2 + 2B_0 B_{1x} \end{aligned}$$

and this can be plugged into the perturbed magnetic pressure to obtain the desired result.

(f) Given the results shown above, the steps should be straightforward.

(g) The steps should be straightforward, to obtain

$$\omega^2 \;=\; -gk_\perp \left[\frac{\rho_0^{(+)} - \rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}} \right] \; .$$

(h) Apologies for defining x and y directions in a different way here than in the lecture notes.

Interchange: If we restrict $k_x = 0$ and $k_y \neq 0$, then the magnetic terms in the dispersion relation disappear. The dispersion relation is identical to the non-magnetized R–T instability.

Undular: If we restrict $k_y = 0$, then we can write $k_x = k_\perp \neq 0$. If we also define the Alfvén speed in the lower region as $V_{\rm A} = B_0^{(-)}/\sqrt{4\pi\rho_0^{(-)}}$, then the dispersion relation becomes

$$\omega^2 \;=\; -gk_\perp \left[\frac{\rho_0^{(+)} - \rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}} \right] \;+\; k_\perp^2 V_{\rm A}^2 \left[\frac{\rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}} \right] \; .$$

If $\rho_0^{(+)} < \rho_0^{(-)}$, the system is always stable. If $\rho_0^{(+)} > \rho_0^{(-)}$, it's only stable when the second (tension) term on the RHS is larger in magnitude than the first (buoyancy) term. That occurs for large values of $k_\perp$, which is the same as small values of the "wavelength" or field-line curvature.

(i) The math is a bit involved, since the dispersion relation reduces to a quadratic equation,

$$\omega^2(\rho_+ + \rho_-) - 2\omega k(\rho_+ u_+ + \rho_- u_-) + k^2(\rho_+ u_+^2 + \rho_- u_-^2) + gk(\rho_+ - \rho_-) \;=\; 0$$

where I hope the notational shorthand is clear. An instability occurs when ω has an imaginary component, which happens when the discriminant of the quadratic formula is negative. This criterion indeed boils down to the expression given in class,

$$k \;>\; \frac{g(\rho_-^2 - \rho_+^2)}{\rho_+\rho_-(u_+ - u_-)^2} \; .$$
