Liouville's Theorem

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Supplement to Chapter 3 of *Astrophysical Processes* (AP)
by Hale Bradt, Camb. U. Press. 2008
(www.cambridge.org/us/catalogue/catalogue.asp?isbn=9780521846561)

What we learn in this Supplement

Liouville's **theorem** tells us that the density of points representing particles – "**representative points**" (RP) – in six dimensional (6-D) ***x, p*** phase space is conserved as the RP propagate if certain conditions are met. Specifically, any **forces** encountered by the particles **must be conservative and differentiable** (i.e., "smooth"); they must not change the total (kinetic plus potential) energies of the particles nor can they be abrupt forces such as those arising from collisions. More formally, the forces must be **divergence free in momentum space** (p-divergence = 0). The phase-space density is known as the **distribution function** f (m^{-3} (N s)$^{-3}$); see Section 3.3 of AP.

The theorem is first illustrated graphically and justified from Newton's second law for one-dimensional motion. Formally, the theorem follows from consideration of the **continuity equation** applied to six-dimensional phase space**, the Boltzmann equation**, which takes into account abrupt collisions, and the **Vlasov (or collisionless Boltzmann) equation**. Finally, one examines the **rate of change of the distribution function** (the total derivative), which describes the density change in 6-D phase space as one follows the RP. The Vlasov equation tells us it is equal to zero. Liouville's theorem is thus proven.

Forces giving rise to energy losses due to **radiation** and **dissipation** do not satisfy the p-divergence requirement, but **magnetic forces** and (Newtonian) gravitational forces do.

The free propagation through phase space of the RP of a group of **photons** emitted by a photon source is **illustrated graphically**. The volume occupied by their RP in phase space is conserved and hence so is the phase-space density. Thus, Liouville's theorem is obeyed. The **conservation of specific intensity**, $I(\nu, \theta, \phi)$ with units W m^{-2} Hz^{-1} sr^{-1}, a basic quantity in astronomy, follows directly from this.

Solutions to the problems are available on the associated password-protected portion of the CUP website for this text (see URL above).

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1 Introduction

Liouville's theorem underlies all of astronomy! It describes a fundamental characteristic of photons as they propagate freely through space. The conservation of brightness from source to detector is a direct consequence of Liouville's theorem.

In astronomy and space physics, one often encounters groups or "swarms" or streams of particles or photons. Methods to describe these collectively have been developed by some of the great physicists and mathematicians. These approaches extend beyond that of Newton who successfully described the behavior of a single particle in a force field, under non-relativistic conditions ($v << c$).

In this supplement, we use the concepts of phase space and phase-space density (the distribution function f) introduced in Section 3.3 of *Astrophysical Processes* (AP; by Hale Bradt, Camb U. Press 2008). Here we introduce the continuity equation, the Boltzmann equation, the Vlasov equation, and finally Liouville's theorem. These concepts are used in many branches of physics, e.g. kinetic theory, plasma physics, and fluid dynamics as well as astrophysics. We have demonstrated in Section 3.3 of AP that the specific intensity, $I(\nu, \theta, \phi)$ with units W m^{-2} Hz^{-1} sr^{-1}, a fundamental quantity in astronomy, is directly related to the phase-space density f; see (3.26) of AP.

Liouville's theorem tells us that the density of points representing particles in 6-D phase space is conserved as one follows them through that space, given certain restrictions on the forces the particles encounter. Applied to photons, this is the theoretical underpinning of the equivalence of surface brightness and specific intensity (see (3.27) of AP) or the conservation of specific intensity as one follows the photons through space.

In Section 2, we develop the concept of phase space and motions of "representative points (RP)" therein. In Section 3, we develop Liouville's theorem. In Section 4, we examine the effect of frictional and magnetic forces. In Section 5, we demonstrate graphically that photons traveling in space obey Liouville's theorem.

2 Representative points (RP) in phase space

In this section, we first present a few relations from special relativity that permit us to develop relations that are correct for relativistic particles, i.e., those that move close to the speed of light. This is essential because a photon is the extreme case of a relativistic particle and photons are intrinsic to most astrophysics. We then expand on our discussion of phase space and the distribution function in Section 3.3 of AP and exhibit graphical examples of Liouville's theorem in a one-dimensional (1-D) world.

Relativistic Particles

Einstein's special theory of relativity addresses the kinematics and dynamics of relativistic particles, those with speed v approaching c, the speed of light. The basic ideas and equations of special relativity are developed in Sections 7.2 and 7.3 of AP. The relations of special relativity reduce to those of Newtonian mechanics at low speeds, $v << c$. Here we present a few of the those relations.

In special relativity, the energy U of a particle of mass m is related to its rest energy mc^2 and its speed v as,

$$U = \gamma \; mc^2 \qquad \text{(Relativistic energy)} \qquad (1)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \qquad \text{(Lorentz factor)} \qquad (2)$$

which is known as *Lorentz factor* γ. At speed zero, the relativistic energy is equal to the rest energy, $U \rightarrow mc^2$.

The momentum $\boldsymbol{p}$, a vector, is given in the theory by

$$\boldsymbol{p} = \gamma \, m\boldsymbol{v} \qquad \text{(Momentum)} \qquad (3)$$

which shows that $\boldsymbol{p}$ and $\boldsymbol{v}$ are parallel, as in the non-relativistic case; this is the non-relativistic relation enhanced by the factor γ. Since γ is a measure of energy (1) which can increase without limit, the momentum can also increase without limit as the energy is increased, even though the speed is limited to $v < c$ in special relativity.

A relation between U and $p \equiv |\boldsymbol{p}|$ is obtained by evaluating the quantity $U^2 - c^2p^2$ with (1) and (3) while invoking (2) to eliminate $\boldsymbol{v}$,

$$U^2 - (pc)^2 = (mc^2)^2 \; . \qquad (4)$$

We find itequal to the square of the rest mass energy mc^2 of the particle, which is invariant under relativistic (Lorentz) transformations. The left-hand side of (4) is the square of the four vector [$c\boldsymbol{p}$,U], which is also therefore a Lorentz invariant; see discussion of (7.20) in AP.

The momentum (3) depends on velocity directly through the term $\boldsymbol{v}$ and also through the term γ (2). Thus a relation between $\boldsymbol{v}$ and $\boldsymbol{p}$ may be obtained from (3) by eliminating γ with (2), squaring and solving for $\boldsymbol{v}$,

$$\boldsymbol{v} = \frac{c\,\boldsymbol{p}}{\sqrt{p^2 + m^2c^2}} = \frac{c^2\boldsymbol{p}}{U} \qquad \text{(Velocity - momentum)} \qquad (5)$$

The second equality follows from (4). This is (3.22) of AP.

The components of $\boldsymbol{v}$ will be $v_i = cp_i/(p^2 + m^2c^2)^{1/2}$ where the denominator is simply U/c and $p^2 = p_x{}^2 + p_y{}^2 + p_z{}^2$. Thus the x component of velocity depends on the y and z components of momentum as well as the x component! At speeds $v << c$, one has from (3), $p << mc$, and then (5) yields the usual (non-relativistic) expression, $\boldsymbol{v} = \boldsymbol{p}/m$.

A simple relation between $\boldsymbol{v}$ and U is developed later in this chapter (54). For the record, it is $\boldsymbol{v} = \boldsymbol{\nabla}_p U$, where $\boldsymbol{\nabla}_p$ is the del operator in momentum space and $\boldsymbol{\nabla}_p U$ is called the *p-gradient* of *U*.

Phase space

Phase space is a 6-dimensional (6-D) space with the usual three spatial dimensions x, y, z and also three dimensions of momenta: p_x, p_y, p_z. Here we develop briefly the concept of phase space and that of phase-space density. See also Section 3.3 of AP.

Momentum space

The momentum $\boldsymbol{p}$ can be illustrated in terms of its cartesian components $p_x,\ p_y,\ p_z$ (Fig. 1a). A rectangular volume element $dV_{mom} \equiv d^3p$ in this space is described by the product $dp_x\, dp_y\, dp_z$,

$$dV_{mom} = dp_x\, dp_y\, dp_z \equiv d^3p \quad . \qquad \text{(Volume element; momentum space)} \qquad (6)$$

In spherical coordinates, the volume element in momentum space is (from Fig. 1c),

$$\Rightarrow \qquad dV_{mom} = p^2 \sin\theta\, d\phi_p\, d\theta_p\, dp \;\equiv\; d^3p \qquad \text{(Spherical coordinates)} \qquad (7)$$

The angles are subscripted as θ_p, ϕ_p because they specify angles of the momentum vector $\boldsymbol{p}$ in momentum space rather than the position angles of a position vector $\boldsymbol{x}$ for a particle in physical (x, y, z) space.

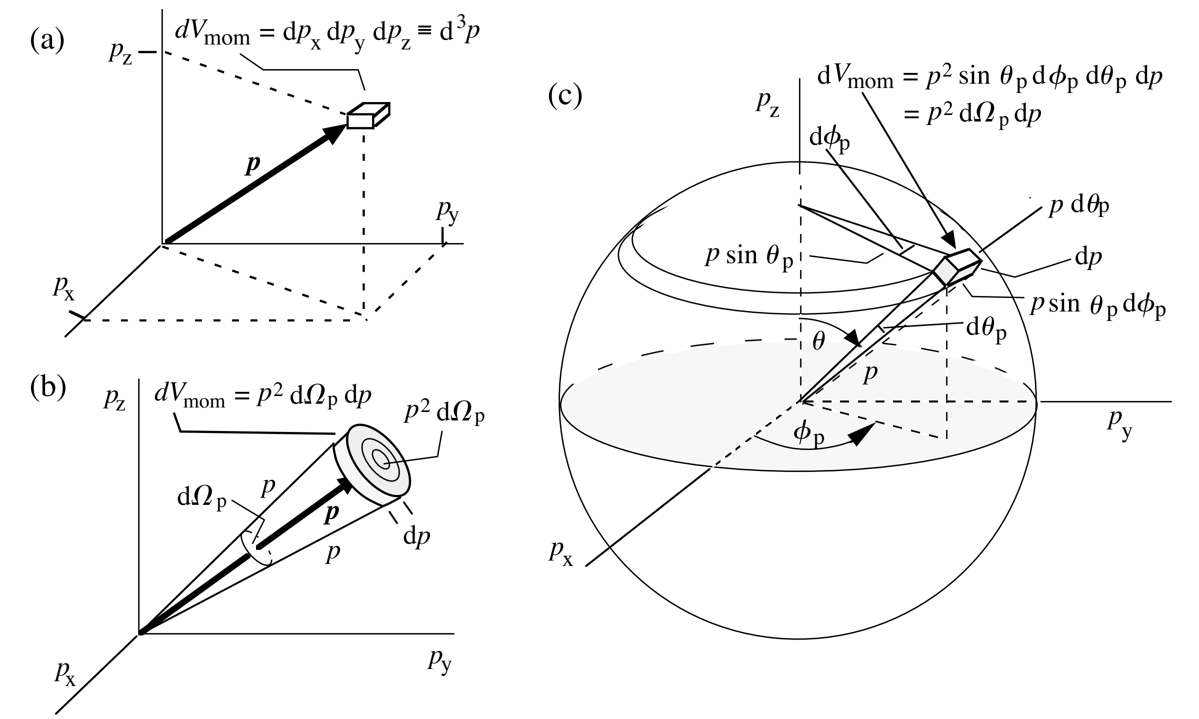

Figure 1: Volume elements in momentum space. (a) The element in cartesian coordinates. Each of the particles in the element have vector momentum close to $\boldsymbol{p}$. The components of the vector are measured on the axes. (b) The element for a group of particles each with momentum $\boldsymbol{p}$ and directions within a cone. (c) The volume element in spherical coordinates.

The element of solid angle element in physical space $d\Omega = \sin\theta\, d\phi\, d\theta$. Analogously, in momentum space,

$$d\Omega_p = \sin\theta_p\, d\phi_p\, d\theta_p \qquad \text{(Solid angle in mom. space)} \qquad (8)$$

which allows one to rewrite (7) more compactly,

$$dV_{mom} = p^2\, d\Omega_p\, dp \qquad (9)$$

The expression $p^2\, d\Omega_p$ is simply the area occupied by the element on the sphere in Fig. 1b. This is a special case of a conical volume element.

Phase-space density

Consider a sample of particles streaming through space. At some given instant of time one can note the spatial position and momentum of any given particle. The 6 coordinates, x, y, z, p_x, p_y, p_z (or x_i, p_i) of a particle locate a point in *phase space*. Each particle can be represented similarly. As time progresses and forces are applied to a group of particles, the points representing them move to new positions in this six dimensional space. We call these "representative points" (RP).

A particle density in phase space at any given position and time can be determined by counting the RP within the 6-dimensional volume element defined by $x \to x + dx$, $y \to y + dy$, $z \to z + dz$, $p_x \to p_x + dp_x$, $p_y \to p_y + dp_y$, $p_z \to p_z + dp_z$,

$$dV_{phase} = dx\, dy\, dz\, dp_x\, dp_y\, dp_z \equiv d^3x\;\; d^3p$$ (Volume element in phase space; m^3 (N s)^3) (10)

where we note that units of momentum may be written as newton seconds (from $\boldsymbol{F} = d\mathbf{p}/dt$).

The distribution function f may be defined with

$$dN = f(\mathbf{x}, \mathbf{p}, t)\;\; d^3x\;\; d^3p\;\; ,$$ (Number in volume element $d^3x\;\; d^3p$ at $\boldsymbol{x}, \boldsymbol{p}$) (11)

where dN is the dimensionless number of particles at time t in the aforementioned volume element. The particle number dN divided by the size of the 6-D volume element equals f, which may therefore be treated as the density of RP at the position in question, $\boldsymbol{x}, \boldsymbol{p}$. This density is a function of the six coordinates and time t; it is called the *phase-space density* or the *distribution function f*:

$$\Rightarrow \qquad f(x, y, z, p_x, p_y, p_z, t) \equiv f(\boldsymbol{x}, \boldsymbol{p}, t)\;.$$ (Distribution function; m^{-3} (N s)$^{-3}$) (12)

The distribution function is thus the number of RP per unit 6-D phase space.

The function $f(\boldsymbol{x}, \boldsymbol{p}, t)$ describes the density of RP in phase space in the limit of very small volume elements. It can apply to an infinitesimal volume element because the volume of the element and the number of particles in the element scale proportionally. However, the differential volume element must be large enough to contain sufficient numbers of particles for the density to be evaluated.

The utility of the distribution function stems in large part from its Lorentz invariance; see Section 7.8 of AP.

Derived quantities

The distribution function f is a very useful quantity. It is directly related to the particle specific intensity J as shown in AP (3.23),

$$J = p^2 f$$ Particle specific intensity; $s^{-1}\, m^{-2}\, J^{-1}\, sr^{-1}$ (13)

and to the photon (energy) specific intensity I_ν (3.26),

$$I_\nu = \frac{h^4 \nu^3}{c^2} f$$ Energy specific intensity; $W\, m^{-2}\, Hz^{-1}\, sr^{-1}$ (14)

It also significant that all possible measurements of numbers, energies, and directions of travel of an incoming flux of particles can be described as an integral over the distribution function, for example the particle density n in physical space, (3.15) of AP or (20) below, the particle flux (3.16) or the bulk velocity,

$$\boldsymbol{v}_{\mathrm{b}} = \langle \boldsymbol{v} \rangle_{\mathrm{av}} = \iiint \boldsymbol{v} \frac{f}{n}\, \mathrm{d}^3 p \qquad \text{(Bulk velocity; m s}^{-1}\text{)} \qquad (15)$$

.

The velocity vector $\boldsymbol{v}$ in the integrand of (15) is the velocity of the particles in a given cell of momentum space; it is thus a function of $\boldsymbol{p}$ as given by (5). The integration averages $\boldsymbol{v}$, weighted by the density in momentum space f/n, over all directions to yield the bulk velocity $\boldsymbol{v}_{\mathrm{b}}$.

Motion in phase space

We comment on different types of forces and give graphical examples of representative points (RP) moving in a 2-D phase space., x, p_x.

Forces, positions and momenta

Particles may be acted upon by forces that change their trajectories, either abruptly, e.g., through collisions, or smoothly through magnetic, electric or gravitational fields. A "smooth" force is one that produces an infinitesimal change in momentum $\Delta\boldsymbol{p}$ in an infinitesimal interval of time Δt. That is, it is differentiable with respect to the position, momentum and time variables.

The path followed by a given particle depends upon its initial momentum and initial x, y, z position and also upon the forces imposed upon it as it travels. The RP of a group of particles with similar positions and momenta will occupy a small region in phase space, and they will propagate more or less together as the associated particles encounter similar forces along their paths.

We will find that the density of RP in phase space remains constant if the force is *conservative*, i.e. it does not change the mechanical (kinetic plus potential) energy of a particle. A force that depends only on position is conservative while forces that depend on momentum generally are not. The magnetic force is an excepion as it does not change the momentum magnitude; it is a conservative force.

Constant velocity in a 2-dimensional phase space

The tracks followed by RP in a 6-D space can not be plotted simply on a 2-D piece of paper. Accordingly one may choose to plot the two coordinates, x, p_x, of the RP in a one-dimensional world wherein particles are constrained to move only in the x direction.

Consider the RP in the 2-D phase space that occupy a differential rectangular volume with a range of momenta δp and range of position δx (Fig. 2a).

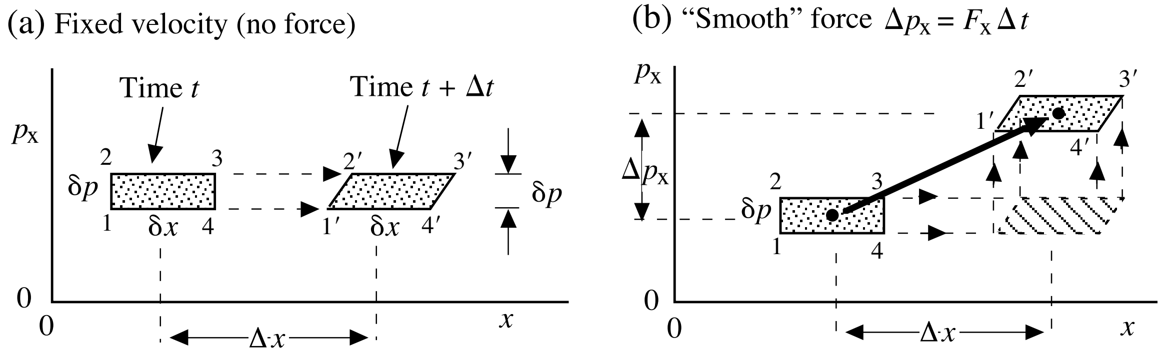

Figure 2: Two-dimensional phase space (x, p_x). (a) No forces encountered by particles. A group of representative points (RP) occupying a differential rectangular area evolves from a rectangle to a parallelogram wherein the RP with the most momentum travel the farthest. (b) Conservative constant "smooth" force that depends on neither position nor momentum. The resultant (vertical) momentum change $\Delta p_x = F\Delta t$ is superposed on the horizontal motion due to the initial velocity to yield final positions 1′–4′. In both cases, the area of the region containing the RP is preserved. If no particles are lost (e.g., through collisions), the density of RP in phase space is conserved.

Suppose there are no forces acting on these particles. How will they move? Consider the four sample RP located at the corners of the rectangle, RP 1, 2, 3, and 4. Since each of the particles has a positive momentum in the x direction, each will be displaced to a new, greater x position, according to,

$$\Delta x = v_x \Delta t = \frac{p_x}{m} \Delta t \qquad \text{(Position displacement; non-relativistic, } v \ll c\text{)} \qquad (16)$$

or, more generally, for particles of all speeds up to $v = c$, from (5),

$$\Delta x = v_x \Delta t = \frac{c p_x}{\sqrt{p^2 + m^2 c^2}} \Delta t \qquad \text{(Position displacement; all speeds } v \le c\text{)} \qquad (17)$$

The particles do not change their momenta because there are no forces on them. The particles 1 and 4 have identical momenta initially. Thus, their RP each move the same distance in the time Δt; their spacing remains constant as they move. The same is true of RP 2 and 3, but they move farther than 1 and 4 because the particle momentum is greater. (For extremely relativistic particles, the distance traveled becomes independent of p_x, $\Delta x \to c\,\Delta t$.) The RP at other locations in the rectangle will move a distance that is related to the individual momenta; see (17).

For the non-relativistic case (16), the distance traveled is linear in p_x. The result is that all the RP in the original rectangle come to occupy a region shaped as a parallelogram at time $t + \Delta t$. Since the base length and the height of the rectangle and the parallelogram are the same, the areas of the two shapes are identical. As these RP continue to move in phase space at constant velocity, the parallelogram will become more and more tilted, but it will always have the same area. In the relativistic case (17), the left and right sides of the rectangle would become curved according to (17). Since the left and right sides would have the same shape, the area in phase space would still be preserved.

The areas as defined here contain the same RP before and after the move. Their number is preserved. This together with the area conservation just demonstrated tells us that the density of RP is conserved as one follows the RP in phase space – for the no-force case,

Constant force in a 2-dimensional phase space

Now, suppose a group of RP in a small element of phase space (e.g. a parallelogram) moves under the influence of a constant force F_x that depends on neither position nor momentum. Thus in the time Δt, all the RP in the parallelogram will experience the same displacement in the p_x direction,

$$\Delta p_x = F_x \, \Delta t \tag{18}$$

This moves all the RP upward by the same amount in phase space. This preserves the shape of the parallelogram and its area, as illustrated in Fig. 2b.

This vertical motion Δp_x is superposed on the rightward motion Δx during Δt, which arises from the velocity at the beginning of the time interval (Fig. 2b) just as in the no-force case. For differential motions, the force does not materially affect the distance traversed because the distance change varies only as $(\Delta t)^2$; recall that, for $v << c$, $\Delta x = at^2/2$ where $a = F_x/m$ and m is the particle mass. In other words, the additional momentum Δp_x developed during this interval has no time to cause develop a significant modification to the position displacement. The true differential motion (diagonal line in Fig. 2b) can thus be approximated as the illustrated two step process.

The net result of the motions in x and p_x, both of which conserve area, is that the area in phase space area is conserved. A sequence of such displacements will yield macroscopic displacements in both x and p_x with the area containing the RP remaining the same.

The conservation of area has been illustrated here for the 2-D phase space x, p_x. In a six-dimensional space, one could in principle apply the same graphical argument to a six-dimensional volume element. Thus one might not be surprised to find that a six-dimensional volume element containing a group of RP would maintain its volume as the RP move through phase space. The element might change shape, but its volume would remain constant. Since the number of RP in the volume is conserved, their density is conserved.

We illustrated this for a constant force. Forces can depend on position and momentum. A formal proof given below demonstrates that the phase space density is conserved if the force is conservative (see above), or more formally, if the "p-divergence" of the force equals zero.

3 Liouville's theorem

The constancy of the density of RP in phase space as one follows the RP, as illustrated above, is known as *Liouville's theorem.* The density function is known as the distribution function $f(\boldsymbol{x}, \boldsymbol{p}, t)$. Thus,

$f(\boldsymbol{x}, \boldsymbol{p}, t)$ = constant at phase-space positions that follow a group of representative points (RP) as time progresses.

or equivalently, as we shall demonstrate,

$$\Rightarrow \quad \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \qquad \text{(Liouville's theorem)} \qquad (19)$$

This, as we have stated and shall see below, is valid if the forces are conservative and differentiable.

Liouville's theorem will be derived formally. We begin with a general conservation law, the *continuity equation*, which is applied to phase space to yield the Boltzmann equation. In turn, this is modified to produce the collisionless Boltzmann equation (or the Vlasov equation), which may be restated as Liouville's theorem.

Continuity equation

The concept of *continuity* is presented here. This is a formal way to relate the rate of change of density of particles at a point in space to the velocities, density gradients, and forces *under the assumption that the number of particles is conserved*. We carry the concept from physical space (x, y, z), to momentum space, and finally to 6-D phase space (x_i, p_i).

Physical-space continuity

First we present the concept of continuity in ordinary physical space. This is a *macroscopic* description of flowing matter, such as fluid flow. The motion of any small volume element of the fluid is a superposition of individual atomic motions which may be quite random, e.g. from thermal motions, in the frame of reference of an element of the fluid.

Consider a closed surface fixed in physical (x, y, z) space, Fig. 3a. Particles are streaming through the region with a *bulk velocity* $\boldsymbol{v}_{\mathrm{b}}(\boldsymbol{x})$, which can vary in magnitude and direction with x, y, z position and with time. In doing so, some of the stream will pass through the volume enclosed by the closed surface. The density of particles at any position is designated $n(\boldsymbol{x})$ (m^{-3}), which is related to the distribution function as (3.15),

$$n(x,y,z,t) = \iiint f \, \mathrm{d}^3p \qquad (\mathrm{m}^{-3}\text{; number density in } x,y,z \text{ space}) \qquad (20)$$

The total number of particles within the volume is,

$$N = \iiint_{\text{vol}} n \, \mathrm{d}V$$ (Total number of particles within closed surface) (21)

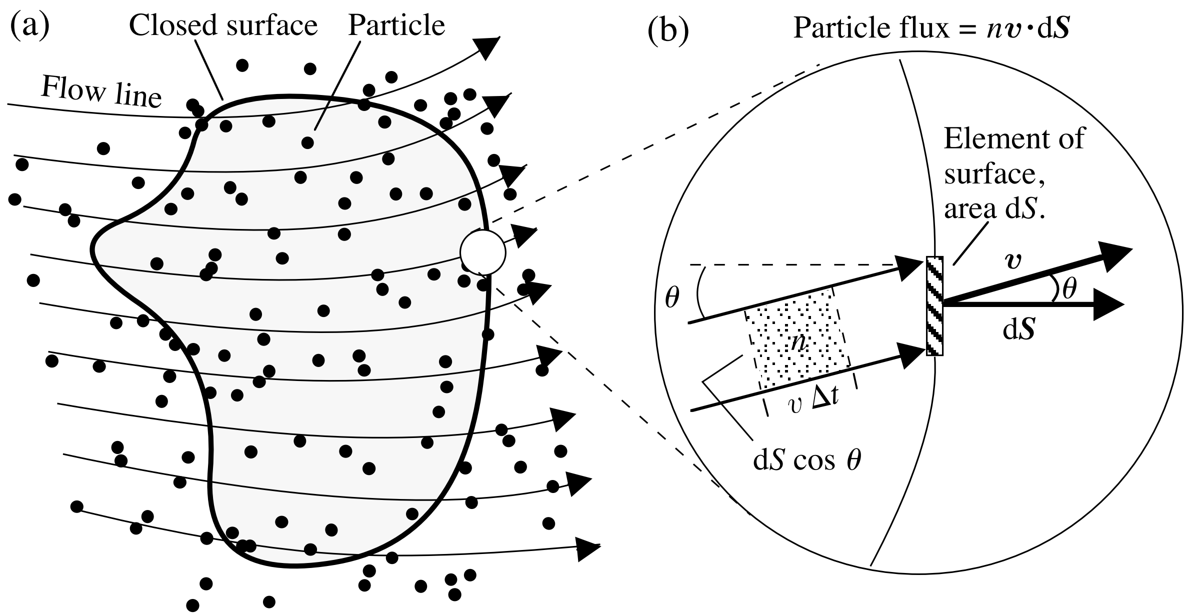

Figure 3. Flow of particles through a closed surface. The particles in the speckled area pass through an element **d*S*** (from the inside to the outside) in time Δt. Their number is $nv\Delta t \, \mathrm{d}S \cos\theta$. The number passing through unit area in unit time, i.e., the particle flux, is thus $nv \cos\theta$, or in vector notation, $n\boldsymbol{v}\cdot\mathbf{d}\boldsymbol{S}$. This sketch could equally well represent the flow of RP in phase space.

The flux of particles (number/m^2-s) passing outward through any given position of the surface at some given time is obtained as a function of velocity $\boldsymbol{v}$ and density n from the geometry of Fig. 3b. The flux is seen to be the dot product of the normal component of the velocity and the density at each point on the surface, $n \, \boldsymbol{v}_\mathrm{b}\cdot\mathrm{d}\boldsymbol{S}$, where d$\boldsymbol{S}$ is the element of surface area, with the positive direction being outward. Integration of this quantity over the entire surface yields the total rate at which particles cross the surface in the outward direction. This is equivalent to the negative rate of change of the number N of particles within the volume, $\partial N/\partial t$.

$$\Rightarrow \quad \frac{\partial N}{\partial t} = -\oint\!\!\oint_{\substack{\text{closed}\\ \text{surface}}} n \, \boldsymbol{v}_\mathrm{b}\cdot\mathrm{d}\boldsymbol{S}$$ (Continuity equation; integral form) (22)

The partial differentiation indicates that the variation with time of number N is evaluated at a fixed set of positions, i.e. at the positions surrounded by the closed surface.

The continuity equation can be written in differential form by making use of *Gauss's theorem*,

$$\oint\!\!\oint \boldsymbol{A}\cdot\mathbf{d}\boldsymbol{S} = \iiint \operatorname{div} \boldsymbol{A} \, \mathrm{d}V$$ (Gauss's theorem) (23)

for an arbitrary vector field $\boldsymbol{A}(\boldsymbol{x}, t)$, which can vary with position $\boldsymbol{x}$ and time t. The left side is an integration of the flux of $\boldsymbol{A}$ over a closed surface, and the right side is an integration of "divergence

A" over the volume enclosed by the same closed surface. Apply Gauss's theorem to the right side of (22), apply (21) to the left side,

$$\iiint \frac{\partial n}{\partial t}\, dV = -\iiint \text{div}\,(n\boldsymbol{v}_b)\, dV \tag{24}$$

The two integrals are over the same volume, namely that enclosed by the surface, and this volume can have any arbitrary shape or be at any arbitrary location. Hence, they can be equal in general only if the integrands themselves are equal. The differential form of the continuity equation is thus

$$\Rightarrow \qquad \frac{\partial n}{\partial t} = -\,\text{div}\,(n\boldsymbol{v}_b) \qquad \text{(Continuity equation; differential form)} \tag{25}$$

The left term is the time rate of change of the particle density at a fixed position, and the right term represents, qualitatively, the "source" or "sink" of the vector $n\boldsymbol{v}_b$. Since particles are conserved by assumption, the particle density at any point in space can increase only if there is more flux entering the differential region than leaving it, i.e. the divergence is negative. This can occur, for example, as a high-density bubble of particles flows into a region of space. A positive divergence represents a net outward flux, which leads to a reduction of n.

Introduce the definition of the *del operator*,

$$\boldsymbol{\nabla} \equiv \hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} \qquad \text{(Del operator)} \tag{26}$$

where $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are unit vectors in the x, y, z directions respectively. The divergence of some vector $\boldsymbol{A} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}} + A_z\hat{\boldsymbol{k}}$ is defined as the dot product $\boldsymbol{\nabla}\cdot\mathbf{A}$,

$$\text{div}\,\boldsymbol{A} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{(Divergence defined)} \tag{27}$$

where the partial derivative $\partial/\partial x$ specifies that only variation with respect to x is considered; all other variables are held constant, e.g., $y, z,$ and t. We thus can write (25) in the del notation,

$$\frac{\partial n}{\partial t} = -\,\boldsymbol{\nabla}\cdot(n\boldsymbol{v}_b) \tag{28}$$

where the number density n and bulk velocity $\boldsymbol{v}_b$ are both macroscopic quantities and are functions of x, y, z, t.

Phase-space continuity

Let us now move to a microscopic description of the motions of the individual particles. In the presence of known forces, one can describe each particle sufficiently to define its future motion with the 6 coordinates of position and momentum. Each particle can be represented as a point in this space, and they must flow smoothly through the space if the forces are smooth (e.g. no collisions, or they are differentiable). A closed surface can be constructed in this six-dimensional space and the density defined as f (RP per unit phase-space volume).

One can then construct a continuity equation for this space just as we did to arrive at (28). The result is a simple generalization wherein the temporal variation of the distribution function at a fixed position in phase space depends on a six-dimensional divergence,

$$\Rightarrow \qquad \frac{\partial f}{\partial t} = -\nabla\cdot(f\boldsymbol{v}) - \nabla_p\cdot(f\boldsymbol{F}) \qquad \text{(Continuity equation in phase space)} \qquad (29)$$

The rightmost term of (29) may be called the "p-divergence" of $f\boldsymbol{F}$. The symbol ∇_p is the del operator in momentum coordinates; the variables p_i replace the x_i in (26),

$$\nabla_p \equiv \hat{\boldsymbol{i}}\frac{\partial}{\partial p_x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial p_y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial p_z} \qquad \text{(Gradient operator with momentum coords.)} \qquad (30)$$

Finally, the force $\boldsymbol{F}$ in the rightmost term replaces $\boldsymbol{v}$ in the central term, because a "velocity" in momentum space is a force; compare $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{v}$ and $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = \boldsymbol{F}$.

Since the right-most term of (29) is a scalar (as are all divergences), it is invariant to rotations of the frame of reference. Thus we can choose the directions $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ in $\boldsymbol{p}$ space to be the same as those in ordinary (physical) x, y, z space. The force may thus be described in the usual manner as

$$\boldsymbol{F} = F_x\hat{\boldsymbol{i}} + F_y\hat{\boldsymbol{j}} + F_z\hat{\boldsymbol{k}} \qquad (31)$$

The expression (29) is the desired differential continuity equation in 6-D space. Its validity depends upon the particles (or fluid) being conserved and the forces being differentiable.

Boltzmann equation

p–divergence forces

The six-dimensional continuity equation (29) can be expanded by the rules of differentiation,

$$\frac{\partial f}{\partial t} = -\boldsymbol{v}\cdot\nabla f - f\,\nabla\cdot\boldsymbol{v} - \boldsymbol{F}\cdot\nabla_p f - f\,\nabla_p\cdot\boldsymbol{F} \quad . \qquad \text{(Continuity equation in phase space)} \qquad (32)$$

If necessary, write out the components of (29) to see that (32) follows from (29). Here, we see the two divergences $\nabla\cdot\boldsymbol{v}$ and $\nabla_p\cdot\boldsymbol{F}$ and have introduced the *gradient* ∇f and $\nabla_p f$. In the spatial case, the gradient is

$$\mathbf{grad}\, f \equiv \nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \qquad \text{(Definition of gradient)} \qquad (33)$$

Note that the gradient is a vector. In momentum coordinates, $\nabla_p f$ is simply (33) with the x_i replaced by the p_i.

$$\nabla_p f = \frac{\partial f}{\partial p_x}\hat{\boldsymbol{i}} + \frac{\partial f}{\partial p_y}\hat{\boldsymbol{j}} + \frac{\partial f}{\partial p_z}\hat{\boldsymbol{k}} \qquad (34)$$

The $\nabla\cdot\boldsymbol{v}$ in the third term of (32) is equal to zero because, in phase space, the partial derivative with respect to a spatial variable (e.g. $\partial/\partial y$) implies that all other coordinates are held constant, including the momentum variables p_i. In our case, the arguments of the spatial derivatives in the divergence are the components of $\boldsymbol{v}$, which are functions only of the components p_i (5). Thus the partial derivatives with respect to the spatial variables, e.g, $\partial v_x/\partial x$, are zero. In vector notation, one says that the divergence of velocity is zero,

$$\nabla\cdot\boldsymbol{v} = 0 \qquad \text{(Identity)} \qquad (35)$$

The second term on the right side of (32) is thus identically equal to zero with no physical assumptions required to make it so.

In contrast, we choose to apply a physical restriction so that the fourth term on the right side may be set to zero. Here we have derivatives with respect to p_i of the force components F_i. The fourth term will therefore go to zero if the forces do not depend on particle momentum. (Recall that this was a condition in our graphical presentation above.) In vector notation, the restriction is more general, namely that the p-divergence (in momentum space) of the force must equal zero,

$$\nabla_{\mathrm{p}}\cdot\boldsymbol{F} = 0 \qquad \text{(p-divergence-free force)} \qquad (36)$$

Collisions added

With this knowledge, one can write a generalized continuity equation by including a collision term $(\delta f/\delta t)_{\mathrm{coll}}$ that incorporates not only abrupt collisions but any force for which $\nabla_{\mathrm{p}}\cdot\boldsymbol{F} \neq 0$. This enables us to drop the $\nabla_{\mathrm{p}}\cdot\boldsymbol{F}$ term in (32) and to write without loss of generality the *Boltzmann equation,*

$$\Rightarrow \qquad \frac{\partial f}{\partial t} = -\,\boldsymbol{v}\cdot\nabla f \;-\; \boldsymbol{F}\cdot\nabla_{\mathrm{p}} f \;+\; \left(\frac{\delta f}{\delta t}\right)_{\mathrm{coll}} \qquad \text{(Boltzmann equation)} \qquad (37)$$

In this context, the forces $\boldsymbol{F}$ in (37) are only those that satisfy $\nabla_{\mathrm{p}}\cdot\boldsymbol{F} = 0$, i.e., they are differentiable conservative forces.

The Boltzmann equation has wide utility in fluid mechanics and kinetic theory.

Vlasov equation

*Boltzmann equation without collisions**

If the collision term in (37) is set again to zero, one obtains the *collisionless Boltzmann equation*, also known as the *Vlasov equation*,

$$\Rightarrow \qquad \frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \boldsymbol{F}\cdot\nabla_{\mathrm{p}} f = 0 \;. \qquad \text{(Vlasov equation = Liouville's theorem)} \qquad (38)$$

This equation *does* explicitly require that the p-divergence term in (32) equal zero. The forces $\boldsymbol{F}$ in (38) must be differentiable and conservative. The collision term has been omitted under the assumption that collisional effects are negligible. This is acceptable in highly rarified gases (e.g., "collisionless" plasmas, a cosmic-ray "gas" in outer space, or a gas of photons in outer space).

Preview of Liouville's theorem

In the particulate view (in contrast to the fluid view), the velocity $\boldsymbol{v}$ represents the rate of change of the coordinates of the particles, $\mathrm{d}x_i/\mathrm{d}t$ where the subscripts designate the 3 dimensions, x, y, and z. Similarly, the force $\boldsymbol{F}$ represents the rate of change of the momentum coordinates, $\mathrm{d}p_i/\mathrm{d}t$. We can thus express (39) as follow,

$$\Rightarrow \qquad \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \sum_i \frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} = 0 \qquad \text{(Vlasov equation = Liouville's theorem)} \qquad (39)$$

You may recognize the left side of (38) as the *total* derivative, $\mathrm{d}f/\mathrm{d}t$. We will demonstrate below that the total derivative is a measure of the density of RP in phase space as one follows a local group of RP through phase space. Equation (39) or (38), being equal to zero, $\mathrm{d}f = 0$, thus tells us that the phase space density is conserved as one follows the RP. Equation (38) *is thus a statement of Liouville's theorem*. If the reader readily accepts these statements, he or she may wish to skip ahead to (44). Otherwise, continue on for discussion of the terms in the Vlasov equation and the physical significance of the total derivative.

Density gradient and speed: physical space

Let us pause a minute to gain some insight into the Vlasov equation in its component form (39) and the partial time derivative of the distribution function.

Consider the motion in the x direction of a cylinder of RP of cross sectional area 1 m^2 (Fig. 4a). Let the gas have a spatial density gradient in the x direction $\partial f/\partial x$ = 5 RP m^{-3} (N s)$^{-3}$/m; that is, the phase-space density increases to the right by 5 units per meter. The unit of volume in $\boldsymbol{x}, \boldsymbol{p}$ phase space is m^3 (N s)^3. Also, let this gas have an x component of velocity $\mathrm{d}x/\mathrm{d}t$ = 3 m/s as shown in the figure, with no motion in the other five directions of phase space.

The temporal variation of the density of RP at a fixed position in space is obtained by the partial $\partial/\partial t$ because it holds all other coordinates fixed. The differential volume element (Fig 4a) thus stays fixed in position while the gas moves to the right at 3 m/s. The time rate of change of density at the fixed position of the volume element decreases. It is equal to the negative of the product of the density gradient and the speed,

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = -15\,\frac{\text{particles m}^{-3}\,(\text{N s})^{-3}}{\text{s}}. \qquad (40)$$

The product of derivatives reproduces one term of the first summation of (39), or equivalently of (38), and hence illustrates its significance. If there were finite components of the gradient and velocity in the y and z directions, the density would be affected by all three terms of the summation.

The application of a differentiable force in the x direction does not change our graphical description of this differential expression. A force would change the speed v only after a significant passage of time; recall $v = v_0 + at$ where a is the acceleration.

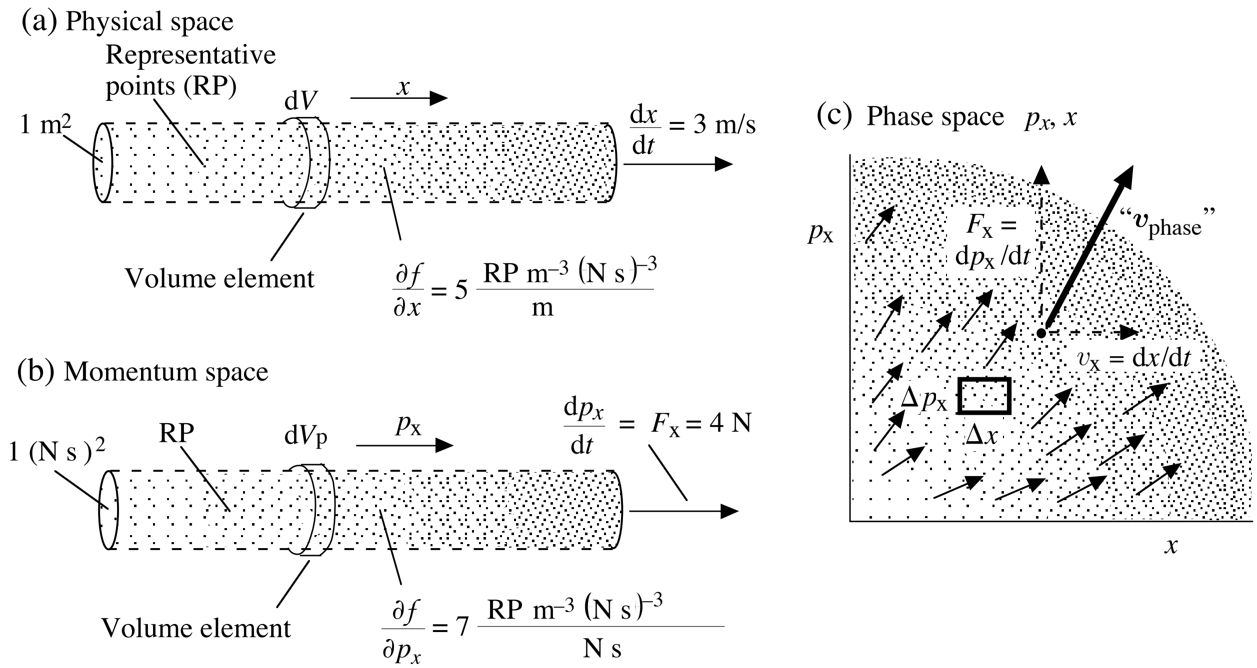

Figure 4: Density gradients and velocities in phase space as incorporated into the Vlasov equation (collisionless Boltzmann equation). (a) Representative points (RP) shown in *x, y, z* space, the "spatial" part of phase space. The cylinder has a gradient of RP density, increasing to the right, and it is moving to the right. The product of the gradient and speed yield the time dependence of the density $\partial f/\partial t$ at the fixed location of the differential volume element d*V*. (b) RP in momentum space. The geometry and formalism are exactly the same as in (a). This yields an expression for $\partial f/\partial t$ at a fixed position in momentum space. (c) Area element $\Delta x \Delta p_x$ at fixed position in two-dimensional phase space. The "phase-space velocity" of the RP is toward the upper right with components dx/dt, the spatial velocity, and dp_x/dt, the force. The value of $\partial f/\partial t$ at the fixed rectangular element depends on both of these "velocities" and the gradients of density (indicated by density of dots).

Density gradient and speed: momentum space

Now, let us look at momentum space, and in particular p_x (Fig. 4b). Hold the other five components constant. Here we have an exact analog to the spatial case. There may be a density gradient in the p_x direction, e.g. $\partial f/\partial p_x = 7$ RP m^{-3}(N s)$^{-3}$/(N s), that is, the density of RP increases with momentum. Let *all* the RP move to the right with "speed" 4 (N s)/s (Fig. 4b). In momentum space, this "velocity" is a time rate of change in momentum dp_x/dt which is a force, $F_x = dp_x/dt =$ 4 N. The force is applied to *all* the particles, regardless of their momenta. This motion leads to a time rate of change of the density (in momentum space) at fixed p_x,

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial p_x}\frac{dp_x}{dt} = -28\,\frac{\text{RP m}^{-3}\,(\text{N s})^{-3}}{\text{s}} \qquad (41)$$

Again the product of derivatives is one term of the second summation of (39). If the gradient and force had *y* and *z* components, all three terms in the summation would contribute to $\partial f/\partial t$.

The total change of f in six dimensions would be due to gradients and velocities in all six dimensions, as given in (38). The gradients and velocities in a two-dimensional phase space, x, p_x, are shown in Fig. 4c. It is apparent that the number of particles in the area $\Delta x\, \Delta p_x$ is due to the flow and gradients in both directions.

This provides, we hope, some physical insight into the meaning of the products in the Vlasov equation and of the partial time derivative of the distribution function, $\partial f/\partial t$. The Vlasov equation is basically an illustration of continuity in 6-D phase space with particle conservation and only conservative differentiable forces.

Rate of change of distribution function

The determination of the total rate of change of the distribution function $\mathrm{d}f/\mathrm{d}t$ is the final step in our quest for Liouville's theorem.

Two positions in phase space

The quantity f is defined as the phase-space density at time t in a volume element at $\boldsymbol{x}, \boldsymbol{p}$. Now, consider the density at a nearby position $\boldsymbol{x}'$, $\boldsymbol{p}'$ and at a slightly different (later) time t', where the positions and times are separated by the differentials $\mathrm{d}\boldsymbol{x}$, $\mathrm{d}\boldsymbol{p}$, $\mathrm{d}t$ (Fig. 5). This new position could, in principle, be at any arbitrary nearby point.

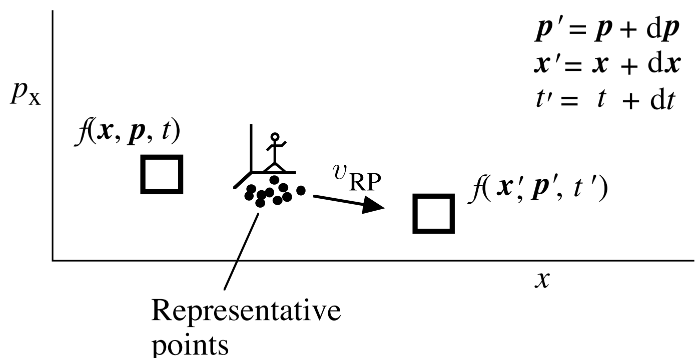

Figure 5: Observer frame of reference in two-dimensional phase space following the RP. The relative phase-space densities f are compared at two differentially separated positions by means of the total derivative $\mathrm{d}f/\mathrm{d}t$, which specifies that these positions be on the path followed by the RP. The total time derivative turns out to be zero (see text), that is $\mathrm{d}f/\mathrm{d}t = 0$, thereby demonstrating that the phase-space density f does not change with time if one follows the RP.

Compare the two positions/times by means of the definition of the total differential,

$$f' - f \rightarrow \mathrm{d}f = \sum \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i + \sum \frac{\partial f}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial f}{\partial t}\,\mathrm{d}t \qquad \text{(Compare two positions)} \qquad (42)$$

where again we sum over the components of the spatial and momentum coordinates. The last term gives the temporal change at a fixed position (fixed $\boldsymbol{x}, \boldsymbol{p}$) while the other six terms yield the differential changes due to the translation to the new position in phase space. Next, divide by the time interval $\mathrm{d}t$ to obtain the total change per unit time,

$$\frac{f'-f}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = \sum \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \sum \frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} + \frac{\partial f}{\partial t} \;. \qquad \text{(Total derivative)} \qquad (43)$$

which is the total derivative.

The velocity components $\mathrm{d}x_i/\mathrm{d}t$ and $\mathrm{d}p_i/\mathrm{d}t$ are the changes of phase-space coordinates divided by the time interval between the two measurements. Thus they are the velocity components of the frame of reference of the observer who travels in phase space from the unprimed position to the primed position in time dt. Since we have defined the x_i to be the coordinates of the RP in phase space, these velocity components are also the velocity components of the RP.

The *total* derivative df/dt (43) thus *is* the rate of change of density f in a frame of reference that follows the RP in phase space. This is the quantity addressed by Liouville's theorem. The question is, then, what is its value?

Liouville's theorem, finally!

The right hand side of (43) is equal to the left side of the Vlasov equation (39). In turn, the Vlasov equation tells us that these terms sum to zero. Thus we find that the time rate of change of RP density in phase space, *as one follows the RP*, is zero,

$$\Rightarrow \qquad \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \;. \qquad \text{(Liouville's theorem; first version)} \qquad (44)$$

Equation (44) is a statement of Liouville's theorem that particle density in phase space is constant as one follows the particles. This is the result we anticipated in (19) and previewed in (39). There is yet another form of the theorem that we now present.

Conservation of phase-space volume

The statement of Liouville's theorem (44) was that the particle density in phase space is conserved as one follows the RP. This was derived from the continuity equation which is based on the conservation of particles, or equivalently the conservation of RP in phase space. The conservation of RP and the conservation of RP density together tell us that the volume occupied by the RP in phase space is also conserved. This conservation of phase-space volume is the other statement of Liouville's theorem.

Let us write this out formally. The conservation of RP within the six-dimensional element is, from (11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(\boldsymbol{x}, \boldsymbol{p}, t)\;\mathrm{d}^3x\;\mathrm{d}^3p\right] = 0 \qquad \text{(RP number conserved)} \qquad (45)$$

which can be expanded,

$$\mathrm{d}^3x\,\mathrm{d}^3p\,\frac{\mathrm{d}}{\mathrm{d}t}\big(f(\boldsymbol{x}, \boldsymbol{p}, t)\big) + \big(f(\boldsymbol{x}, \boldsymbol{p}, t)\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{d}^3x\,\mathrm{d}^3p\big) = 0 \qquad (46)$$

The density conservation, $\mathrm{d}f/\mathrm{d}t = 0$ (Liouville's theorem) given in (44), tells us that the first term of (46) is zero. The phase-space density f may take on arbitrary values, so that

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}^3x\,\mathrm{d}^3p\right) = 0 \qquad \text{(Liouville's theorem; second version.)} \qquad (47)$$

Thus, as claimed, the volume occupied by the RP sample is conserved as one follows the particles. This is the aforementioned other statement of Liouville's theorem.

The conditions for the validity of (44) and (47) are that the forces be p-divergent free and differentiable (e.g. no collisions). The former requirement is equivalent to requiring that the forces be conservative, that is, dependent only on position. This usually means the forces are independent of momentum, but, in fact, the forces can depend on momentum if the p-divergence (taking into account all three momentum variables) is zero. Magnetic forces are an example of this as we discuss below.

4 Special forces

We have shown that Liouville's theorem is valid if the force is divergence free in momentum coordinates, $\boldsymbol{\nabla}_p \cdot \boldsymbol{F} = 0$. Here we examine two cases: friction or radiation forces that do not satisfy Liouville's theorem and magnetic forces, which do satisfy it.

Radiation and dissipation

Let the particles experience frictional forces such that they are all slowed with the highest-momentum particles slowing the most. The effect of this is to move particles to lower momenta with the higher momenta particles being moved the most. This serves to squash the volume occupied in momentum space (Fig. 2). (It does not affect the spread in x for particles of any given momentum.) Thus the phase-space volume is not conserved, and Liouville's theorem is not satisfied.

Consider a friction force $\boldsymbol{F}_{\text{fric}} = F_x\hat{\boldsymbol{i}} \propto -bp_x\hat{\boldsymbol{i}}$ where b is a constant. The divergence of the force is therefore,

$$\boldsymbol{\nabla}_p \cdot \boldsymbol{F}_{\text{fric.}} = \frac{\partial F_x}{\partial p_x} \propto -\frac{\partial bp_x}{\partial p_x} = -b \neq 0 \qquad \text{(p-divergence of friction force is non-zero)} \qquad (48)$$

This more formally indicates that particles acted on by such a force will not obey Liouville's theorem. Losses of energy due to radiation of energy by the particles (e.g. synchrotron radiation; Ch. 8 of AP) will have the same effect.

Magnetic force

Another special case is the magnetic force. In this case, the force does depend upon the momentum,

$$\boldsymbol{F}_B = q\,\boldsymbol{v} \times \boldsymbol{B}. \qquad \text{(Magnetic force)} \qquad (49)$$

Ordinarily a momentum-dependent force would not yield p-div = 0. However, this particular force where the force is normal to the velocity is an exception. Here we demonstrate formally that the magnetic force is p-divergence free ($\boldsymbol{\nabla}_p \cdot \boldsymbol{F}_B = 0$).

The p-divergence of (49) is

$$\boldsymbol{\nabla}_p \cdot \boldsymbol{F}_B = \boldsymbol{\nabla}_p \cdot (q\boldsymbol{v} \times \boldsymbol{B}) \equiv q\boldsymbol{B} \cdot (\boldsymbol{\nabla}_p \times \boldsymbol{v}) - q\boldsymbol{v} \cdot (\boldsymbol{\nabla}_p \times \boldsymbol{B}) \qquad (50)$$

where we used a vector identity to expand the divergence. The quantities in parentheses (right terms) are the "p-curl $\boldsymbol{v}$" and "p-curl $\boldsymbol{B}$" in momentum space, respectively, where the p-curl operation is defined as

$$(\text{curl } \mathbf{A})_p \equiv \nabla_p \times \boldsymbol{A} \equiv \left(\frac{\partial A_z}{\partial p_y} - \frac{\partial A_y}{\partial p_z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial A_x}{\partial p_z} - \frac{\partial A_z}{\partial p_x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial A_y}{\partial p_x} - \frac{\partial A_x}{\partial p_y}\right)\hat{\boldsymbol{k}} \quad \text{(Curl in mom. space)} \quad (51)$$

Both of these curl terms on the right side of (50) are zero. The curl $\boldsymbol{B}$ term is zero simply because the magnetic field is an externally imposed field that depends on position but does not depend upon the momentum of the individual particle. As a consequence, all the derivatives with respect to momentum components are zero.

The (curl $\boldsymbol{v}$)$_p$ term in (50) goes to zero because $\boldsymbol{v}$ is a p-gradient of a scalar function, and the curl of a gradient is identically zero (Prob. 41). Consider the p-gradient of the relativistic energy U,

$$(\mathbf{grad}\ U)_p \equiv \nabla_p U = \frac{\partial U}{\partial p_x}\hat{\boldsymbol{i}} + \frac{\partial U}{\partial p_y}\hat{\boldsymbol{j}} + \frac{\partial U}{\partial p_z}\hat{\boldsymbol{k}} \quad \text{(p-grad)} \quad (52)$$

where, from (4),

$$U = c\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2c^2} \quad \text{(Relativistic particle energy)} \quad (53)$$

Evaluation of (52) yields (Prob. 41) $\nabla_p U = c^2\, \boldsymbol{p}/U$, which from (5) equals $\boldsymbol{v}$, giving the useful and relativistically correct relation between $\boldsymbol{v}$ and U,

$$\Rightarrow \quad \boldsymbol{v} = \nabla_p U \quad (\boldsymbol{v} \text{ as fcn of } U) \quad (54)$$

Thus $\boldsymbol{v}$ indeed is the p-gradient of a scalar, and the p-curl of $\boldsymbol{v}$ is

$$\nabla_p \times \boldsymbol{v} = \nabla_p \times \nabla_p U = 0 \quad (\text{curl}_p\ \boldsymbol{v} = 0) \quad (55)$$

as substitution of the components of (52) into (51) readily confirms.

Both terms on the right side of (50) are thus zero for the magnetic force. Hence the magnetic force is p-divergence free,

$$\Rightarrow \quad \nabla_p \cdot \boldsymbol{F}_B = \nabla_p \cdot (q\, \boldsymbol{v} \times \boldsymbol{B}) = 0 \quad \text{(Magnetic force is divergence free in mom. space)} \quad (56)$$

and Liouville's theorem ($\mathrm{d}f/\mathrm{d}t = 0$) is valid for the magnetic force.

Particles emerging from a slit in a magnetic field

The conservation of phase space volume for particles in a magnetic field may be demonstrated with the example of Fig. 6. Negatively charged particles emerge from the slit on the right which has width Δx. The particles all have the same magnitude of momentum but have a small spread of directions $\Delta\theta$ about the vertical. A uniform magnetic field directed out of the paper pervades the region. This causes the particles to travel in circular trajectories of radius R. The figure shows the two extreme and central circular trajectories that all emerge from the center of the slit. The emerging particles have many other trajectories, those that emerge at intermediate angles and at different positions.

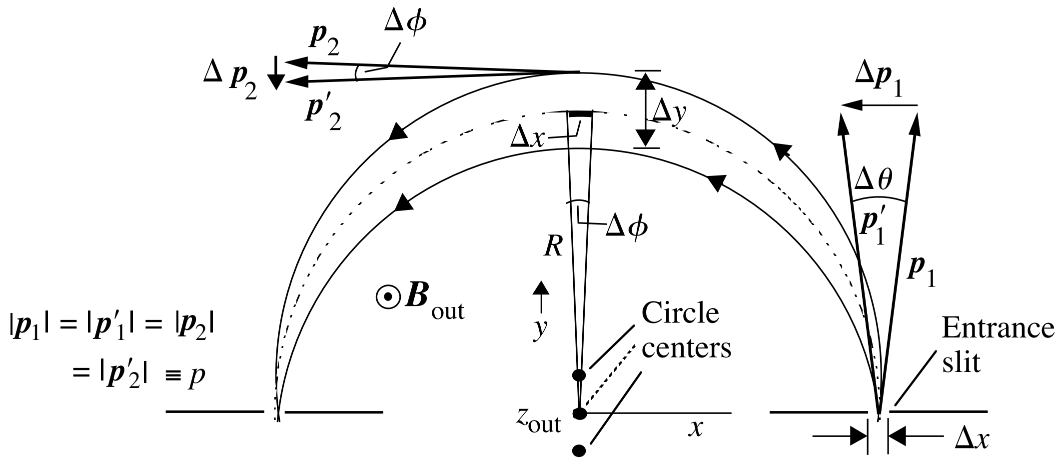

Figure 6: Demonstration of validity of Liouville's theorem for a magnetic force. Charged particles leave the right slit with a large spread of directions (transverse momenta $\Delta\boldsymbol{p}_1$) and small range of lateral positions Δx. At the top of the trajectory the spatial spread Δy has become quite large while the momentum spread $\Delta\boldsymbol{p}_2$ has become quite small. The four-dimensional phase space volume thus remains constant.

Let us look at the phase-space volume of a small group of particles traveling in the x, y plane. (We suppress the z coordinate.) The phase-space has four coordinates, x, y, p_x, p_y, and the volume element is:

$$\Delta V_{\text{phase}} = \Delta x\;\Delta \text{y}\;\Delta p_x\;\Delta p_y \tag{57}$$

At the entrance slit, the momentum spread is equal to the magntude of $\Delta\boldsymbol{p}_1$, or $\Delta p_x = p\;\Delta\theta$ and the spatial spread is simply Δx. The two x components of the phase space volume at the slit thus give,

$$\Delta V_{\text{phase,x}} = \Delta x\;\Delta p_x = \Delta x\;\; p\Delta\theta \;\;=\;\; p\;\Delta\theta\;\Delta x \qquad (\Delta V \text{ in } x \text{ direction at slit}) \tag{58}$$

At the top of the trajectories, the initial spread of momentum $\Delta p_x = |\boldsymbol{\Delta p}_1|$ has been converted into a large spread of spatial positions Δy. The vertical spread of the extreme circle centers (upper and lower black dots) for these trajectories is $R\Delta\theta$, and hence $\Delta y = R\Delta\theta$.

The other dimension of interest at the center at the top of the trajectories is the vertical spread of momentum, Δp_y. The trajectories shown are all horizontal at this position because their centers are displaced vertically from one another (in the limit of small $\Delta\theta$). However a small spread of momentum does arise from the trajectories (not shown) that emerge from other positions within the slit, e.g., at the edges. These trajectories will be circles displaced horizontally from those shown, with a spread of Δx. This results in a small spread of propagation directions $\Delta\phi = \Delta x/R$ at the top of the central trajectory (see geometry in Fig. 6). This corresponds to a vertical spread of momentum, $\Delta p_y = p\Delta\phi = p\;\Delta x/R$. The two y components of the phase-space volume at the top thus give

$$\Delta V_{\text{phase,y}} = \Delta y\ \Delta p_{\text{y}} = R\Delta\theta\ \ p\Delta x/R = p\ \Delta\theta\ \Delta x \qquad \text{(Volume at top)} \qquad (59)$$

The phase-space volumes of (58) and (59) are equal.

One can also show that the phase-space components omitted in (58) and (59) yield equal products at the two positions. At the slit, a range Δy represents the small spread of vertical positions occupied by our group of particles, and a range Δp_{y} represents a range of momentum magnitudes. To first order, the initial spread in y position becomes a similar spread in x position at the top, and the initial spread in p_{y} becomes a similar spread in p_{x} at the top. Thus the missing components also balance. We therefore find that the four-dimensional phase-space volume (57) is the same at the slit as at the top, which demonstrates Liouville's theorem.

The particles continue on and refocus at the left slit, but with small deviations from their configuration at the right slit. Close examination (Prob. 42) shows that again the phase-space volume is conserved. Finally, Liouville demands that the phase-space volume remain constant for our group of particles all along their trajectories, not just at the points discussed here.

Cosmic rays in the Galaxy

Particles that wend their way along magnetic field lines in the Galaxy and in the earth's magnetic field thus obey Liouville's theorem. The phase-space density of a group of particles does not change. The conditions on this are that particles not be lost to a volume element by collision or radiation processes. These conditions are met in large part for cosmic-ray ions in the Galaxy spiraling around the interstellar magnetic fields and by solar particles streaming into the earth's magnetic field.

The particle specific intensity J ($\text{s}^{-1}\ \text{m}^{-2}\ \text{J}^{-1}\ \text{sr}^{-1}$) is directly related to the phase-space density f as $J = p^2 f$ (13). Thus, if Liouville's theorem tells us that the phase space density is constant if the forces encountered are conservative, the particle specific intensity is also conserved. This condition is largely satisfied in interstellar space. Thus, satellite measures of the cosmic-ray intensity J directly tell us conditions (specifically the quantity J) far away in distant parts of the Galaxy from which the cosmic rays may have originated.

5 Photons in phase space

Photons are a particularly simple example of particles traversing the spaces between stars. They travel at the speed of light and in "straight" lines (in a vacuum in the absence of gravity). It is instructive, as an example, to follow in phase space a group of photons emitted from a distant diffuse source. We will demonstrate graphically that they occupy an unchanging volume in phase space; i.e., Liouville's theorem is satisfied.

A photon is a relativistic particle with zero rest mass, $m = 0$. Its velocity-momentum relation is, from (5),

$$\boldsymbol{v} = \frac{c\boldsymbol{p}}{\sqrt{p^2 + m^2c^2}} \underset{m=0}{\rightarrow} c\frac{\boldsymbol{p}}{p} \qquad \text{(Velocity-mom. relation for photon)} \qquad (60)$$

Photons leaving an extended source

The photons that leave the surface of a source occupy larger and larger volumes of ordinary (physical) space as they recede from the source; consider the vertical spread of the photons as they move to the right in Fig. 7. In contrast, as one moves farther and farther from the source, the spread of angles of photon trajectories from the source at a given position in space become smaller and smaller. Thus the spread of the vertical components of the momenta at any given position also becomes less and less, $\Delta p_2 < \Delta p_1$. These two factors lead to the conservation of area in x, p_x space occupied by a specified group of photons as they travel outward, as we demonstrate below.

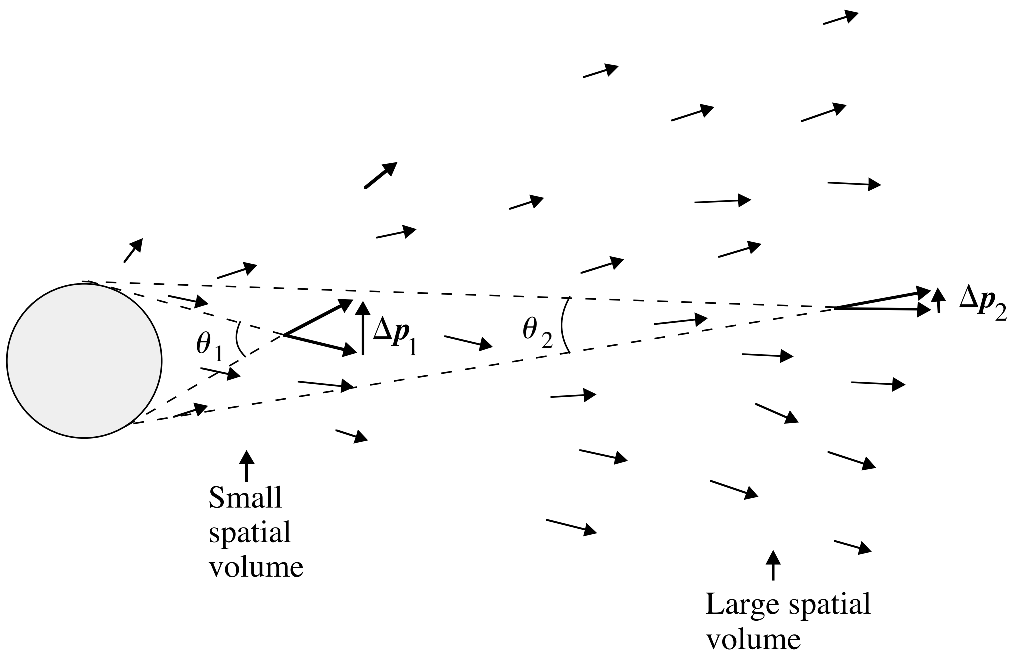

Figure 7: Photons leaving an extended source. As a group of photons travels outwards, the range of physical space occupied by the photons increases while the range transverse momentum at each position in physical space decreases. When mapped in phase space (Fig. 8), these changes cancel to yield conservation of the phase space area occupied by the photons. If no photons are lost, the phase-space density remains at a fixed value as one follows the photons.

In Fig. 8, we examine explicitly a particular group of photons, namely all those that leave the source A during a time Δt and intercept a (mathematical) surface B at a horizontal distance D_B and of vertical extent x_B.

Transverse position and momentum

For simplicity, the angle subtended by the surface B from the source is taken to be small. That is, the angle between each of the several rays and the horizontal is small, $\theta << 1$. Also, let all the photons have approximately the same magnitude of momentum. The momentum of a photon is simply its energy divided by the speed of light,

$$p = \frac{h\nu}{c} \qquad \text{(N s; momentum of photon)} \qquad (61)$$

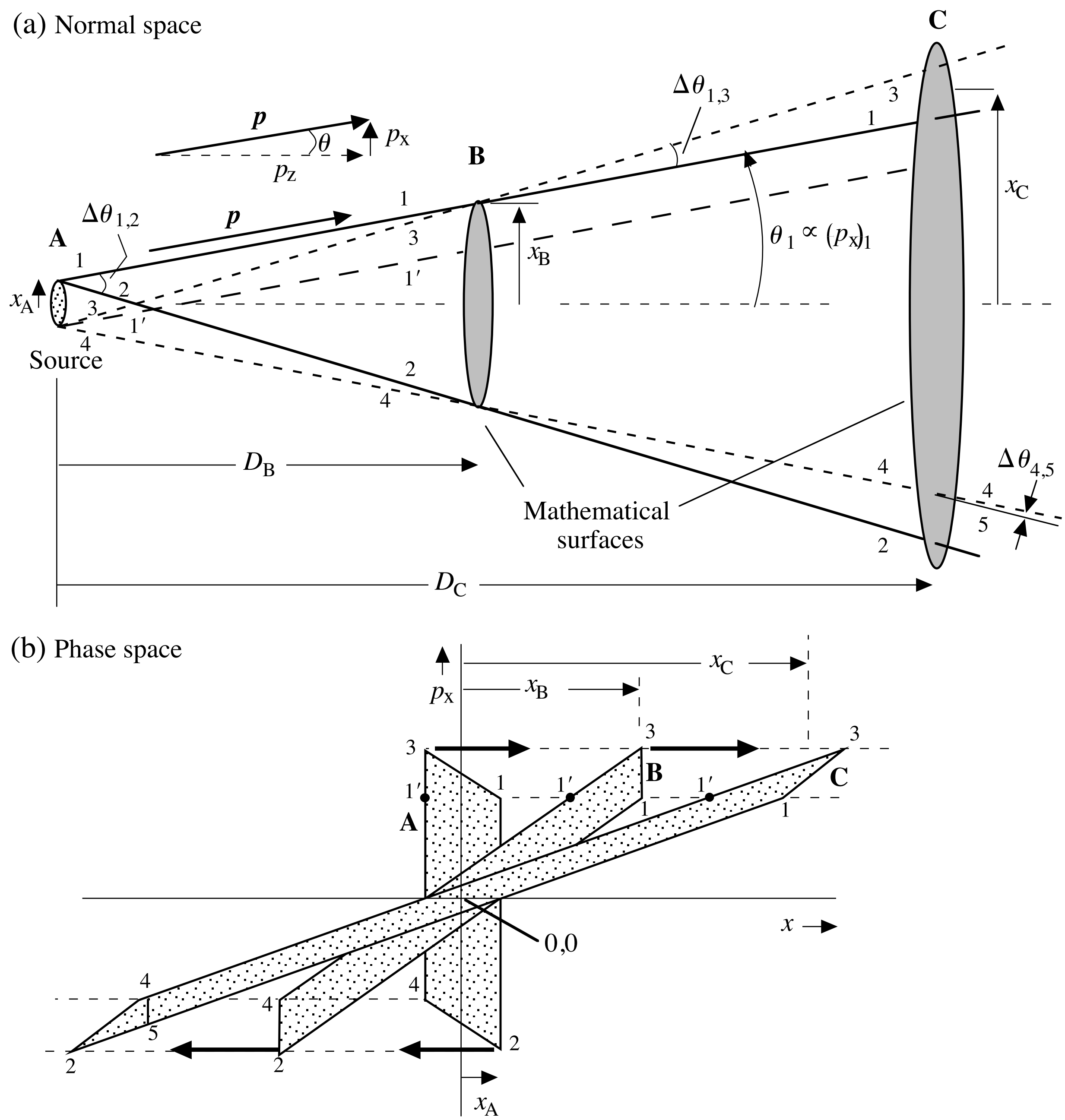

Figure 8: (a) Photons tracks 1–4 representing the extremes of x position and angle for all photons leaving the source A and traversing surface B with $|x| < x_B$. The angle of a track from the horizontal represents the transverse momentum p_x of the photons that follow that track. The track "5" to the right originates at the top of the source A. The track 1′, parallel to track 1, originates at the bottom of the source. (b) The locations, in two-dimensional phase space, of the points representing photons (RP) on tracks 1 – 4 at the three locations A, B, and C. At each location, they form a parallelogram of area equal to the parallelograms at the other locations. Since the RP move at constant velocity, they move only horizontally in phase space. As the range of positions increases, the range of momentum at each x position decreases. The area is thereby preserved.

The component of momentum in the transverse direction p_x (up in the figure) and the position x in the same direction constitute one plane (x, p_x) of the phase space. It is this plane that we consider. The other transverse plane (y, p_y; out of page) would yield similar results. The z, p_z plane (left-right) yields a trivial result since the motion is strictly horizontal at constant speed, similar to that described for Fig. 2a.

The angle from the horizontal axis is proportional to the transverse momentum p_x, in the limit of small angles; specifically $p_x \approx p\theta$. It is positive for the rays with a positive (rising) slope and negative for those with negative slope. The tracks of four photons 1–4 are shown; they leave the source A and pass through the surface B at distance $|x_B|$ from the axis. The four rays include the extremes of angle and position x of the group of photons we wish to follow, namely all those intersecting surface B. Photons 1 and 3 arrive at the same position on the surface enroute to the more distant surface C. Tracks 2 and 4 also intersect at surface B. Track 1′ is parallel to track 1.

The possibility of conservation of volume in phase space is seen again in Fig. 8a. The photons we consider are constrained at the source to the small distance $2x_A$, but at each point on the source, the range of transverse momenta is quite large because the angle between rays 1 and 2, $\Delta\theta_{1,2}$, is quite large (within our constraint of small angles). At the surface B, the range of distance $2x_B$ is greater, but the range of momenta at a given position (angle $\Delta\theta_{1,3}$) is smaller. Similarly, at C, the range of positions is greater ~$2x_C$, while the range of momenta at each position is smaller (e.g. $\Delta\theta_{4,5}$). In each case, the area occupied by the particles in x, p_x phase space at each of the three positions A, B, and C must be plotted and compared quantitatively.

Areas in x, p_x phase space

Distribution at source

The x, p_x coordinates in phase space of the four photons as they leave A are indicated in Fig. 8b. They are the corners of parallelogram A. Note that 1 and 2 each have positive x (top of source) while 1 has positive momentum (rising) and 2 has negative momentum (descending). Also note that the difference in the momenta p_x of 1 and 3 is quite small because the angle $\Delta\theta_{1,3}$ is small,

$$\Delta\theta_{1,3} = \frac{2\,x_A}{D_B} \tag{62}$$

In contrast the momentum difference between 1 and 2 is much larger because $x_B > x_A$, since

$$\Delta\theta_{1,2} = \frac{2\,x_B}{D_B} \tag{63}$$

The parallelogram A includes all the points (RP) representing the photons that leave the source from all positions between its top and bottom $|x| < x_A$ with tracks that intercept B at $|x| < x_B$.

The surface area S_A of parallellogram A in the p_x, x plane (Fig. 8b) constitutes the "volume" of the 2-dimensional plane of phase space we are considering. The surface area of a parallelogram is its base times its height (distance normal to the base to the opposite side). If the base is taken to be the side 1-2, the base length is $p\Delta\theta_{1,2}$, and the height is $2x_A$. The area S_A is thus,

$$S_A = 2x_A\, p\Delta\theta_{1,2} \;=\; 4p\,\frac{x_A x_B}{D_B} \qquad \text{(N s m; Phase space occupied at A)} \tag{64}$$

where we used (63).

Distribution at surface B

Now follow the four photons to position B. There is no force, as in Fig. 2a, so the vertical velocity components p_x of the photons remain constant; i.e. the RP move horizontally in our phase space (Fig. 8b). RP1 and RP3 move to the right (greater x), the latter moving faster because of its greater transverse momentum p_x. Thus, photon RP3 eventually overtakes RP1, and by definition this takes place at the surface B. At this time t_B, the two points have the same value of $x = +x_B$. Similarly, RP2 and RP4, with negative momenta, move to negative x and reach $x = -x_B$ together.

At surface B, the RP of the four photons again form the corners of a parallelogram, and this parallelogram contains all of our RP, i.e., those whose photons intercept B at $x < |x_B|$. Take the base of the parallelogram to be side 1–3 (length $p\,\Delta\theta_{1,3}$) and the height to be the perpendicular distance, $2x_B$. The surface area S_B is thus

$$S_B = 2x_B\, p\Delta\theta_{1,3} = 4p\,\frac{x_B\, x_A}{D_B} \qquad \text{(Phase space occupied at B)} \qquad (65)$$

where we used (62). The area is equal to the area at A (64). This illustrates the validity of Liouville's theorem for the case where photons are moving radially out from different portions of a source, for small angles θ.

The horizontal extent of a parallelogram at fixed p_x in Fig. 8b (e.g., 1–1′) represents the range of position at a given transverse momentum. This dimension remains constant as the RP propagate. (The photon tracks 1 and 1′ in Fig. 8a illustrate this range; the latter track originates at the bottom of the source and is parallel to track 1.)

Distribution at surface C

Finally, follow the photons to surface C (Fig. 8a). Since there is no force on them, they continue to move horizontally in the phase diagram. Photons 1 and 3 arrive at C with different values of x; photon 3 is at a greater x because of its greater momentum. We set the displacement x_C to be midway between the positions of 1 and 3.

The area of the parallelogram C is the product of the "height" $2x_C$ and the "base" 4–5. (This product can be shown to equal the product of the actual base and height of the parallelogram.) The "base" 4–5 represents the range of momenta at a given position x at surface C. Consider the position 4 at C in Figs. 8a,b. The range of momenta found at this one position is indicated by angular range $\Delta\theta_{4,5}$ defined by tracks 4 and 5 where track 5 projects back to the top of the source at A (not drawn). This angle is $\Delta\theta_{4,5} = 2x_A/D_C$, and the range of momentum is thus $\Delta p_{4,5} = p\,\Delta\theta_{4,5} = p\,2x_A/D_C$. The surface area of the parallelogram is thus, for our small angle approximations,

$$S_C = 2\,x_C\, p\Delta\theta_{4,5} = 4p\,\frac{x_C\, x_A}{D_C} = 4p\,\frac{x_B\, x_A}{D_B} \qquad \text{(Phase space at C)} \qquad (66)$$

where we invoke the relation $x_B/D_B \approx x_C/D_C$ from the geometry of Fig. 8a.

Here again, we see that the x,p_x phase space occupied is the same at C as at surfaces A and B. As stated after (61), the other dimensions also yield conservation of area. Since the phase space area is conserved in each of the 3 2-D planes, the 6-D volume is also be conserved.

Consistency with Liouville's theorem

This again demonstrates consistency with Liouville's theorem. Note that we have assumed no forces and have restricted our discussion to small angles θ, i.e., for photons near the axis of symmetry. For larger angles and for force fields, we rely on the more general derivations given earlier.

The underlying assumption in this discussion is that the individual photons in the sample are neither destroyed nor otherwise abruptly removed from the phase space element occupied by the sample, for example, by collisions. Given the conservation of photons (and hence RP) in the sample and also the conservation of phase space volume as demonstrated in this example, it follows that the density of RP in phase space must also be conserved as one follows the RP.

This conservation directly yields the conservation of specific intensity of radiation as it transits to us from a distant astronomical source if redshifts, absorption and scattering are negligible; see discussion in Section 3.3 of AP.

Problems

2 Particles in phase space

Problem 21 (a) Demonstrate that the energy-momentum relation (4) follows from (1) and (3) where the definition of γ (2) must also be invoked. (b) Show that the velocity-momentum relation (5) follows from (2) and (3). Since $\boldsymbol{v}$ is always parallel to $\boldsymbol{p}$, you may carry out your algebra with scalar quantities, $p \equiv |\boldsymbol{p}|$ and $v \equiv |\boldsymbol{v}|$, or the square of the magnitudes, e.g., $p^2 \equiv \boldsymbol{p}\cdot\boldsymbol{p}$.

Problem 22. A gas of N particles occupies a volume V. The probability of finding a particle of a gas in a given volume element d^3x is independent of its x, y, z position. The probability of finding a particle in a given momentum state is independent of the direction of motion but does depend on the magnitude of the momentum, i.e., the motions are isotropic. The distribution function is $f(\boldsymbol{x}, \boldsymbol{p}, t) = A \exp(-ap^2)$. (a) Integrate the distribution function according to (20) to obtain the number density n in x, y, z space. (b) The function f given here is the Maxwell-Boltzmann distribution if $a = (2mkT)^{-1}$, where k is Boltzmann's constant, m is the particle mass, and T is the temperature. If the gas contains N particles in an (x, y, z) volume V, what is the normalization factor A? [Ans. $A(\pi/a)^{3/2}$; $(N/V)(2\pi mkT)^{-3/2}$]

3 Liouville's theorem

Problem 31. (a) Demonstrate graphically in a two-dimensional x, p_x space how a friction force $F = -c_1 p_x$ changes the phase space volume of a group of representative points (RP) where c_1 is a constant. In other words, how would you modify Fig. 2b? Is the area occupied by the RP conserved? Find an expression for the fractional area change per unit time in terms of given quantities. Calculate the p-divergence of $\boldsymbol{F}$ and comment. (b) (b) Repeat (a) for the smooth force, $F_x = c_2\, x$, that varies only with position.. [Ans: $-c_1$, $-c_1$; 0; 0]

Problem 32. A gas is characterized by the Maxwell-Boltzmann distribution function $f(\boldsymbol{x}, \boldsymbol{p}, t) = A \exp\lfloor -ap^2 \rfloor$ where $p^2 = p_x^2 + p_y^2 + p_z^2$. There is no dependence on position; i.e., the density in x, y, z space is uniform. The effect of collisions between molecules may be characterized by a "friction" force, $\boldsymbol{F} = -b\boldsymbol{p}$, applied to each particle in the direction opposed to its motion and with magnitude proportional to momentum; that is, b is a constant. No other collision term is needed. (a) Use the continuity equation (32) to find $\partial f/\partial t$, the rate of change of the distribution function at a fixed position in phase space under the influence of the given force, in terms of f, b, and p. (b) Find $\mathrm{d}f/\mathrm{d}t$, the rate of change of the distribution function as one follows a group of RP. (c) Do your answers to (a) and (b) seem reasonable? Consider Problem 31a (if you did it) and plots similar to Fig. 4. Comment on signs and magnitudes. Note that f decreases with p unlike in Fig. 4b. (d) Find the same quantities, $\partial f/\partial t$ and $\mathrm{d}f/\mathrm{d}t$, if the force is dependent on position only; specifically it is $\boldsymbol{F} = -k\,\boldsymbol{r}$. The motion is harmonic. The function f remains the same as before. Again discuss whether your results seem reasonable. [Ans. $bf(3 - 2ap^2)$; $3bf$; —; $-2akf(\boldsymbol{r}\cdot\boldsymbol{p})$, 0]

Problem 33. It can be demonstrated that any function f of the constants of motion alone is a solution to the Vlasov equation. As an example, show that the one dimensional Vlasov equation can be satisfied by the following distribution function (phase-space density),

$$f = e^{-aE} \quad \text{where} \quad E = \frac{kx^2}{2} + \frac{p_x^2}{2m}$$

What is the condition on the force F_x for the equation to be satisfied? Consider only non-relativistic motion. (b) Can you justify the value of $\partial f/\partial t$ you obtain? Hint: consider the particle density distribution and the motion of a group of particles in the x, p_x plane, where the axes are adjusted to be $x(k/2)^{1/2}$ and $p_x/(2m)^{1/2}$; make a sketch. [Ans. $-kx$; circular track]

4 Special forces

Problem 41 (a) Carry out the missing steps in the text to demonstrate that the velocity $\boldsymbol{v}$ is identically equal to the p-gradient of the relativistic energy (54). Use the given relativistically correct expressions. (b) Demonstrate that, in general, the curl of a gradient is zero. Use an arbitrary scalar function $\phi(x,y,z)$ and cartesian coordinates.

Problem 42. Consider again the argument that the charged particles emitted from a slit in the presence of a uniform magnetic field obey Liouville's theorem; see Fig. 6. In particular compare the areas in x, p_x space at the right and left slits. (The x direction is horizontal in Fig. 6.) Let a group of particles with momentum p leave the right slit uniformly distributed across its width Δx at the same time with a small range of directions, $\Delta\theta \ll 1$, that is constant across the slit. All trajectories are circles of radius R and the slit centers are separated by $2R$. (a) Consider first particles leaving the center of the right slit. A vertical beam will pass through the center of the left slit, which is sufficiently wide to encompass all arriving particles. Calculate the x shift at the left slit if the emission angle at the right slit is at the extreme value of $+\Delta\theta/2$ from the vertical; sketch your geometry. What is the x shift for $-\Delta\theta/2$? What is the value of p_x in each case. (b) Make an x, p_x plot and locate with dots the x and p_x positions at the two slits of the central and two extreme (in angle) beams emerging from the center of the right slit. Then add dots for the three beams ($\theta = 0, \pm\Delta\theta/2$) emerging from each of the two extremes of position, again for each of the two slits. Finally connect the dots to illustrate the regions occupied by all the particles at the two slits; use your result from part (a). Compare the areas at the two slits. [Ans. $\Delta x' = R(\Delta\theta)^2/4$, $p_x = \pm p\Delta\theta/2$; areas match]

5 Photons in phase space

Problem 51. Consider the surfaces A and B in Fig. 8a. Let surface B be the emitter, and consider all the leftward-moving photons that intercept surface A. (a) Draw, as in Fig. 8a, the tracks of the four extreme photons that encompass these photons as they proceed from surface B to A; omit the surface C of Fig. 8. Draw in x, p_x phase space, analogously to Fig. 8b, the regions occupied by the RP of all the photons intercepting surface A when they are at the two surfaces, B and A. (b) Determine that the area occupied by the RP in this phase space is the same at the two positions, B and A. Again, assume small angles.