General Philosophy

The two exams in this course are meant to be helpful in preparation for Comps 1. The text that will be at the top of the exam should be close to the following:

> **GUIDELINES:** Closed-book, closed-notes. Feel free to use the sheet of useful facts appended to the end of the exam. Each problem is worth one third of the exam grade.

The "useful facts" all are taken directly from the useful-formula sheet distributed at the start of the semester. As I'm sure you've seen by looking at old Comps 1 exams, there seem to be two universal steps involved in solving these kinds of problems:

(a) Interpreting all the verbiage to figure out what basic principles of physics/astronomy you're being asked to recall.

(b) Recalling and applying those basic principles.

These problems *aren't* primarily concerned with your ability to solve equations or crunch numbers, but those are tools in your toolbox that are often necessary to employ, in order to get through step (b) above. In the subset of my own problems where numerical answers (in real units) are desired, I aim for all calculations to be straightforward to do *without* a calculator. You can always make 10%-level approximations such as $3 \times 0.3 \approx 1$.

Midterm Exam:	Lecture sets 1–6
Final Exam:	Lecture sets 8–12
Comps 1 Exam:	Everything in this document.

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1. Intro/Review Material

- Know the definition of the advective derivative D/Dt and what its two components mean.
- For the three basic types of PDEs (advection/diffusion/wave equations) know them when you see them, and know the meanings of their coefficients. For the 1D diffusion equation, know that, for a delta-function initial condition, the solution is a gaussian with an r.m.s. width (i.e., standard deviation) of $\sigma = \sqrt{2Dt}$.
- E&M: In your favorite set of units, know how the Lorentz force depends on $\mathbf{E}$ and $\mathbf{B}$, and know at least the two Maxwell's equations with divergences. (If I can't memorize Ampere & Faraday, I won't ask you to, either.) Still, it's important to understand the physical meanings of all four Maxwell's equations.

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2. Random Walks & the Langevin Equation

- The basic nature of random walks should be understood by realizing that the following three statements are saying essentially the same thing, for $N \gg 1$:

    1. If it would have taken N steps, in a straight line, to reach a given distance from your starting location, then it really takes about N^2 steps to get there while random-walking.
    2. If the time is fixed and your step-size is ℓ, then you can reach a distance of ℓN by walking in straight line, but only a shorter distance $\ell\sqrt{N}$ by random-walking.
    3. Random walks behave in a similar way as solutions to the diffusion equation, with the diffusion coefficient given by $D \sim \ell^2/\tau$ (where ℓ and τ are the individual step-sizes in length and time).

- When N is not $\gg 1$, the above ideas about diffusion aren't quite so true any more. At short times, random-walking particles move "ballistically" ($r \sim t$), and only at longer times do they move diffusively ($r \sim t^{1/2}$).

- The Langevin equation: Don't worry about the actual equation, but know what kind of a system is being described by it. A key feature of the system is that it can only come into thermal equilibrium when the source of the random motions is "balanced" by a non-zero source of dissipation/drag.

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3. Plasmas & Coulomb Collisions

- Know that plasmas like to maintain quasi-neutrality. If charges become separated, an $\mathbf{E}$-field is created that forces them back together quickly and "shorts itself out."

- Know how a charged particle's Larmor gyrofrequency depends on the magnetic field.

- For random collisions in a gas of number density n with mean inter-particle speed v, you should understand how the mean-free-path $\ell_{\rm mfp}$ and mean time between collisions $\tau_{\rm coll}$ can be written as

$$\ell_{\rm mfp} = \frac{1}{n\sigma} \ , \qquad \tau_{\rm coll} = \frac{1}{n\sigma v}$$

  where σ is the collisional cross section.

- Understand how Coulomb collisions give rise to typical cross sections much larger than if the particles were impacting one another directly. You don't need to memorize the expressions for the impact parameters $b_{\rm min}$ and $b_{\rm max}$ (or the plasma parameter Λ), but understand what they mean in general.

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4. Kinetic Theory & Vlasov/Boltzmann Equation

- Make sure you understand the basics of what the phase-space distribution function $f(\mathbf{r}, \mathbf{p}, t)$ describes, and that its most important moments are:

$$n = \int d^3\mathbf{p}\ f\ , \qquad n\mathbf{u} = \int d^3\mathbf{p}\ f\,\mathbf{v}\ , \qquad U = \frac{3}{2} n k_\mathrm{B} T = \int d^3\mathbf{p}\ f\ \left(\frac{1}{2} m v^2\right)$$

  i.e., number density n, bulk flow velocity $\mathbf{u}$, thermal energy U, and temperature T.

- Know that the idea of "no particles being created or destroyed" is equivalent to the Vlasov/ Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

  and also know that Coulomb collisions can make that right-hand side $\neq 0$, as long as we're working with an appropriately large-scale or ensemble-averaged f.

- Know that, in a system undergoing frequent collisions, the most likely end-state is thermal equilibrium with all species having the same flow speed $\mathbf{u}$, same temperature T, and a Maxwell-Boltzmann distribution of individual random velocities/momenta...

$$f_\mathrm{MB}(\mathbf{p}) \propto \exp\left[-\frac{(\mathbf{p} - \mathbf{p_0})^2}{2 m k_\mathrm{B} T}\right]$$

  where $\mathbf{p}_0 = m\mathbf{u}$ for sub-relativistic speeds.

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5. Fluid Moments & Magnetohydrodynamics (MHD)

- For a gas in Maxwell-Boltzmann equilibrium, the pressure is a scalar, with $P_\mathrm{gas} = n k_\mathrm{B} T$.

- For systems that vary on space/time scales that are big compared to the particle microphysics (i.e., Larmor gyrations, Debye screening, and collisional mean-free paths), the MHD conservation equations are important to know:

  Mass conservation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

  Momentum conservation: $$\rho \frac{D\mathbf{u}}{Dt} + \nabla P_\mathrm{gas} - \rho \mathbf{g} - \frac{1}{c} \mathbf{J} \times \mathbf{B} = 0$$

  Thermal energy conservation: $$\frac{D}{Dt}\left(P_\mathrm{gas}/n^{5/3}\right) = 0 \qquad \text{(adiabatic limit)}$$

  Magnetic induction equation: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + D_\mathrm{B} \nabla^2 \mathbf{B}$$

  It's also useful to understand the physical meaning of each term in these equations.

- In MHD, one can also write the current density $\mathbf{J}$ as being proportional to $\nabla \times \mathbf{B}$, so that the $\mathbf{J} \times \mathbf{B}$ force can be broken up into two terms interpretable as magnetic tension and magnetic pressure. The exact terms don't need to be memorized, except perhaps for the magnetic pressure, which is a key component of the plasma β ratio:

$$P_{\rm mag} = \frac{B^2}{8\pi} \quad \text{(in Gaussian/cgs units)} \ , \qquad \beta = \frac{P_{\rm gas}}{P_{\rm mag}}$$

  $\beta \ll 1$: magnetic field is important (i.e., it tells the fluid how to flow).
  $\beta \gg 1$: magnetic field is unimportant (i.e., it gets pushed around).

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6. Applications of Ideal MHD

- A so-called "force-free" magnetostatic equilibrium is one with $\mathbf{J} \times \mathbf{B} = 0$. There are two ways to make that happen:

  1. $\mathbf{J} = 0$ describes a potential field (i.e., $\mathbf{B} = -\nabla\psi$) which tends to be a minimum magnetic-energy state. One obtains the field by solving Laplace's equation ($\nabla^2\psi = 0$).
  2. If $\mathbf{J}$ is parallel to $\mathbf{B}$, then the field is twisted and can be described by $\nabla \times \mathbf{B} = \alpha\mathbf{B}$, where the torsion parameter α must be constant along field lines.

- Ideal MHD waves—with small (linear) amplitudes, adiabatic pressure fluctuations, and sinusoidal oscillations in space/time—break up into two categories. Define a plane that contains the wavevector $\mathbf{k}$ and the background magnetic field $\mathbf{B}_0$, and:

  1. Fluctuations OUT of the plane are **Alfvén waves,** which are incompressible (i.e., no density perturbations) and driven by magnetic tension, like a plucked guitar string. The only non-zero amplitudes are perpendicular to the background magnetic field:

  $$\frac{u_{1\perp}}{V_{\rm A}} = \frac{B_{1\perp}}{B_0} \qquad \text{and the Alfvén speed defined as} \qquad V_{\rm A} = \frac{B_0}{\sqrt{4\pi\rho_0}} \ .$$

     For Alfvén waves propagating parallel to $\mathbf{B}_0$, their phase speed is $V_{\rm A}$.
  2. Fluctuations IN of the plane are **magnetosonic waves,** which are compressible and driven by a combination of the gas-pressure gradient and the magnetic $\mathbf{J} \times \mathbf{B}$ force. In the limit of $B_0 \rightarrow 0$ they become acoustic waves, with longitudinal oscillation amplitudes given by

  $$\frac{u_1}{c_{s0}} = \frac{\rho_1}{\rho_0} \qquad \text{and the sound speed defined as} \qquad c_{s0} = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

     and $\gamma = 5/3$ for a monatomic ideal gas. Sound waves propagate parallel to $\mathbf{k}$ with a phase speed c_{s0}. For nonzero magnetic fields, the properties of magnetosonic waves depend on $V_{\rm A}$ and c_{s0} and the angle between $\mathbf{k}$ and $\mathbf{B}_0$.

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7. Resistive MHD & Plasma Physics Beyond MHD

- Not in exam.

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8. Lagrangian Dynamics & Two-Body Keplerian Motion

- Understand the relationship between work, kinetic energy, & gravitational potential energy:

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = K_2 - K_1 = U_1 - U_2$$

  where $K = \frac{1}{2}mv^2 = \frac{1}{2}m|\dot{\mathbf{r}}|^2$ and $U = -\frac{GMm}{|\mathbf{r}|}$ and $\mathbf{F} = -\nabla U$

  for a particle of mass m at a displacement $\mathbf{r}$ from a another body of mass M.

- You won't have to apply the Euler-Lagrange equations, but it's good to know the overall motivation for why they work (i.e., the principle of least action).

- For two-body Keplerian orbits, one can understand nearly everything from the conservation of energy and angular momentum:

$$E = K + U = \frac{1}{2}mv^2 - \frac{Gm_1 m_2}{r} = \text{constant} \quad \text{and} \quad \ell = mr^2\dot{\theta} = \text{constant}$$

  where $r = |\mathbf{r}_1 - \mathbf{r}_2|$ and the reduced mass $m = m_1 m_2/(m_1 + m_2)$. For a circular orbit, $\dot{\theta}$ is the angular velocity:

$$\Omega = \sqrt{\frac{G(m_1 + m_2)}{r^3}} \ .$$

  This can also be derived by solving the energy equation for v (i.e., the vis-viva equation),

$$v^2 = G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)$$

  and noting that for a circular orbit, we have both $r = a$ and $v = v_\theta = r\Omega$.

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9. Dynamics: Three-Body Problem & Tides

- For the circular restricted 3-body problem, know the qualitative features of *Roche lobes:* i.e., how they're defined, what's their frame of reference, and where are the 5 Lagrange points.

- Know that the tidal force on a planet (mass M_p, radius R_p) due to a more massive star (mass M_*, distance D from planet) is given essentially by the difference between the star's gravity on either sides of the planet. For $R_p \ll D$, differences become differentials...

$$\Delta g_{\text{tide}} \approx \frac{dg_{\text{star}}}{dr}\Delta r \approx \frac{d}{dr}\left(\frac{GM_*}{r^2}\right) R_p \approx \frac{2GM_*}{D^3} R_p$$

  and one can compare this to the planet's self-gravity ($g_{\text{self}} = GM_p/R_p^2$) to assess whether the planet will or won't be disrupted/distorted by the tidal force.

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10. Dynamics: N-Body Stellar Dynamics in Galaxies & Clusters

- Regarding collisions, know that stars in a galaxy behave somewhat similarly to charged particles in a plasma...
    - "Strong" collisions are defined by the smallest impact parameters (recall $b_{\rm min}$).
    - Sometimes, strong collisions result in "gravitational focusing" that causes stars to collide with one another.
    - "Weak" collisions are *NOT* limited by a Debye-shielding $b_{\rm max}$. Impact parameters out to the size of the system need to be taken into consideration.

- For collisionless systems, the distribution of mass $\rho(\mathbf{r})$ determines the gravitational potential Φ via the Poisson equation:

$$\nabla^2\Phi \,=\, 4\pi G\rho$$

where for a test particle m, the relevant definitions are all interconnected...

$$\mathbf{F} = -\nabla U \left\{ \begin{array}{c} \text{force } \mathbf{F} \\ \text{potential energy } U \end{array} \right\} \quad \boxed{\begin{array}{c} \mathbf{F} = m\mathbf{g} \\ U = m\Phi \end{array}} \quad \left\{ \begin{array}{c} \text{acceleration } \mathbf{g} \\ \text{potential } \Phi \end{array} \right\} \mathbf{g} = -\nabla\Phi$$

- Stable orbits in the equatorial plane of a cylindrically symmetric potential $\Phi(R, z)$ are defined by a circular speed

$$V_{\rm circ} \,=\, \Omega R \,=\, \sqrt{R\frac{\partial\Phi}{\partial R}} \ .$$

Know that perturbations around these circular orbits are described by higher derivatives of Φ (i.e., epicycle frequencies & Oort constants), but details don't need to be memorized.

- Time-steady systems with large numbers of gravitational particles ($N \gg 1$) obey a *virial theorem* that is essentially given by

$$2\mathcal{E}_{\rm K} + \mathcal{E}_{\rm G} \,=\, 0$$

where $\mathcal{E}_{\rm K}$ is the total kinetic energy, and $\mathcal{E}_{\rm G}$ is the total gravitational potential energy.

- For gas clouds (and/or stars) with mass M, radius R, and constant temperature T, one can estimate these quantities as

$$\mathcal{E}_{\rm K} = \int dV\ U_{\rm therm} \,=\, \int dV\ \frac{3}{2}P_{\rm gas} \,\approx\, \frac{3}{2}\frac{M\,k_{\rm B}T}{\mu m_{\rm H}} \qquad \text{and} \qquad \mathcal{E}_{\rm G} \approx -\frac{GM^2}{R} \ .$$

If the cloud is in virial equilibrium (i.e., $|\mathcal{E}_{\rm G}| = 2\mathcal{E}_{\rm K}$), it may be stable.
If $|\mathcal{E}_{\rm G}| > 2\mathcal{E}_{\rm K}$, it's likely to collapse via the Jeans instability.

- For a galaxy cluster of total mass M, one can write

$$\mathcal{E}_{\rm K} \,\approx\, \frac{1}{2}Mv^2 \qquad \text{and} \qquad \mathcal{E}_{\rm G} \,\approx\, -\frac{GM^2}{r}$$

where r and v are statistical estimates of the size and velocity dispersion of the cluster. Using the virial theorem and solving for M gives a "virial mass" of the cluster.

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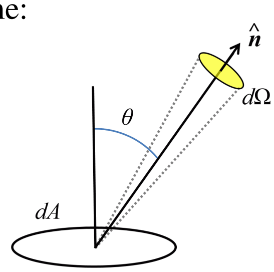
11. Radiation: Definitions, Radiative Transfer, & Gray Atmospheres

- It's always good to be rock-solid on the units of flux (energy/area/time) and luminosity (i.e., power = energy/time). Any Comps question may presume that you know how to convert between fluxes & astronomical magnitudes.

- Specific intensity I_ν describes the distribution of photons in 3D physical space and 3D momentum space. The latter is broken up into

$$|\mathbf{p}| = \frac{h\nu}{c} = \frac{h}{\lambda} \qquad \text{and} \qquad \hat{\mathbf{n}} = \frac{\mathbf{p}}{|\mathbf{p}|} .$$

  I_ν describes how much photon energy is flowing through a particular area, in a particular direction (i.e., into a particular solid angle bin), per unit frequency (i.e., energy "bin"), per unit time:



$$I_\nu(\hat{\mathbf{n}}) = \lim \frac{dE}{(dA\,\cos\theta)\,d\Omega\,d\nu\,dt}$$

  (in the limit of $dA \to 0$, $d\Omega \to 0$, $d\nu \to 0$, $dt \to 0$)

  Standard units of I_ν: erg/s/cm^2/sterad/Hz

  In vacuum, I_ν is constant along any given ray.

- Solid-angle moments of the radiation field:

  Mean intensity: $$J_\nu = \frac{1}{4\pi}\int d\Omega\; I_\nu(\hat{\mathbf{n}})$$ If I_ν is isotropic, then $I_\nu = J_\nu$.

  Vector flux: $$\mathbf{F}_\nu = \int d\Omega\; \hat{\mathbf{n}}\; I_\nu(\hat{\mathbf{n}})$$ If I_ν is isotropic, then $\mathbf{F}_\nu = 0$.

- The equation of radiative transfer comes from the Boltzmann/Vlasov equation for photons. For a system that is time-steady, non-relativistic, and plane-parallel (with spatial variation in z only), it's given by

$$\mu\frac{dI_\nu}{dz} = j_\nu - \chi_\nu I_\nu = \chi_\nu(S_\nu - I_\nu)$$

  where $\mu = \cos\theta$, the angle pointing $\hat{\mathbf{n}}$ away from the z axis. The opacity χ_ν (units: 1/length) describes how gas attenuates an incoming beam, and essentially the photon mean-free-path $\ell_{\rm mfp} = 1/\chi_\nu$. Often, one sees $\chi_\nu = \kappa_\nu\rho$, where κ_ν is the "absorption coefficient" (cm^2/g). Know the definitions of emissivity j_ν and the source function S_ν.

- A slab of gas of thickness dz has an incremental **optical depth** $d\tau_\nu = \pm\chi_\nu\,dz$, where the sign depends on the geometry. One typically integrates over the system to obtain its total optical depth τ_ν. Useful limiting cases include:
  - Optically thin ($\tau_\nu \ll 1$): "transparent" and low density.
  - Optically thick ($\tau_\nu \gg 1$): "opaque" and high density. $I_\nu \to$ isotropic.
  - Photosphere ($\tau_\nu \sim 1$): "visible surface" that provides majority of photons observed.

A slab with incident light shining on it with I_ν^* has light coming out the other side with

$$I_\nu = I_\nu^* e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$$

and you should understand the thin & thick limiting cases.

- In strict thermodynamic equilibrium (STE), everything is isotropic, there is zero flux, and $I_\nu = S_\nu = B_\nu$, where the latter is the equilibrium Planck function,

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

which you probably don't need to memorize, but it is important to know that

$$B = \int_0^\infty d\nu \; B_\nu(T) = \frac{\sigma T^4}{\pi} \quad \text{where } \sigma = \text{the Stefan-Boltzmann constant.}$$

Some stellar atmospheres are in local thermodynamic equilibrium (LTE), where I_ν can be anisotropic, but the locally-generated photons are thermal, with $S_\nu = B_\nu$.

- In a gray (frequency-independent), plane-parallel, & LTE stellar atmosphere, the flux is constant, and we also have $J = B$. Be familiar with how the Eddington approximations lead to the gray temperature distribution:

$$T(\tau) = T_{\text{eff}} \left[\frac{3}{4} \left(\tau + \frac{2}{3} \right) \right]^{1/4}$$

with $F = \sigma T_{\text{eff}}^4$, and understand how the optically thick limit is like a hot "greenhouse."

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12. Radiation: Non-Ideal Cases, Spectral Lines, Ionization, Irradiation

- All moments of the radiation field from a spherical source of radius R, viewed from $r \gg R$, fall off inverse-squarely with radius, with

$$F = \pi I (R/r)^2 ,$$

and where I is the emitted intensity from any point on the spherical surface.

- Spectral lines: it's probably a good idea to know the 5 processes that cause bound electrons to transition up or down, and how these correspond to the A, B, C rate coefficients.

- Ionization/recombination: it's also a good idea to know the identities of the major processes (PI, RR, CI, 3BR) and to have an idea about which ones are dominant in different limiting regimes (Saha, coronal, nebular).

- Irradiated planetary atmospheres: understand basics of energy-balance equilibrium.

- Radiation pressure: understand the physical basis of the various effects described in the lecture notes (e.g., Eddington factor, Poynting-Robertson drag).
