A selection of hopefully useful formulae

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The inspiration (and some of the LaTeX!) for this document comes from the NRL Plasma Formulary, excellently curated by Joseph Huba. Generally, I add to this document when I find myself repeatedly looking up something that I probably should have memorized. C'est la vie!

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."

— Eugene Wigner (1960)

Other Sources of Useful Data

NRL Plasma Formulary .... http://www.nrl.navy.mil/ppd/content/nrl-plasma-formulary
Fundamental physical constants from NIST .... http://physics.nist.gov/cuu/Constants/
Zombeck's *Handbook of Space Astronomy & Astrophysics* ... http://ads.harvard.edu/books/hsaa/
Wolfram Alpha (just type a question or equation) .... http://www.wolframalpha.com/

1 Mathematics

1.1 On the Art of Approximation

You may be used to solving problems that have exact solutions. At some point, however, we run out of those, and we must rely increasingly on approximation/assumption...

$$\text{It's an } \textbf{art} \text{ to figure out} \begin{cases} \text{what to simplify} \\ \text{what to neglect} \\ \text{what to flat-out ignore} \end{cases}$$

Hopefully, by seeing it done in upper-level courses, you'll start to get a feel for doing it yourself. It takes a while...

$=$	the "exact equality" will often give way to
$\approx$	"is approximately equal to," or sometimes even
$\sim$	"very roughly equal to" (within an order of magnitude!?)
$\propto$	and sometimes we just care about which quantities are "proportional to" one another, ignoring normalizing constants.

Mahajan (2018) says a lot more about the mind-decluttering power of $\sim$ ("twiddle").

1.2 Trigonometric Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \quad , \quad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \quad , \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Undoing the inverse...

$$\sin\left(\cos^{-1}x\right) = \cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$$
$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1+x^2}} \quad , \quad \cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}$$
$$\tan\left(\sin^{-1}x\right) = \frac{x}{\sqrt{1-x^2}} \quad , \quad \tan\left(\cos^{-1}x\right) = \frac{\sqrt{1-x^2}}{x}$$

1.3 Vector Identities

Notation: f, g, are scalars; $\mathbf{A}$, $\mathbf{B}$, etc., are vectors.

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}=\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}=\mathbf{B}\times\mathbf{C}\cdot\mathbf{A}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}=\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\times\mathbf{B})\times\mathbf{A}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C}-(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg)=\nabla(gf)=f\nabla g+g\nabla f$

(7) $\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla f$

(8) $\nabla\times(f\mathbf{A})=f\nabla\times\mathbf{A}+\nabla f\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B})=(\nabla\mathbf{B})\cdot\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f=\nabla\cdot\nabla f$

(14) $\nabla^2\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f=0$

(16) $\nabla\cdot\nabla\times\mathbf{A}=0$

Also, if vectors $\mathbf{A}$ & $\mathbf{B}$ depend on time t, then

$$\frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B}) \;=\; \mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial t}+\frac{\partial\mathbf{A}}{\partial t}\cdot\mathbf{B} \qquad \frac{\partial}{\partial t}(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{A}\times\frac{\partial\mathbf{B}}{\partial t}+\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{B}$$

1.4 Coordinate Systems

Conversions between Cartesian and Cylindrical (sometimes ρ or ϖ or R are used for r)

$$\begin{aligned} x &= r\cos\phi & r &= \sqrt{x^2+y^2} \\ y &= r\sin\phi & \phi &= \tan^{-1}(y/x) \\ z &= z & z &= z \end{aligned}$$

Conversions between Cartesian and Spherical

$$\begin{aligned} x &= r\cos\phi\sin\theta & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\phi\sin\theta & \theta &= \cos^{-1}(z/r) \\ z &= r\cos\theta & \phi &= \tan^{-1}(y/x) \end{aligned}$$

Vector & Differential Operators in CARTESIAN Coordinates

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{e}_x (A_y B_z - A_z B_y) + \mathbf{e}_y (A_z B_x - A_x B_z) + \mathbf{e}_z (A_x B_y - A_y B_x)$$

$$\nabla f = \mathbf{e}_x \frac{\partial f}{\partial x} + \mathbf{e}_y \frac{\partial f}{\partial y} + \mathbf{e}_z \frac{\partial f}{\partial z} \qquad \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times \mathbf{F} = \mathbf{e}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$[(\mathbf{A} \cdot \nabla)\mathbf{B}]_i \ = \ A_x \frac{\partial B_i}{\partial x} + A_y \frac{\partial B_i}{\partial y} + A_z \frac{\partial B_i}{\partial z} \qquad \text{for } i = x, y, z$$

Volume element $\qquad dV \ = \ dx\, dy\, dz$

Vector & Differential Operators in CYLINDRICAL Coordinates (r, ϕ, z)

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\phi = \frac{1}{r} \frac{\partial f}{\partial \phi}; \quad (\nabla f)_z = \frac{\partial f}{\partial z}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r} \frac{\partial A_r}{\partial \phi}$$

Laplacian

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2}{r^2} \frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi + \frac{2}{r^2} \frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2}$$

$$(\nabla^2 \mathbf{A})_z = \nabla^2 A_z$$

Components of $(\mathbf{A} \cdot \nabla)\mathbf{B}$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_r = A_r \frac{\partial B_r}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_r}{\partial \phi} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_\phi = A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_\phi}{\partial \phi} + A_z \frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_z = A_r \frac{\partial B_z}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_z}{\partial \phi} + A_z \frac{\partial B_z}{\partial z}$$

Divergence of a tensor

$$(\nabla \cdot \mathbb{T})_r = \frac{1}{r} \frac{\partial}{\partial r}(r T_{rr}) + \frac{1}{r} \frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} - \frac{T_{\phi\phi}}{r}$$

$$(\nabla \cdot \mathbb{T})_\phi = \frac{1}{r} \frac{\partial}{\partial r}(r T_{r\phi}) + \frac{1}{r} \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{\phi r}}{r}$$

$$(\nabla \cdot \mathbb{T})_z = \frac{1}{r} \frac{\partial}{\partial r}(r T_{rz}) + \frac{1}{r} \frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z}$$

Volume element $\quad dV = r\, dr\, d\phi\, dz$

Vector & Differential Operators in SPHERICAL Coordinates (r, θ, ϕ)

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta}; \quad (\nabla f)_\phi = \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{1}{r \sin\theta} \frac{\partial A_\theta}{\partial \phi}$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi)$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r} \frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta}$$

Laplacian

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Laplacian of a vector

$$(\nabla^2\mathbf{A})_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta A_\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2\mathbf{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\nabla^2\mathbf{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}$$

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\theta = A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi} + \frac{A_\theta B_r}{r} - \frac{\cot\theta A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} + \frac{A_\phi B_r}{r} + \frac{\cot\theta A_\phi B_\theta}{r}$$

Divergence of a tensor

$$(\nabla\cdot\mathbb{T})_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi} - \frac{T_{\theta\theta}+T_{\phi\phi}}{r}$$

$$(\nabla\cdot\mathbb{T})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta T_{\phi\phi}}{r}$$

$$(\nabla\cdot\mathbb{T})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{\phi r}}{r} + \frac{\cot\theta T_{\phi\theta}}{r}$$

Volume element $\quad dV = r^2\,\sin\theta\,dr\,d\theta\,d\phi$

1.5 Special Functions & Series Expansions

Binomial Series

$$\text{For } |x| \ll 1, \qquad (1+x)^{\alpha} \;=\; 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

Exponential Functions

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$10^x \;=\; \exp(2.30259x) \qquad (\ln 10 = 2.30259)$$

Full-width at half-maximum (FWHM):

$$\text{For} \quad y \;=\; \exp\left[-\frac{x^2}{2\sigma^2}\right] \;=\; \exp\left[-\left(\frac{x}{V_{1/e}}\right)^2\right] \;, \qquad \text{the FWHM} \;=\; V_{1/e} 2\sqrt{\ln 2} \;\approx\; 1.66511 V_{1/e} \;.$$

Dirac Delta Function

$$\int_{-\infty}^{+\infty} dx\; f(x)\,\delta(x-a) \;=\; f(a) \qquad\qquad \int_{-\infty}^{+\infty} dx\; f(x)\,\delta'(x-a) \;=\; -f'(a)$$

$$\delta[g(x)] \;=\; \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|} \qquad \text{where } x_i \text{ are the roots of } g(x).$$

$$\text{Limiting forms:} \qquad \delta(x) \;=\; \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\frac{\epsilon}{x^2+\epsilon^2}\right) \qquad \delta(x) \;=\; \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon\sqrt{\pi}}e^{-x^2/\epsilon^2}\right)$$

Identities:

$$\delta(-x) \;=\; \delta(x) \qquad \delta(x-a) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\; e^{i\omega(x-a)} \qquad x\,\delta(x) \;=\; 0 \qquad x\,\delta'(x) \;=\; -\delta(x)$$

3D vector delta function:

$$\int_V d^3\mathbf{r}\;\delta(\mathbf{r}-\mathbf{r}_0) \;=\; \begin{cases} 1, & \text{if } V \text{ contains } \mathbf{r}=\mathbf{r}_0, \\ 0, & \text{if } V \text{ does not contain } \mathbf{r}=\mathbf{r}_0. \end{cases} \qquad\qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \delta(\mathbf{r}_0-\mathbf{r})$$

$$\text{Cartesian:} \qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$\text{Cylindrical:} \qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \frac{1}{r}\,\delta(r-r_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0)$$

$$\text{Spherical:} \qquad \delta(\mathbf{r}-\mathbf{r}_0) \;=\; \frac{1}{r^2}\,\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)$$

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\right) \;=\; -\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \qquad\qquad \nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\right) \;=\; -4\pi\,\delta(\mathbf{r}-\mathbf{r}_0)$$

Error Function

$$\mathrm{erf}(x) \;=\; \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \qquad\qquad \mathrm{erfc}(x) \;=\; \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2} \;=\; 1 - \mathrm{erf}(x)$$

For small arguments ($x \ll 1$),

$$\mathrm{erf}(x) \;\approx\; \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \right)$$

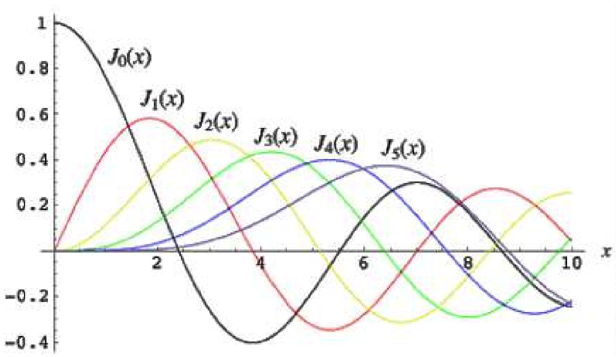
For large arguments ($x \gg 1$),

$$\mathrm{erf}(x) \;\approx\; 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{1}{x} - \frac{1}{2x^3} + \frac{3}{4x^5} - \frac{15}{8x^7} + \cdots \right)$$

The Chandrasekhar (1943) function:

$$G(x) = \frac{1}{2x^2} \left[\mathrm{erf}(x) \;-\; \frac{2x}{\sqrt{\pi}} e^{-x^2} \right] \qquad \text{small } x \;\left(G \approx \frac{2x}{3\sqrt{\pi}} \right) \;, \quad \text{large } x \;\left(G \approx \frac{1}{2x^2} \right)$$

Bessel Functions of the First Kind



For small arguments ($x \ll \sqrt{n+1}$) and indices $n > 0$,

$$J_n(x) \;\approx\; \frac{1}{\Gamma(n+1)} \left(\frac{x}{2} \right)^n$$

For large arguments ($x \gg |n^2 - 1/4|$),

$$J_n(x) \;\approx\; \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right)$$

$$J_{-n}(x) \;=\; (-1)^n J_n(x) \qquad\qquad \int_0^x du\; u\; J_0(u) \;=\; x\, J_1(x) \qquad\qquad \frac{d}{dx} \left[x^n J_n(x) \right] \;=\; x^n J_{n-1}(x)$$

$$\frac{2n}{x} J_n(x) \;=\; J_{n-1}(x) + J_{n+1}(x) \qquad\qquad 2 \frac{dJ_n}{dx} \;=\; J_{n-1}(x) - J_{n+1}(x)$$

Modified Bessel Functions

For small arguments ($z \to 0$) and fixed n indices,

$$I_n(z) \;\approx\; \frac{(z/2)^n}{\Gamma(n+1)} \qquad\qquad I_0(z) \;\approx\; 1 + \frac{z^2}{4} + \frac{z^4}{64} + \cdots$$

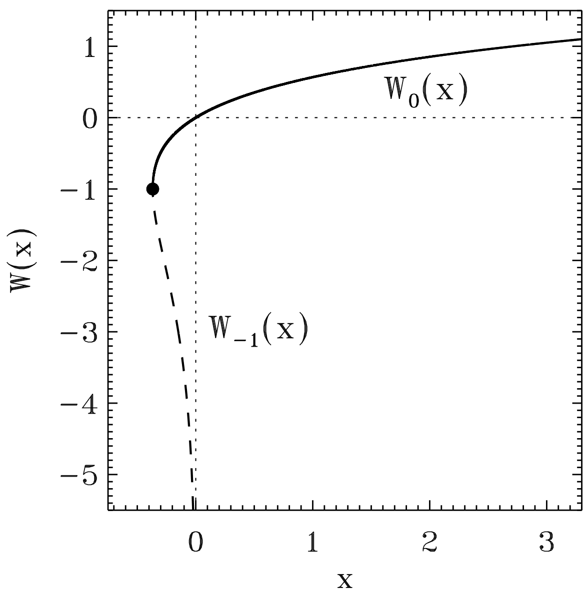
$$K_n(z) \;\approx\; \frac{\Gamma(n)}{2(z/2)^n} \qquad\qquad K_0(z) \;\approx\; -\ln z$$

Lambert W Function

The Lambert W function is defined as the multivalued inverse of the function xe^x. Equivalently, the multiple branches of W are the multiple roots of the equation

$$W(z)e^{W(z)} = z \ ,$$

where z is in general complex. There are an infinite number of solution branches, labeled by convention by an integer subscript: $W_k(z)$, for $k = 0, \pm 1, \pm 2, \ldots$ If z is a real number x, the only two branches that take on real values are $W_0(x)$ and $W_{-1}(x)$. These branches are shown in the accompanying plot.



Numerous formulae for the differentiation, integration, and series expansion of W are given by Corless et al. (1996) and Valluri et al. (2000). One useful result is given here. Near the branch cut point at $x = -1/e$, $W_0 = W_{-1} = -1$, and the two real branches can be approximated to lowest order by

$$\begin{aligned} W_0(x) &\approx -1 + \sqrt{2 + 2ex} \\ W_{-1}(x) &\approx -1 - \sqrt{2 + 2ex} \ . \end{aligned}$$

A useful way for expressing the solutions to a standard family of transcendental equations in terms of the Lambert W function is to note that the equation

$$\ln(A + Bx) + Cx = \ln D \ ,$$

where A, B, C, and D do not depend on x, has the exact solution

$$x = \frac{1}{C} \, W \left[\frac{CD}{B} \exp\left(\frac{AC}{B} \right) \right] - \frac{A}{B} \ .$$

The choice of solution branch usually depends on physical arguments or boundary conditions.

1.6 Derivatives and Integrals

Definite Integrals of Gaussians

$$\int_0^\infty dx\, e^{-(x/\sigma)^2} \;=\; \frac{\sigma\sqrt{\pi}}{2} \qquad\qquad \int_0^\infty dx\, x\, e^{-(x/\sigma)^2} \;=\; \frac{\sigma^2}{2}$$

$$\int_0^\infty dx\, x^2\, e^{-(x/\sigma)^2} \;=\; \frac{\sigma^3\sqrt{\pi}}{4} \qquad\qquad \int_0^\infty dx\, x^3\, e^{-(x/\sigma)^2} \;=\; \frac{\sigma^4}{2}$$

$$\int_0^\infty dx\, x^4\, e^{-(x/\sigma)^2} \;=\; \frac{3\sigma^5\sqrt{\pi}}{8} \qquad\qquad \int_0^\infty dx\, x^5\, e^{-(x/\sigma)^2} \;=\; \sigma^6$$

$$\int_0^\infty dx\, x^6\, e^{-(x/\sigma)^2} \;=\; \frac{15\sigma^7\sqrt{\pi}}{16} \qquad\qquad \int_0^\infty dx\, x^n\, e^{-(x/\sigma)^2} \;=\; \frac{\sigma^{n+1}}{2}\Gamma\left(\frac{n+1}{2}\right)$$

$$\int_{-\infty}^\infty dx\, e^{-ax^2+bx} \;=\; e^{b^2/4a}\sqrt{\frac{\pi}{a}}$$

Definite Integrals relevant to the Planck function

$$\int_0^\infty dx\, \frac{x^n}{e^x-1} \;=\; \zeta(n+1)\,\Gamma(n+1) \;=\; \zeta(n+1)\, n! \qquad \text{(for integer } n\text{)}$$

$\zeta(n)$ is the Riemann zeta function, which ≈ 1 for $n \gtrsim 3$. In that case, $\int_0^\infty dx\, \frac{x^n}{e^x-1} \;\approx\; n!$

Integration by Parts

$$\int_a^b u\, dv \;=\; [u\, v]_a^b \;-\; \int_a^b v\, du \qquad\qquad \int u(x)\,\frac{dv}{dx}\, dx \;=\; u(x)\, v(x) \;-\; \int v(x)\,\frac{du}{dx}\, dx$$

and a vector version comes from Gauss' divergence theorem (Binney & Tremaine, eqn B.45):

$$\int d^3\mathbf{r}\; g\,\nabla\cdot\mathbf{F} \;=\; \oint g\mathbf{F}\cdot d^2\mathbf{S} \;-\; \int d^3\mathbf{r}\; (\mathbf{F}\cdot\nabla)g$$

1.7 Differential Equations

For a 1st order linear equation of the form

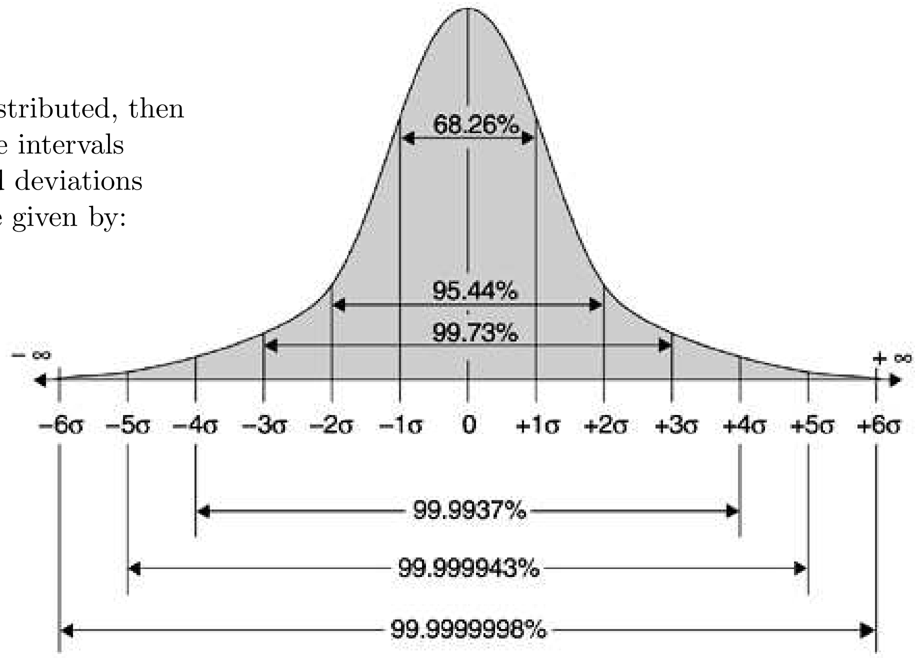
$$\frac{dy}{dx} + P(x)y(x) \;=\; Q(x)$$

use an *integrating factor*

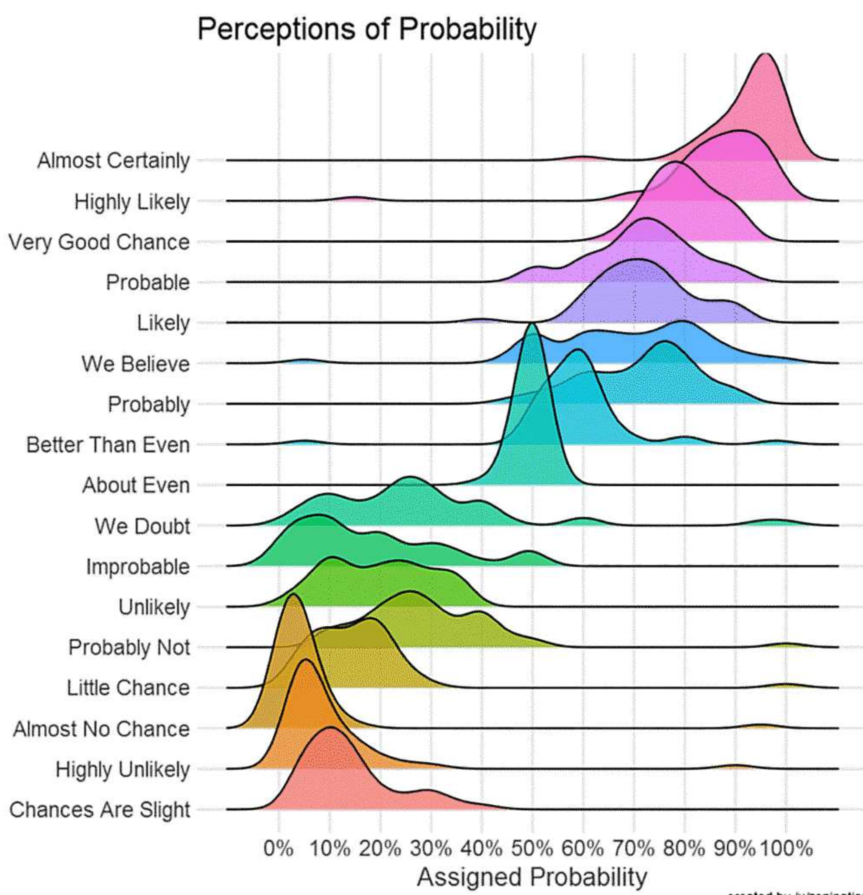
$$\mu(x) \;=\; \exp\left[\int^x P(x')\, dx'\right] \qquad \text{and the solution is} \qquad y(x) = \frac{1}{\mu(x)}\left[\int^x Q(x')\mu(x')\, dx' \;+\; C\right]\;.$$

1.8 Probability and Statistics

If events are normally distributed, then the traditional confidence intervals (in units of $\pm N$ standard deviations away from the mean) are given by:



Perceptions of probability associated with common phrases (github link):



Permutations and Combinations

Consider a pile of n books that you want to read. How many different ways are there to order them? There are n ways of choosing the first one in your list. There are then $n-1$ ways of choosing the second one (because there's now one fewer book to choose from), $n-2$ ways of choosing the third one, and so on. The total number of uniquely ordered arrangements, or **permutations,** is thus given by

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \quad = \quad n! \qquad (\text{“}n \text{ factorial.”})$$

For $n \gg 1$, Stirling's approximation gives

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad .$$

What if you don't have enough time to read all n books? If you have enough time to read only r books (where $r \leq n$), how many ways can you order them? Like before, you start with n ways of choosing the first one, $n-1$ ways of choosing the second one, and so on. But you stop when you reach the rth book, for which there are $n-r+1$ options. This number of permutations is denoted

$${}_nP_r \;=\; n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \;=\; \frac{n!}{(n-r)!} \;.$$

Another way of thinking about that last version of ${}_nP_r$ is the following: Because you're only reading r books, that means there are $(n-r)$ books that you *won't* be reading. Thus, you can divide out the $(n-r)!$ ways that those books could have been ordered from the overall total ($n!$).

Lastly, what if you wanted to compute the number of unique subsets of r books that can be extracted from the larger pool of n books, but *without regard to their ordering?* You first compute ${}_nP_r$, then divide it by the number of possible orderings of the r books that you will read (i.e., $r!$). This gives the number of **combinations,**

$${}_nC_r \;=\; \frac{n!}{(n-r)!\;r!} \;=\; \binom{n}{r}$$

These are also called binomial coefficients, since

$$(x+y)^n \;=\; \sum_{r=0}^{n} \binom{n}{r} \, x^{n-k} y^k \;.$$

The above explanations were derived from similar ones by Spiegel (1975) and Arbuckle (2008).

2 Physics

Metric Prefixes

Multiple	Prefix	Symbol	Multiple	Prefix	Symbol
10^{-1}	deci	d	10	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E
10^{-21}	zepto	z	10^{21}	zetta	Z
10^{-24}	yocto	y	10^{24}	yotta	Y

Physical Constants (SI)

Speed of light in vacuum .... $c = 2.99792458 \times 10^{8}$ m s^{-1}
Newton's gravitation constant .. $G = 6.67384 \times 10^{-11}$ N m^{2} kg^{-2}
Boltzmann's ideal gas constant . $k_{\mathrm{B}} = 1.3806488 \times 10^{-23}$ J K^{-1}
Stefan-Boltzmann constant ..... $\sigma = 5.670373 \times 10^{-8}$ W m^{-2} K^{-4}
Radiation pressure constant .... $a = 4\sigma/c = 7.5657314 \times 10^{-16}$ J m^{-3} K^{-4}
Planck's constant .... $h = 2\pi\hbar = 6.62606957 \times 10^{-34}$ J s
Permittivity of free space .... $\epsilon_0 = 8.8542 \times 10^{-12}$ C^{2} N^{-1} m^{-2}
Permeability of free space ...... $\mu_0 = 4\pi \times 10^{-7}$ N s^{2} C^{-2}
Energy associated with 1 eV ... $E_{\mathrm{eV}} = 1.602176565 \times 10^{-19}$ J

Atomic Constants (SI)

Electron charge .... $e = 1.602176565 \times 10^{-19}$ C
Electron mass .... $m_e = 9.10938291 \times 10^{-31}$ kg
Proton mass .... $m_p = 1.672621777 \times 10^{-27}$ kg $\approx$ 1836 m_e
Neutron mass .... $m_n = 1.674927351 \times 10^{-27}$ kg $\approx$ 1839 m_e
Atomic mass unit .... 1 u $= m(^{12}\mathrm{C})/12 = 1.660538921 \times 10^{-27}$ kg
Bohr radius .... $a_0 = \hbar^2/(m_e e^2) = 5.2917721092 \times 10^{-11}$ m
Classical electron radius .... $r_e = e^2/(\hbar c) = 2.8179403267 \times 10^{-15}$ m
Thomson cross section .... $\sigma_{\mathrm{T}} = (8\pi/3) r_e^2 = 6.652458734 \times 10^{-29}$ m^{2}

Astronomical Constants (SI)

Solar mass .... $M_\odot = 1.989 \times 10^{30}$ kg
Solar radius .... $R_\odot = 6.963 \times 10^{8}$ m
Solar luminosity .... $L_\odot = 3.83 \times 10^{26}$ J s^{-1}
Solar effective temperature ..... $T_{\mathrm{eff}} = 5770$ K
Solar surface gravity .... $g_\odot = 273.79$ m s^{-2} $\quad \log g_\odot$ (cgs) $= 4.4374$
Earth's mass .... $M_\oplus = 5.9736 \times 10^{24}$ kg $\approx$ $3 \times 10^{-6} M_\odot$
Earth's radius .... $R_\oplus = 6371$ km
Astronomical unit .... 1 AU $= 1.495978707 \times 10^{11}$ m $\approx$ 215 $R_\odot$
Parsec .... 1 $pc = 3.085677 \times 10^{16}$ m $=$ 3.2616 light years

Physical Constants (cgs)

Speed of light in vacuum	$c = 2.99792458 \times 10^{10}$ cm s^{-1}
Newton's gravitation constant ..	$G = 6.67384 \times 10^{-8}$ cm^3 g^{-1} s^{-2}
Boltzmann's ideal gas constant .	$k_{\rm B} = 1.3806488 \times 10^{-16}$ erg K^{-1}
Stefan-Boltzmann constant	$\sigma = 5.670373 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}
Radiation pressure constant	$a = 4\sigma/c = 7.5657314 \times 10^{-15}$ erg cm^{-3} K^{-4}
Planck's constant	$h = 2\pi\hbar = 6.62606957 \times 10^{-27}$ erg s
Energy associated with 1 eV ...	$E_{\rm eV} = 1.602176565 \times 10^{-12}$ erg

Atomic Constants (cgs)

Electron charge	$e = 4.803243 \times 10^{-10}$ esu
Electron mass	$m_e = 9.10938291 \times 10^{-28}$ g
Proton mass	$m_p = 1.672621777 \times 10^{-24}$ g $\approx$ $1836\, m_e$
Neutron mass	$m_n = 1.674927351 \times 10^{-24}$ g $\approx$ $1839\, m_e$
Atomic mass unit	1 u $= m(^{12}{\rm C})/12 = 1.660538921 \times 10^{-24}$ g
Bohr radius	$a_0 = \hbar^2/(m_e e^2) = 5.2917721092 \times 10^{-9}$ cm
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Astronomical Constants (cgs)

Solar mass	$M_\odot = 1.989 \times 10^{33}$ g
Solar radius	$R_\odot = 6.963 \times 10^{10}$ cm
Solar luminosity	$L_\odot = 3.83 \times 10^{33}$ erg s^{-1}
Solar effective temperature	$T_{\rm eff} = 5770$ K
Solar surface gravity	$g_\odot = 2.7379 \times 10^4$ cm s^{-2} $\quad \log g_\odot = 4.4374$
Earth's mass	$M_\oplus = 5.9736 \times 10^{27}$ g $\approx$ $3 \times 10^{-3}\, M_{\rm J}$ $\approx$ $3 \times 10^{-6}\, M_\odot$
Earth's radius	$R_\oplus = 6.371 \times 10^8$ cm
Jupiter's mass	$M_{\rm J} = 1.8986 \times 10^{30}$ g $\approx$ $320\, M_\oplus$ $\approx$ $10^{-3}\, M_\odot$
Jupiter's radius	$R_{\rm J} = 6.9911 \times 10^9$ cm
Astronomical unit	1 AU $= 1.495978707 \times 10^{13}$ cm $\approx$ $215\, R_\odot$
Parsec	1 $pc = 3.085677 \times 10^{18}$ cm $=$ 3.2616 light years
Convenient mass loss unit	1 $M_\odot$ yr^{-1} $= 6.3029 \times 10^{25}$ g s^{-1}

Converting between SI (mks) and Gaussian (cgs) Units

In the table below, {3} = 2.99792458 (exactly)

Physical quantity	Symbol	Amount in SI/mks	= Amount in Gaussian/cgs
Force	$\mathbf{F}$	1 newton (N)	= 10^5 dyne
Presure	P	1 pascal (Pa)	= 10 dyne cm^{-2} = 10 erg cm^{-3}
Energy	E	1 joule (J)	= 10^7 erg
Power, Luminosity	L	1 watt (W) = 1 J s^{-1}	= 10^7 erg s^{-1}
Energy flux	$\mathbf{F}$	1 W m^{-2}	= 10^3 erg s^{-1} cm^{-2}
Charge	q	1 coulomb (C)	= $\{3\} \times 10^9$ statcoul
Charge density	ρ_c	1 C m^{-3}	= $\{3\} \times 10^3$ statcoul cm^{-3}
Current	I	1 ampere (A)	= $\{3\} \times 10^9$ statamp
Current density	$\mathbf{J}$	1 A m^{-2}	= $\{3\} \times 10^5$ statamp cm^{-2}
Electric potential	V	1 volt (V)	= $10^{-2}/\{3\}$ statvolt
Electric field	$\mathbf{E}$	1 V m^{-1}	= $10^{-4}/\{3\}$ statvolt cm^{-1}
Magnetic induction	$\mathbf{B}$	1 tesla (T)	= 10^4 gauss (G)
Magnetic flux	Φ	1 weber (Wb) = 1 T m^2	= 10^8 maxwells (Mx) = 10^8 G cm^2

For a general way to convert equations from one system to another, see the NRL Plasma Formulary or Weibel (1968). If only magnetic quantities are present (i.e., no charges or electric fields), it's often sufficient to just replace μ_0 by 4π.

Specific Examples:

Expressions in SI

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$|\mathbf{E}| = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}\ , \qquad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left\{ \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right\}$$

$$U = \frac{\epsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0}\ , \qquad \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

$$V_{\rm A} = \frac{B}{\sqrt{\mu_0 \rho}}\ , \qquad P_{\rm mag} = \frac{B^2}{2\mu_0}$$

Expressions in Gaussian

μ_0 and ϵ_0 not used

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

$$|\mathbf{E}| = \frac{q}{r^2}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho_c\ , \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{4\pi \mathbf{J}}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$U = \frac{|\mathbf{E}|^2}{8\pi} + \frac{|\mathbf{B}|^2}{8\pi}\ , \qquad \mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$$

$$V_{\rm A} = \frac{B}{\sqrt{4\pi\rho}}\ , \qquad P_{\rm mag} = \frac{B^2}{8\pi}$$

Unit Conversions

Changing from one set of units to another is enabled by thinking of them as ratios that get multiplied together in a chain. Example:

$$\frac{\textbf{55 }\cancel{\textbf{miles}}}{\cancel{\textbf{hour}}} \times \frac{5280\ \cancel{\text{feet}}}{1\ \cancel{\text{mile}}} \times \frac{12\ \cancel{\text{inches}}}{1\ \cancel{\text{foot}}} \times \frac{2.54\ \cancel{\text{cm}}}{1\ \cancel{\text{inch}}} \times \frac{1\ \text{m}}{100\ \cancel{\text{cm}}} \times \frac{1\ \cancel{\text{hour}}}{3600\ \text{s}} = \textbf{24.59 m/s}$$

Significant Figures

You should already know how to count up the number of significant digits in a quantity. In scientific notation, it's usually assumed that every digit given prior to the exponential is significant (e.g., 4.1800×10^7 has five significant digits).

- When combining two quantities (e.g., adding, multiplying), the answer should be given with the *least* number of significant digits from the initial quantities.
- However, when working through multi-step calculations, it's useful to keep at least one more significant digit in the intermediate results than will be needed in the final answer.
- If asked to guess or approximate, the final answer should only have (at most) two significant digits.

Special Units

Solar physics:

1 X/M/C flare $\approx 10^{33}/10^{32}/10^{31}$ erg ... 1 microflare $\approx 10^{27}$ erg ... 1 nanoflare $\approx 10^{24}$ erg.

Radiative transfer:

$$1 \text{ Jansky (Jy)} = 10^{-26} \frac{\text{W}}{\text{m}^2 \text{ Hz}} = 10^{-23} \frac{\text{erg}}{\text{cm}^2 \text{ s Hz}} \qquad (\text{specific flux } F_\nu)$$

$$1 \text{ Rayleigh} = \frac{10^6}{4\pi} \frac{\text{photons}}{\text{cm}^2 \text{ s sr}} \qquad (\text{total intensity } I)$$

Plasma Physics

The Langmuir plasma frequency of species j (Gaussian units) is defined as

$$\omega_{pj}^2 \equiv \frac{4\pi q_j^2 n_j}{m_j} .$$

For protons, $\dfrac{\omega_{pp}}{c} = \dfrac{\Omega_p}{V_{\rm A}}$. For a pure proton-electron plasma, $\dfrac{\omega_{pp}^2}{\Omega_p} = \dfrac{\omega_{pe}^2}{|\Omega_e|}$.