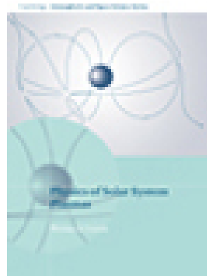

Physics of Solar System Plasmas

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Book DOI: http://dx.doi.org/10.1017/CBO9780511529467

Online ISBN: 9780511529467

Hardback ISBN: 9780521352802

Paperback ISBN: 9780521611947

Chapter

4 - Magnetohydrodynamics pp. 90-158

Chapter DOI: http://dx.doi.org/10.1017/CBO9780511529467.007

Cambridge University Press

4
Magnetohydrodynamics

In Chapter 3, we studied how single charged particles move in specified electric and magnetic fields, and we then applied our knowledge of single particle motion to the radiation belt and ring current plasma. However, the fields in some situations depend too much on the particle distributions to be readily specified and must be found self-consistently using the charged particle distribution functions. Often, it is not necessary to have complete information about the distribution functions in a system. In fact, it is usually sufficient to know only a few of the velocity moments of the distribution function, as derived in Chapter 2. In Chapter 4, we will adopt the "fluid" picture of a plasma, introduced in Chapter 2, and further refine it to obtain an analytical tool useful for studying space plasma phenomena. This analytical tool is called magnetohydrodynamics (or MHD for short). We cannot adequately cover in one chapter all the material that would be desirable to know about this subject and so the reader is encouraged to consult one or more of the references listed in the bibliography at the end of this chapter.

4.1 Two-fluid plasma

Let us consider a plasma consisting of two species: electrons (e) with mass m_e and a single ion species (i) with mass m_i. The continuity equations for electrons and ions are given by (see Equation (2.40))

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = S_e \tag{4.1}$$

and

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = S_i, \tag{4.2}$$

where $n_s, \mathbf{u}_s, S_s$ ($s = e, i$) refer to the number density, flow velocity, and net source, respectively, for species s. The respective mass densities can be expressed as $\rho_e = m_e n_e$ and $\rho_i = m_i n_i$.

The momentum equation for a plasma species s was given by Equation (2.61). We assume that both the electron and ion thermal pressure tensors are isotropic, with scalar pressures given by p_e and p_i, respectively. The electron and ion momentum equations are

$$n_e m_e \left[\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla\right]\mathbf{u}_e = -n_e e[\mathbf{E} + \mathbf{u}_e \times \mathbf{B}] - \nabla p_e + n_e m_e \mathbf{g} - \sum_{t \neq e} n_e m_e \nu_{et}(\mathbf{u}_e - \mathbf{u}_t) \tag{4.3}$$

and

$$n_i m_i \left[\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla\right]\mathbf{u}_i = +n_i e[\mathbf{E} + \mathbf{u}_i \times \mathbf{B}] - \nabla p_i + n_i m_i \mathbf{g} - \sum_{t \neq i} n_i m_i \nu_{it}(\mathbf{u}_i - \mathbf{u}_t) - P_i(m_i \mathbf{u}_i - m_n \mathbf{u}_n). \tag{4.4}$$

Recall from Chapter 2 that $\mathbf{g}$ is the acceleration due to gravity and ν_{st} is the effective momentum transfer collision frequency between species s and t. The "frictional" term includes a sum over all species but does not include self-collisions (i.e., $s \neq t$). For electrons, one must consider electron–ion and electron–neutral collisions. Equation (4.4) includes the "mass-loading" term, but the viscosity terms have been omitted. P_i is the production rate of ions due to the ionization of a neutral species with mass m_n and velocity $\mathbf{u}_n$.

The energy equations for electrons and ions can be found from Equation (2.73), (2.78), or (2.79) with $s = e$ or i. However, these equations will not be reproduced here but will be provided in convenient forms as they are required later in the chapter.

The $\mathbf{E}$ and $\mathbf{B}$ fields that are present in Equations (4.3) and (4.4) are macroscopic fields and can be found from Maxwell's equations with macroscopically defined source terms (see Chapter 2, Section 2.5).

The full two-fluid equations are difficult to use, although they have been occasionally used for space physics problems such as solar wind outflow from the Sun. Usually these equations are further approximated.

4.2 Plasma oscillations

4.2.1 Waves

Waves in plasmas can be studied using either the fluid equations or the Vlasov equation. However, the fluid approach is much easier to carry out and gives a better physical picture of the nature of wave propagation. We restrict ourselves to the fluid approach in this book, although it should be noted that wave growth or damping must be treated with the Vlasov equation approach. Before dealing with waves in plasmas, let us review some properties of waves in general.

Any general wave train (as specified by the wave part of the electric field $\mathbf{E}_1(\mathbf{x}, t)$, for example) can be represented by the sum of many plane waves with a range of frequencies. Formally, $\mathbf{E}_1(\mathbf{x}, t)$ is given by the inverse Fourier–Laplace transform of $\mathbf{E}_1(\mathbf{k}, \omega)$, where $\mathbf{k}$ is the *wave vector* and ω is the angular *frequency* of a single Fourier–Laplace component:

$$\mathbf{E}_1(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int\int d^3\mathbf{k}\, d\omega \mathbf{E}_1(\mathbf{k}, \omega) e^{i[\mathbf{k}\cdot\mathbf{x} - \omega t]}. \tag{4.5}$$

If a wave train has a very narrow range of frequencies and wavenumbers, ω and $\mathbf{k}$ (for example, a delta function), then we can write Equation (4.5) as

$$\mathbf{E}_1(\mathbf{x}, t) = \mathbf{E}_0 e^{i[\mathbf{k}\cdot\mathbf{x} - \omega t]}, \tag{4.6}$$

where $\mathbf{E}_0$ is a constant amplitude vector, which can be complex (i.e., have real and imaginary parts). Equation (4.6) describes a plane wave. Actually, the measurable electric field for a plane wave is the real part of Equation (4.5) or (4.6). Recall that the exponential function of an imaginary argument is given by

$$e^{ix} = \cos x + i \sin x. \tag{4.7}$$

Hence, if the wave amplitude $\mathbf{E}_0$ is real (although it does not have to be in general), we find that the electric field for our plane wave is

$$\mathbf{E}_1(\mathbf{x}, t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{x} - \omega t). \tag{4.8}$$

The wave vector $\mathbf{k}$ points in the direction of wave propagation. We write $\mathbf{k} = k\hat{\mathbf{x}}$ for a wave propagating in the x direction. The *wavenumber* k can be written in terms of the *wavelength* λ as

$$k = \frac{2\pi}{\lambda}. \tag{4.9}$$

The angular frequency can be expressed as $\omega = 2\pi f$, where f is the wave *frequency* in *cycles/second* or *hertz* (Hz). For $\mathbf{k} = k\hat{\mathbf{x}}$, Equation (4.8) becomes

$$\mathbf{E}_1(x, t) = \mathbf{E}_0 \cos(kx - \omega t). \tag{4.10}$$

$\mathbf{E}_1(x, t)$ varies sinusoidally in time with frequency ω for fixed x and varies sinusoidally in the spatial variable x for a fixed time. The *phase* of the wave can be written

$$phase = [kx - \omega t] = k\left[x - \left(\frac{\omega}{k}\right)t\right] = constant. \tag{4.11}$$

The phase clearly remains constant for a reference frame moving with velocity (ω/k). The *phase speed* of the wave $(\Delta x/\Delta t)$ is thus given by

$$V_{\text{ph}} = \omega/k. \tag{4.12}$$

A wave packet limited in spatial extent ($\infty \gg \Delta x \gg \lambda$) consists of the superposition of "many" plane waves for a range of wavenumbers (and frequencies) and is centered at (k, ω). The wave packet as a whole propagates at the *group velocity*

$$V_g = \frac{\partial \omega}{\partial k}, \tag{4.13}$$

where ω is expressed as a function of wavenumber k. The functional dependence of ω on k depends on the type of wave and is given by the *dispersion relation* for that type of wave:

$$\omega = \omega(k). \tag{4.14}$$

For example, the dispersion relation for electromagnetic wave propagation in a vacuum is

$$\omega^2 = k^2 c^2, \tag{4.15}$$

where c is the speed of light. In this case, we simply have $V_g = V_{ph} = \pm c$, which tells us that electromagnetic waves travel at the speed of light in a vacuum. The propagation of electromagnetic radiation in a plasma is considered in the appendix and in Problem 4.1. An electromagnetic wave does *not* travel at the speed of light in a plasma.

4.2.2 Plasma (Langmuir) oscillations

Plasmas support many different *wave modes* and *Langmuir oscillations/waves* comprise the most important mode. Langmuir oscillations are a high-frequency phenomena primarily involving electrons; ions are relatively massive and are slow to follow the wave motion. Hence, let us assume that the ions are motionless and that the ion density remains uniform: $n_i = n_0$. Let us further assume that the electron and ion fluids are both cold: $T_e = T_i = 0$. Let the magnetic field be zero and we also neglect collisions. The two-fluid equations are reduced to the following simple continuity and momentum equations for electrons:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0 \tag{4.16}$$

and

$$m_e n_e \left[\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla)\mathbf{u}_e \right] = -n_e e \mathbf{E}. \tag{4.17}$$

The electric field is given by Gauss's law with the charge density expressed in terms of the electron and ion densities:

$$\nabla \cdot \mathbf{E} = \frac{e}{\varepsilon_0}(n_i - n_e) = \frac{e}{\varepsilon_0}(n_0 - n_e). \tag{4.18}$$

We now separate all quantities into background (subscript "0") and wave parts (subscript "1") to obtain

$$n_e = n_{e0} + n_{e1}, \tag{4.19a}$$

$$\mathbf{u}_e = \mathbf{u}_{e0} + \mathbf{u}_{e1}, \tag{4.19b}$$

and

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1. \tag{4.19c}$$

We take the background plasma to be quasi-neutral and uniform ($n_{e0} = n_i = n_0$). In this case, we must also have $\mathbf{u}_{e0} = 0$ and $\mathbf{E}_0 = 0$ for the background plasma in order to satisfy Equations (4.16)–(4.18). Note that $\nabla n_{e0} = 0$ and $\partial n_{e0}/\partial t = 0$.

Further progress can be achieved by assuming that the wave amplitude is small:

$$n_e = n_0 + n_{e1} \quad \text{with } |n_{e1}| \ll n_0. \tag{4.20}$$

All wave quantities, relative to the background quantities, are of the order of a parameter $\varepsilon = O(|n_{e1}|/n_0)$, which is assumed to be very small ($\varepsilon \ll 1$).

We can now write the electron continuity and momentum equations as

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot [(n_0 + n_{e1})\mathbf{u}_{e1}] = 0 \tag{4.21}$$

and

$$m_e\left[\frac{\partial \mathbf{u}_{e1}}{\partial t} + \mathbf{u}_{e1} \cdot \nabla \mathbf{u}_{e1}\right] = -e\mathbf{E}_1 \tag{4.22}$$

$$\approx \varepsilon \qquad \approx \varepsilon^2 \text{ (neglect)} \qquad \approx \varepsilon.$$

The magnitudes of the terms in Equation (4.22) are indicated using the parameter ε.

Gauss's law becomes

$$\nabla \cdot \mathbf{E}_1 = \frac{e}{\varepsilon_0}[n_0 - (n_0 + n_{e1})] = -\frac{n_{e1}e}{\varepsilon_0}. \tag{4.23}$$

The net charge density is proportional to the wave density n_{e1}.

We now *linearize* Equations (4.21) and (4.22) by neglecting all terms of order higher than ε, such as the $\nabla \cdot (n_{e1}\mathbf{u}_{e1})$ term in Equation (4.21) and the $\mathbf{u}_{e1} \cdot \nabla \mathbf{u}_{e1}$ term in Equation (4.22). For example, the linearized continuity equation is

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \mathbf{u}_{e1} = 0. \tag{4.24}$$

Our next step is to assume that we have plane waves propagating in the x direction (i.e., $\mathbf{k} = k\hat{\mathbf{x}}$) (see Equation (4.6)). Then we have

$$n_{e1}(x, t) = \tilde{n}_{e1} \exp[i(kx - \omega t)], \tag{4.25}$$

$$\mathbf{u}_{e1}(x, t) = \tilde{u}_{e1}\hat{\mathbf{x}} \exp[i(kx - \omega t)], \tag{4.26}$$

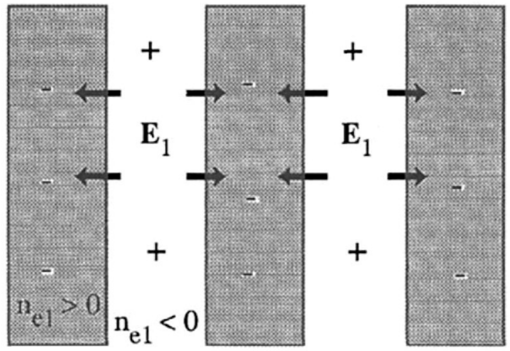

Figure 4.1. Schematic for plane-wave Langmuir wave showing slablike electron density perturbations.

and

$$\mathbf{E}_1(x,t) = \tilde{E}_1\hat{\mathbf{x}}\exp[i(kx-\omega t)], \tag{4.27}$$

where $\tilde{u}_{e1}$, $\tilde{n}_{e1}$, and $\tilde{E}_1$ are plane-wave amplitudes that are independent of x and t. Note that $\mathbf{E}_1$ is parallel to $\mathbf{k}$ because Langmuir waves are longitudinal. One-dimensional (slablike) perturbations in the density n_e (and therefore in the charge density) naturally lead to longitudinal wave electric fields (see Figure 4.1). In the x direction, positive and negative electron density perturbations (n_{e1}) alternate and give rise to negative and positive net charge densities, respectively. The slabs of charge produce a wave electric field ($\mathbf{E}_1$) that points from positive charge to negative charge. This electric field then accelerates the electrons away from regions of excess electron density.

The time derivative of an arbitrary plane-wave quantity, Q, gives

$$\frac{\partial}{\partial t}Q(x,t) = -i\omega Q(x,t). \tag{4.28}$$

And the gradient operator, when applied to plane waves propagating in the x direction, can be written

$$\nabla \Rightarrow +ik\hat{\mathbf{x}}. \tag{4.29}$$

We now substitute the plane-wave expressions for $n_{e1}(x,t)$, $\mathbf{u}_{e1}(x,t)$, and $\mathbf{E}_1(x,t)$ into the linearized versions of Equations (4.22)–(4.24).

The continuity equation becomes

$$-i\omega\tilde{n}_{e1} = -ik\tilde{u}_{e1}n_0, \tag{4.30}$$

the momentum equation becomes

$$-im_e\omega\tilde{u}_{e1} = -e\tilde{E}_1, \tag{4.31}$$

and Gauss's law becomes

$$ik\tilde{E}_1 = -\frac{e}{\varepsilon_0}\tilde{n}_{e1}. \tag{4.32}$$

We use Equations (4.30) and (4.31) to eliminate $\tilde{n}_{e1}$ from Equation (4.32), which becomes

$$ik\tilde{E}_1 = -\frac{e}{\varepsilon_0}\frac{kn_0}{\omega}\left(\frac{e}{im_e\omega}\right)\tilde{E}_1. \tag{4.33}$$

We now divide both sides by $\tilde{E}_1$, rearrange terms, and obtain the following dispersion relation for Langmuir waves in a cold plasma:

$$\omega^2 = \omega_{\text{pe}}^2, \tag{4.34}$$

where we define

$$\omega_{\text{pe}} \equiv \sqrt{\frac{n_0e^2}{\varepsilon_0 m_e}}. \tag{4.35}$$

ω_{pe} is called the *electron plasma frequency*. In cgs units, the electron plasma frequency is given by

$$\omega_{\text{pe}} = \sqrt{\frac{4\pi n_0 e^2}{m_e}}. \tag{4.36}$$

Equation (4.34) describes how the wave frequency varies versus k

$$\omega(k) = \pm\omega_{\text{pe}}. \tag{4.37}$$

In fact, the function $\omega(k)$ is independent of k; and, hence, the group velocity ($V_{\text{g}} = \partial\omega/\partial k$) is zero. In this case, we actually have an oscillation – *a plasma oscillation* – rather than a propagating wave.

When ion motions are allowed, the relevant plasma frequency is given by

$$\omega_{\text{p}} = \sqrt{\omega_{\text{pe}}^2 + \omega_{\text{pi}}^2} \tag{4.38}$$

with

$$\omega_{\text{pi}} \equiv \sqrt{\frac{n_0e^2}{\varepsilon_0 m_i}}$$

Because $m_i \gg m_e$ (unless we assume positrons rather than massive ions – see Problem 4.3), $\omega_{\text{p}} \cong \omega_{\text{pe}}$ is an excellent approximation.

Physically (see Figure 4.1), plasma oscillations occur because the moving electrons have inertia so that they overshoot their equilibrium position (where $\mathbf{E}_1 = 0$) and take a finite time to be decelerated. The deceleration occurs because of the electric field when the electrons pile up in a different location. Electron inertia "opposes" the electric force associated with the pile up of electrons. This is a collective phenomenon (i.e., involving large numbers of electrons acting together) in which a large-scale electron field ($\mathbf{E}_1$) is generated.

A useful expression (in SI units) for the frequency $f_p = \omega_p/2\pi$ is given by

$$f_p \cong 9\sqrt{n_0\,(\mathrm{m}^{-3})} \quad [\mathrm{Hz}]. \tag{4.39}$$

Example 4.1 A typical *solar wind* electron density at 1 AU is

$$n_0 \approx 10^7\,\mathrm{m}^{-3}.$$

Using this density we find from Equation (4.39) that a typical plasma frequency for the solar wind is

$$f_p \approx 3 \times 10^4\,\mathrm{Hz} = 30\,\mathrm{kHz}.$$

The average maximum *ionospheric* electron density at Earth is

$$n_0 \approx 10^{12}\,\mathrm{m}^{-3}.$$

The plasma frequency given by Equation (4.39) for the ionosphere is

$$f_p \approx 9\,\mathrm{MHz}.$$

Electromagnetic waves (these are transverse waves, unlike Langmuir waves, which are longitudinal) cannot propagate in a plasma if their frequency is less than f_p (see the appendix and Problem 4.1). Instead, waves impinging on a plasma medium from the outside reflect at the location in the plasma where $f = f_p$. Recall that the AM (amplitude modulation) radio band is 0.5–1.6 MHz, and hence these waves are reflected from the terrestrial ionosphere. The FM (frequency modulation) radio band has $f > 88$ MHz – well above the maximum ionospheric plasma frequency – explaining why FM waves can propagate right through the ionosphere.

As we have just seen, a cold plasma can "support" Langmuir oscillations but not Langmuir waves that propagate. However, wave propagation is possible in a warm plasma. In a warm plasma ($T_e \neq 0$) the electron pressure gradient force term must be retained in the momentum equation, and the dispersion relation can be rederived (which you will do in Problem 4.2):

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_{te}^2, \tag{4.40}$$

where the *electron thermal speed* is defined by

$$v_{te}^2 \equiv k_B T_e/m_e. \tag{4.41}$$

The frequency ω in Equation (4.40) (unlike in Equation (4.34)) now depends on the wavenumber, and thus the group velocity,

$$V_g = \frac{\partial \omega}{\partial k} = 3\frac{k}{\omega}v_{te}^2 = \frac{3v_{te}^2}{v_{ph}}, \tag{4.42}$$

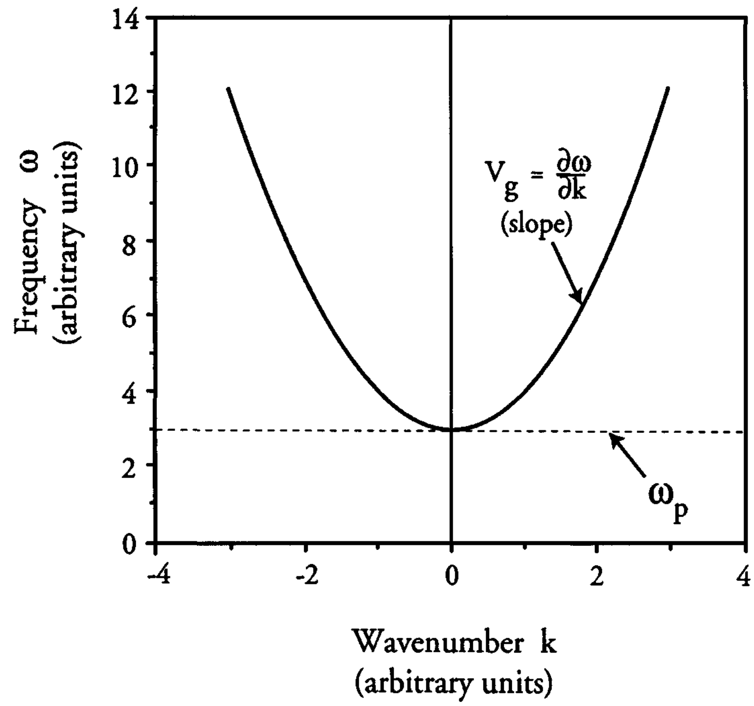

Figure 4.2. Plasma/Langmuir dispersion relation. The group velocity, V_g, is the first derivative of $\omega(k)$.

is nonzero and the phase speed, ω/k, is equal to

$$V_{ph} = \frac{\omega}{k} = \frac{1}{k}\sqrt{\omega_{pe}^2 + 3k^2 v_{te}^2} \approx \frac{\omega_{pe}}{k} \quad \text{(for small } k\text{)}. \tag{4.43}$$

The dispersion relation, $\omega(k)$, is shown in Figure 4.2. The group velocity is just the slope of the function $\omega(k)$; we can clearly see that $V_g = 0$ at $k = 0$. And as $k \to \infty$, $V_{ph} = V_g \to \sqrt{3}v_{te}$. Plasma waves cannot exist with frequencies below ω_{pe}.

4.3 The single-fluid equations and the generalized Ohm's law

4.3.1 Quasi-neutrality and the generalized Ohm's law

If we are dealing with a plasma phenomenon that is both large scale (scale size $L \gg \lambda_D$) and has a relatively low frequency ($\omega \ll \omega_p$), then the plasma is quasi-neutral ($n_e \cong n_i$) on these length and time scales. Most interesting space plasma phenomena satisfy these two criteria. We can then assume that the electron density is equal to the total ion density:

$$n_e = n_i. \tag{4.44}$$

One consequence of Equation (4.44) is that we do not need separate continuity equations for the electron and ion gases; a single continuity equation suffices.

A difficulty arises if we adopt $n_e = n_i$: The *assumed* charge density is zero ($\rho_c = 0$), although the actual charge density must have very small deviations from zero. With zero charge density Gauss's law, Equation (4.18), simply becomes $\nabla \cdot \mathbf{E} = 0$ and is no longer useful for determining the electric field. An alternate means of finding $\mathbf{E}$ is required. The electron momentum Equation (4.3) can be used for this purpose with the assumption $n_e = n_i$. Solving Equation (4.3) for $\mathbf{E}$ we obtain a relation called the *generalized Ohm's law* (GOL):

$$\mathbf{E} = -\mathbf{u}_e \times \mathbf{B} - \frac{1}{n_e e}\nabla p_e + \frac{m_e}{e}\mathbf{g} + \frac{m_e}{e}\sum_{t\neq e}\nu_{\text{et}}(\mathbf{u}_e - \mathbf{u}_t)$$
$$-\frac{m_e}{e}\left[\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right]. \tag{4.45}$$

Equation (4.45) specifies the electric field required to maintain quasi-neutrality in a plasma. We can derive a simpler form of the GOL for a collisionless plasma by neglecting the friction term (i.e., the term that includes the collision frequency ν_{et}) and by neglecting gravity and the electron inertial terms (the latter is the last term), which are proportional to the very small electron mass:

$$\mathbf{E} = -\mathbf{u}_e \times \mathbf{B} - \frac{1}{n_e e}\nabla p_e. \tag{4.46}$$

motional electric field — ambipolar electric field

The *ambipolar electric field* (or *polarization electric field*) is proportional to the gradient of the electron pressure. The *motional electric field* (which is the first term on the right-hand side as indicated) is associated with the frame of reference of the electron gas. The electric field, $\mathbf{E}'$, in a reference frame moving at the electron flow velocity, $\mathbf{u}_e$, does not include this term:

$$\mathbf{E}' = \mathbf{E} + \mathbf{u}_e \times \mathbf{B} = -\frac{1}{n_e e}\nabla p_e. \tag{4.47}$$

Now consider the component of this electric field parallel to the magnetic field, $E_\parallel = \mathbf{E} \cdot \hat{\mathbf{b}}$, where $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$ is the unit vector parallel to $\mathbf{B}$. From Equation (4.46) we find

$$E_\parallel = -\frac{1}{n_e e}(\hat{\mathbf{b}} \cdot \nabla p_e) = -\frac{1}{n_e e}\frac{\partial p_e}{\partial s}, \tag{4.48}$$

where s is the distance along a magnetic field line. If we further assume that the electrons are isothermal with temperature T_e, and if we use the equation of state, Equation (4.48) becomes

$$E_{\parallel} = -\frac{k_B T_e}{e}\frac{1}{n_e}\frac{\partial n_e}{\partial s}. \tag{4.49}$$

Equation (4.49) indicates that a polarization electric field exists as a consequence of gradients along **B** in the plasma density. The electric potential difference between two points along the magnetic field due to the polarization is determined by integrating Equation (4.49). If the electron temperature is independent of distance then this potential difference just depends on the variation of the electron density and is approximately $k_B T_e/e$ over a typical scale length for significant density changes. How can we determine the variation of the electron density n_e as a function of distance s? The electron density is just equal to the ion density n_i, by the assumption of quasi-neutrality, and n_i can be found by solving the fluid conservation equations for ions (we will return to this shortly).

Suppose the ion density as a function of s (i.e., $n_i(s)$) is specified. For example, suppose n_i increases as s increases (Figure 4.3). If no ambipolar field existed, and if we assumed strict charge neutrality initially, an unbalanced pressure gradient force on the electrons would exist and accelerate them so that the electrons would move from the high-density region to the low-density region. A small charge imbalance would then very quickly develop ($n_e \cong n_i$, but not exactly equal) as illustrated in Figure 4.3. When the electric field resulting from the very small, but nonzero, charge density just equaled the ambipolar field as specified by Equation (4.49) (or, actually, by Equation (4.45) if *all* forces on the electron gas are included) then an electron force balance would again be achieved with n_e almost equal to n_i (but not quite – the difference $(n_i - n_e)$ is exceedingly small – see Problem 4.4). The ambipolar, or polarization, electric field thus holds the electron and ion gases together and maintains quasi-neutrality. Theoretically, this electric field could be found using Gauss's law; but, practically, the difference between n_e and n_i is so small (on length scales $L \gg \lambda_D$) relative to the magnitude of n_e that reliably calculating the charge density for real problems is almost impossible.

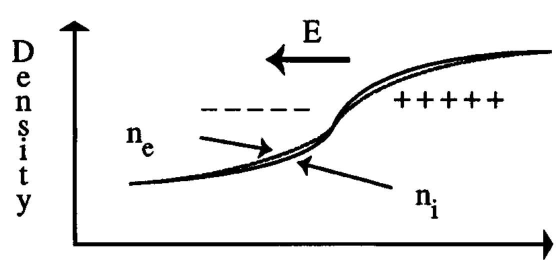

Figure 4.3. Ambipolar polarization electric field associated with electron pressure gradient force (i.e., with the density gradient for isothermal electrons). The Coulomb/electric force due to the minute charge imbalance counteracts the pressure gradient force.

4.3.2 Single-fluid equations

A multi-ion species plasma can be described with the set of Equations (4.1)–(4.4) plus the generalized Ohm's law. The generalized Ohm's law, Equation (4.45), tells us what **E** is in terms of the variables n_e, $\mathbf{u}_e$, and T_e. For *several* singly charged ion species, n_e is equal to the sum of the densities of all ion species (this is a generalization of Equation (4.44)):

$$n_e = n_i \equiv \sum_{\substack{\text{all ion} \\ \text{species } s}} n_s. \tag{4.50}$$

The summation in this equation is over all ion species in the plasma (e.g., H^+, O^+, ...). We can determine the density of each ion species, n_s, and the ion velocities, $\mathbf{u}_s$, from the appropriate continuity and momentum equations. We also need to know the temperature of each ion species, which can be found by using the appropriate energy equation. The electric field in all these fluid equations can be taken from the GOL. Our remaining unknown is the electron velocity $\mathbf{u}_e$, which cannot be found by using the electron momentum equation because that equation was "converted" into the GOL and was instead used to find **E**. The electron flow velocity can be found from the electrical current density and the calculated ion velocities if the magnetic field is known.

At this point, we simplify the problem further by considering just a *single fluid* that combines the electron gas and all the ion species. We start with the definition of the total *mass density*:

$$\rho(x, t) \equiv n_e m_e + \sum_s n_s m_s, \tag{4.51}$$

where the sum is over all ion species. The *center of mass velocity* of the plasma is given by

$$\mathbf{u}(\mathbf{x}, t) \equiv \frac{1}{\rho}\left[n_e m_e \mathbf{u}_e + \sum_s n_s m_s \mathbf{u}_s\right]. \tag{4.52}$$

For a single ion species, we can find expressions for the mass density ρ and the center of mass flow velocity **u**, starting from Equations (4.51) and (4.52):

$$\rho = n_e m_e + n_i m_i = n_e(m_e + m_i) \cong n_e m_i, \tag{4.53}$$

$$\mathbf{u} = \frac{m_e \mathbf{u}_e + m_i \mathbf{u}_i}{m_e + m_i} \cong \mathbf{u}_i. \tag{4.54}$$

The approximate versions of Equations (4.53) and (4.54) are quite accurate because the electron mass is much less than the ion mass ($m_e/m_i \ll 1$). Our description of the plasma also requires knowledge of the current density **J**:

$$\mathbf{J}(x, t) = \sum_s n_s Z_s e \mathbf{u}_s - n_e e \mathbf{u}_e, \tag{4.55}$$

where Z_s is the charge number on ion species s. For a single ion species (with $Z = 1$) Equation (4.55) becomes

$$\mathbf{J} = n_e e(\mathbf{u}_i - \mathbf{u}_e). \tag{4.56}$$

The *single-fluid mass continuity equation* is found by mass-weighting the electron and ion continuity equations (4.1) and (4.2) and using the definitions of ρ and $\mathbf{u}$ to obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = m_i S_i. \tag{4.57}$$

A small source term, $m_e S_e$, was neglected in this equation.

The *single-fluid momentum equation* can be derived by adding together the electron and ion momentum equations; the electric field cancels out during this operation. Note that this is equivalent to substituting the electric field from the GOL, Equation (4.45), into the ion momentum equation (4.4). We have

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right] = -\nabla p + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g} - \rho\bar{\nu}(\mathbf{u} - \mathbf{u}_n) - P_i m_i(\mathbf{u} - \mathbf{u}_n). \tag{4.58}$$

Terms with the charge density ρ_c were neglected, as were terms of order m_e/m_i ($\ll 1$). In the mass-loading term at the end of this equation we have supposed that $m_i = m_n$. The average momentum transfer collision frequency of ions with neutrals is $\bar{\nu} \cong \nu_{in}$, and henceforth this will just be denoted ν. The total thermal pressure is $p = p_e + p_i$; the electron and ion pressures can be found using the appropriate energy equations discussed in Chapter 2.

A slightly more accurate version of the momentum equation (4.58) would also include a term $\rho_c\mathbf{E}$ on the right-hand side with $\mathbf{E}$ specified by the GOL. In this case, we would need the following *charge continuity equation* for ρ_c, which can be derived from Maxwell's equations (Problem 4.5):

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{4.59}$$

Typically, an excellent approximation to the charge continuity equation is given by

$$\nabla \cdot \mathbf{J} = 0. \tag{4.60}$$

The charge continuity equation (4.59) with suitable boundary conditions can be used to find the current density $\mathbf{J}$. The magnetic field must also be specified and we can use Ampère's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{4.61}$$

for this purpose. We have neglected the displacement current in Equation (4.61); this is a good approximation for phenomena with time scales τ such that $\tau^{-1} \ll \omega_p$.

There exists another way of finding the magnetic field, which is often much easier to use, as will be considered in the next section.

One difficulty remains – the electric field given by Equation (4.45) is written in terms of $\mathbf{u}_e$, but we really need it in terms of the variables $\mathbf{u}$ and $\mathbf{J}$. The conversion of Equation (4.45) into a new form of the generalized Ohm's law using $\mathbf{u}$ and $\mathbf{J}$ is messy but straightforward (this can be done as an exercise by the ambitious reader):

$$\mathbf{E} = \underbrace{-\mathbf{u} \times \mathbf{B}}_{\text{motional field}} + \underbrace{\frac{1}{n_e e}\mathbf{J} \times \mathbf{B}}_{\text{Hall term}} - \underbrace{\frac{1}{n_e e}\nabla p_e}_{\text{ambipolar polarization term}} + \underbrace{\eta \mathbf{J}}_{\text{Ohmic term}} + \underbrace{\frac{m_e}{n_e e^2}\left\{\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot [\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J}]\right\}}_{\text{electron inertial term}}. \tag{4.62}$$

The commonly used names for the various terms are indicated.

The *Ohmic resistivity* is given by

$$\eta = \frac{m_e \nu_e}{n_e e^2}, \tag{4.63}$$

where ν_e is the total electron momentum transfer collision frequency (see Chapter 2), which for a single ion species is $\nu_e = \nu_{ei} + \nu_{en}$ (i.e., electron–ion Coulomb collision frequency and electron–neutral collision frequency).

The electron inertial term, typically being quite small relative to other terms in Equation (4.62), is often neglected. The Hall term results from using $\mathbf{u}$ rather than $\mathbf{u}_e$ in the motional electric field term; it is also often neglected. The ambipolar/polarization term was already discussed. In a collisionless plasma, the resistivity is zero ($\eta = 0$); equivalently, the *electrical conductivity* $\sigma = 1/\eta$ is infinite. For the opposite extreme of large η, with $\nabla p_e = 0$ and $\mathbf{B} = 0$, the generalized Ohm's law looks like the "ordinary" Ohm's law:

$$\mathbf{J} = \frac{1}{\eta}\mathbf{E} = \sigma \mathbf{E}. \tag{4.64}$$

If we retain the motional electric field but still neglect the rest of Equation (4.62) we can write Ohm's law as

$$\mathbf{J} = \frac{1}{\eta}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \frac{1}{\eta}\mathbf{E}', \tag{4.65}$$

where $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the electric field in the plasma frame of reference (also see Equation (4.47)).

The GOL has important implications for the time evolution of the magnetic field, as we will see in the next section.

4.4 Magnetic convection–diffusion ("freezing" law)

4.4.1 The magnetic induction equation (convection–diffusion equation)

The time evolution of the magnetic field is given by Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \tag{4.66}$$

The time rate of change of $\mathbf{B}$ depends on the curl of the electric field. You will not be surprised to find that we will use the generalized Ohm's law to supply $\mathbf{E}$. Starting with Equation (4.62), we can again neglect the electron inertial terms. And even when the Hall and pressure gradient terms are not that small, the curls of these terms are usually small. We are left with

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J}, \tag{4.67}$$

which is just Equation (4.65) rearranged.

Combining Equations (4.66) and (4.67) we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \mathbf{J}). \tag{4.68}$$

Using Equation (4.61) (Ampère's law) to eliminate the current density in favor of the magnetic field we obtain the *magnetic convection–diffusion equation*:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{magnetic convection}} - \underbrace{\nabla \times \big(D_B \nabla \times \mathbf{B}\big)}_{\text{magnetic diffusion}}, \tag{4.69}$$

where the *magnetic diffusion coefficient* is given by

$$D_B = \eta/\mu_0 = \frac{m_e \nu_e}{\mu_0 n_e e^2}. \tag{4.70}$$

Equation (4.63) for the resistivity was used in Equation (4.70). The two terms of the right-hand side of Equation (4.69) have been labeled "magnetic convection" and "magnetic diffusion" for reasons you will see below. We can simplify the magnetic diffusion term if $\nabla D_B \times (\nabla \times \mathbf{B}) = 0$ can be assumed, as is true for a medium of uniform resistivity. We can then employ a vector calculus identity and the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ to get

$$\nabla \times \nabla \times \mathbf{B} = \underbrace{\nabla(\nabla \cdot \mathbf{B})}_{\text{equals } 0} - (\nabla \cdot \nabla)\mathbf{B}. \tag{4.71}$$

With the above simplifications, Equation (4.69) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + D_B \nabla^2 \mathbf{B}, \tag{4.72}$$

taking the form of a traditional diffusion-type equation in its last term.

4.4.2 Frozen-in magnetic flux

Let us compare the order of magnitude of the convection term (i.e., first term on the right-hand side) of Equation (4.72) to the order of magnitude of the diffusion term. That is, suppose L is the typical spatial scale over which the variables B and u vary significantly, and suppose that τ is the typical time constant for temporal variation of B. The "dimensional" analysis of Equations (4.69) or (4.72) gives us the following time constant (for $D_B = 0$):

$$\frac{B}{\tau} \sim \frac{u \cdot B}{L} \quad \text{or} \quad \tau \sim L/u. \tag{4.73}$$

For $u = 0$ (no convection), another simple dimensional analysis provides us with a typical magnetic diffusion time constant of

$$\tau \sim L^2/D_B. \tag{4.74}$$

Thus, for a convection-dominated situation, the time scale for evolution of B is directly proportional to the length scale L, whereas for a diffusion-dominated situation, the time scale varies as the square of L (as is typical for diffusion problems of all types).

How do we determine whether magnetic diffusion or convection is more important for a particular plasma regime? We compare the magnitude of the two terms to obtain the *magnetic Reynolds number* R_m:

$$\begin{aligned} R_\mathrm{m} &= \frac{(\text{convection term})}{(\text{diffusion term})} \\ &\sim \frac{uB/L}{D_B B/L^2} \sim \frac{Lu}{D_B}. \end{aligned} \tag{4.75}$$

The magnetic Reynolds number is analogous to the ordinary *Reynolds number*, which is the ratio of the viscosity term to the convection/advection term in the Navier–Stokes equation. The Navier–Stokes equation is a form of the momentum equation that includes viscosity effects (i.e., the viscosity term is the second term on the right-hand side of Equation (2.64)).

From Equation (4.75), we see that R_m is very large ($R_\mathrm{m} \gg 1$) and convection dominates for small values of the magnetic diffusion coefficient D_B (i.e., for low resistivity/high electrical conductivity). For large values of D_B, the magnetic Reynolds number R_m is small ($R_\mathrm{m} \ll 1$) in which case magnetic diffusion (i.e., Ohmic dissipation of currents) dominates the time evolution of the magnetic field. However, notice that even for small values of D_B (as is the case for most space plasmas) a small value of R_m results if the values of L and/or u are small enough. As we shall see in Chapter 8, this can happen in a narrow current sheet such as the one located at the Earth's magnetopause. For $R_\mathrm{m} = 1$, magnetic convection and diffusion are of comparable importance.

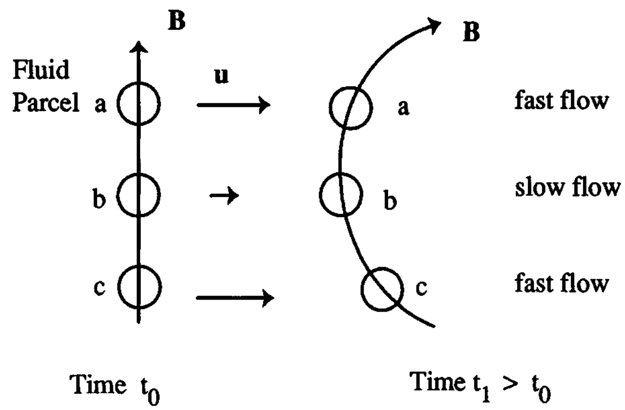

Figure 4.4. Schematic of frozen-in magnetic field. Field line stays attached to particular fluid parcels.

Magnetic flux is frozen into the plasma flow for very large values of the magnetic Reynolds number ($R_m \to \infty$). What is meant by "frozen-in magnetic flux"? The convection–diffusion equation for $D_B = 0$ simply becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{4.76}$$

If Equation (4.76) holds, then the following theorem also holds:

Theorem 4.1 *The magnetic flux through a closed loop within the (infinite conductivity) fluid, and moving with the fluid, remains constant over time.*

The proof of this theorem is not given here but can be found in many plasma physics books such as are listed in the bibliography at end of this chapter (e.g., Siscoe, 1982).

Consider the schematic example shown in Figure 4.4. A field line (or part of a field line) can be thought of as being "tied" to a particular parcel of fluid if $R_m \gg 1$. The field line gets distorted and stretched as the faster fluid with its piece of the field line outruns the slower fluid with its piece of the field line. The concepts of magnetic convection and diffusion can also be illustrated for a one-dimensional geometry, as we shall do in the next section.

4.4.3 One-dimensional convection–diffusion equation

A one-dimensional version of the magnetic convection–diffusion equation (4.72) is easier to understand than the full equation. Assume that the magnetic field is only in the x direction, $\mathbf{B} = B(z, t)\hat{\mathbf{x}}$, and that the flow is only in the z direction with a velocity $\mathbf{u} = u(z)\hat{\mathbf{z}}$ (see Figure 4.5).

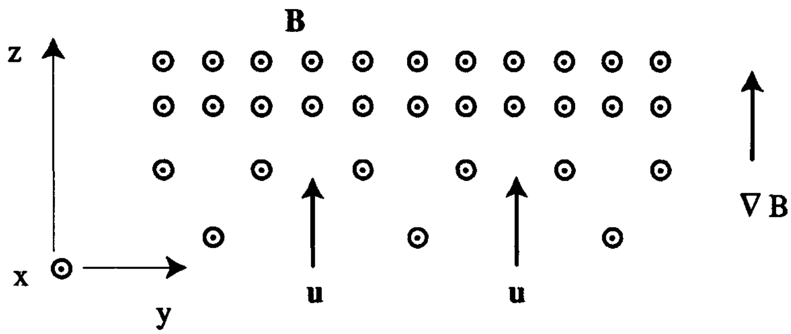

Figure 4.5. Simple one-dimensional geometry for magnetic convection and diffusion.

With this choice of coordinates, the convection–diffusion Equation (4.72) becomes

$$\frac{\partial B(z,t)}{\partial t} = -\frac{\partial}{\partial z}(uB) + D_B \frac{\partial^2 B}{\partial z^2}. \tag{4.77}$$

For $u = 0$ (and for $R_m = 0$), we have purely one-dimensional magnetic diffusion. A localized magnetic field enhancement with scale size L decays with the time constant given by Equation (4.74). In Problem 4.6, you are asked to solve Equation (4.77) (with $\mathbf{u} = 0$) and find the time evolution of B for a given initial profile. You will find that the small-scale variations of B decay most rapidly, as expected from Equation (4.74).

Now let us assume that $R_m \gg 1$, in which case we can neglect the magnetic diffusion term in Equation (4.77). Equation (4.77) then tells us that for a plasma where uB increases with z, the magnetic field at a fixed value of z decreases with time. In Figure 4.6 a steady-state scenario is shown. For this scenario, the plasma slows down and then speeds up as it moves from left to right. The steady-state ($\partial B/\partial t = 0$) solution of Equation (4.77) is obviously

$$u(z)B(z) = constant. \tag{4.78}$$

As the plasma slows down (smaller values of u), the frozen-in field lines "pile up" and B becomes larger. As the plasma speeds up (u increases), the field lines "spread out" and B becomes smaller. A traffic jam on a highway is analogous – as the speed of the traffic slows, cars pile up.

If the resistivity of a plasma is nonzero (i.e., $D_B \neq 0$), then magnetic flux can slip or "thaw." That is, field lines are no longer tightly tied to, or frozen to, particular fluid parcels. In this case, magnetic field enhancements tend to decrease due to diffusion of magnetic flux away from the enhancement. Equivalently, the field enhancement shown in Figure 4.5 is actually created by an electrical current in the y direction. Magnetic diffusion means that the electrical current undergoes Ohmic dissipation.

You should be aware of a complication that occurs for many space plasmas. The collisional resistivity in the solar wind and magnetosphere (but not in the

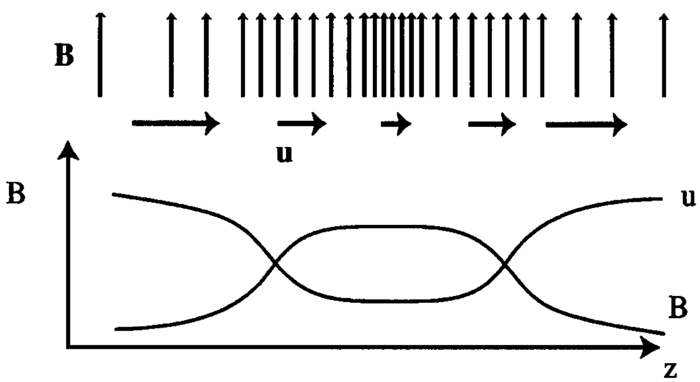

Figure 4.6. Schematic showing frozen-in magnetic field for one-dimensional plasma flow. A steady-state solution, $uB = constant$, is shown.

ionosphere) is virtually zero. And indeed we find that $R_{\mathrm{m}} \gg 1$ *almost* everywhere in these plasma environments. However, near narrow current layers a phenomenon called *magnetic reconnection* takes place that requires $R_{\mathrm{m}} \leq 1$ over a narrow region. However, the magnetic reconnection regions studied in space physics have turned out to be much broader than one would expect using the collisional form of the resistivity (i.e., Equation (4.70)). It seems that some *anomalous resistivity* is required and is thought to originate from microscopic (but still collisionless) plasma processes. A discussion of these processes is outside the scope of this text, but we can still use the convection–diffusion equation, as well as address the phenomenon of magnetic reconnection, by adopting suitable anomalous resistivity coefficients. We discuss magnetic reconnection at the end of this chapter and in Chapter 8.

4.5 The magnetohydrodynamic equations

Let us write again the single-fluid equations (Equations (4.57) and (4.58)) but include the diffusion–convection equation for the magnetic field:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = m_i S_i, \tag{4.79}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g} - \rho \nu (\mathbf{u} - \mathbf{u}_n) - P_i m_i (\mathbf{u} - \mathbf{u}_n), \quad \text{with } p = p_e + p_i, \tag{4.80}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (D_B \nabla \times \mathbf{B}). \tag{4.81}$$

Alternatively, in place of (4.81) we could use Equation (4.60):

$$\nabla \cdot \mathbf{J} = 0. \tag{4.82}$$

Generally, we also need separate energy relations for electrons and ions, such as those given in Chapter 2. However, for now we merely write the polytropic relation (see Equation 2.81) in terms of some reference density and pressure, ρ_0 and p_0, respectively:

$$p/p_0 = (\rho/\rho_0)^\gamma, \tag{4.83}$$

where $\gamma = 5/3$ for an ideal monatomic gas and $\gamma = 1$ for an isothermal gas.

Equations (4.79)–(4.83) together comprise the *equations of magnetohydrodynamics* (i.e., *the MHD equations*). These equations can be used in many ways and with many different approximations. Next we introduce the concept of magnetic pressure.

4.5.1 Magnetic pressure

The term $\mathbf{J} \times \mathbf{B}$ in Equation (4.80) is the *"Maxwell" force (per unit volume)* on a magnetized plasma due to electrical currents. You might recall from your elementary physics course that the force on a length l of a straight wire, carrying current I, in magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = I\mathbf{l} \times \mathbf{B} \quad [N], \tag{4.84}$$

where the direction of $\mathbf{l}$ is the same as that of the current. This is just the "electric motor" force. $\mathbf{J} \times \mathbf{B}$ is the analogous force per unit volume on a plasma.

The current density $\mathbf{J}$ can be eliminated from $\mathbf{J} \times \mathbf{B}$ by using Ampère's law (minus the displacement current):

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}.$$

Thus

$$\begin{aligned}\mathbf{J} \times \mathbf{B} &= \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\mathbf{B} \cdot \nabla\mathbf{B} \end{aligned} \tag{4.85}$$

where a vector calculus identity, which can be found in most electromagnetics textbooks, was used in the final step. Using Equation (4.85) we can rewrite the momentum equation (4.80) as

$$\begin{aligned}\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u} \cdot \nabla\mathbf{u} &= -\nabla\left(p + B^2/2\mu_0\right) + \frac{1}{\mu_0}\mathbf{B} \cdot \nabla\mathbf{B} \\ &\quad + \rho\mathbf{g} - \rho\nu(\mathbf{u} - \mathbf{u}_n) - P_i m_i(\mathbf{u} - \mathbf{u}_n). \end{aligned} \tag{4.86}$$

It is apparent that the quantity $B^2/2\mu_0$ acts on the fluid in the same manner as the thermal pressure p; hence, this quantity is called *magnetic pressure*:

$$p_B = \frac{B^2}{2\mu_0} \quad \left[\text{units of } \frac{\text{N}}{\text{m}^2}\right]. \tag{4.87a}$$

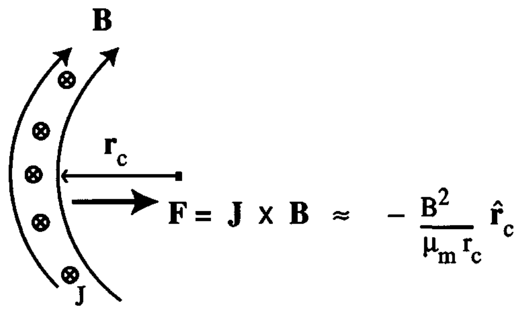

Figure 4.7. Tension force **F** (per unit mass) on palsma due to curved magnetic field lines. Curved field lines imply a current density **J**, as shown, due to Ampère's law. $\mathbf{J} \times \mathbf{B}$ is in the same direction as $-\hat{\mathbf{r}}_c$.

In cgs units the magnetic pressure takes the form

$$p_B = \frac{B^2}{8\pi} \quad [\text{dynes/cm}^2]. \tag{4.87b}$$

B has units of tesla in Equation (4.87a) and Gauss in (4.87b). A gradient in magnetic pressure gives rise to a force on the plasma. The magnetic pressure gradient requires a spatial variation of the magnitude of **B**, with the force pointing from the high-field region to the low-field region. The concept of magnetic pressure will become more clear to you shortly when we consider static force equilibrium for a magnetized plasma. The importance of the thermal pressure term relative to the magnetic pressure term in Equation (4.86) can be approximately judged by the ratio of these two terms, which is given the name of *plasma beta*:

$$\beta \equiv \frac{p}{p_B} = \frac{n_e k_B (T_e + T_i)}{B^2/2\mu_0}. \tag{4.88}$$

The second part of $\mathbf{J} \times \mathbf{B}$ as given by Equation (4.85) (or in Equation (4.86)) represents the force on the plasma due to curvature of field lines. This is the *magnetic tension force*, which we can write very approximately as

$$\frac{1}{\mu_0}\mathbf{B} \cdot \nabla \mathbf{B} \approx \frac{1}{\mu_0}\frac{B^2}{r_c}(-\hat{\mathbf{r}}_c), \tag{4.89}$$

where $\mathbf{r}_c$ is the radius of curvature of the field lines; see Figure 4.7. An analogy can be made to the force on a curved string under *tension*.

4.6 Static equilibrium

An important subclass of problems is one in which the plasma is stationary, or almost stationary: $\mathbf{u} = \mathbf{0}$. In this case, we can greatly simplify the momentum equations (4.80), or (4.86), and obtain the following *static force balance relation*:

$$\mathbf{0} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}. \tag{4.90}$$

We have further assumed that the neutral gas (if it exists) has zero velocity ($\mathbf{u}_n = \mathbf{0}$). Even Equation (4.90) can be further simplified using various approximations.

4.6.1 Hydrostatic balance

We first take the special case of an unmagnetized plasma ($\mathbf{B} = \mathbf{0}$), in which case Equation (4.90) becomes

$$\nabla p = \rho \mathbf{g}. \tag{4.91}$$

This is the *(hydro)static force balance relation*. This relation is especially useful for horizontally stratified atmospheres (e.g., the planets and the Sun), for which the acceleration due to gravity (downward) can be written as $\mathbf{g} = -g\hat{\mathbf{z}}$. We have $g = 9.88\,\mathrm{m/s^2}$ for the Earth. In this case, all quantities are functions only of z (altitude), and $\hat{\mathbf{z}}$ is a unit vector directed up. An equation of state can be used to express the pressure p in terms of the density ρ and temperature T: $p = \rho\tilde{R}T$. $\tilde{R}$ is the gas constant appropriate for the particular fluid/plasma under consideration. Equation (4.91) becomes

$$\begin{aligned}\frac{\partial p}{\partial z} &= -\rho g \\ &= -\frac{g}{\tilde{R}T}p.\end{aligned} \tag{4.92}$$

The solution of (4.92) is simply

$$p(z) = p_0 \exp\left\{-\int_{z_0}^{z} \frac{g\,dz'}{\tilde{R}T(z')}\right\}, \tag{4.93}$$

where p_0 and z_0 are reference values of pressure and altitude, respectively. For an isothermal atmosphere (T = constant) and constant g (true for a limited altitude range), Equation (4.93) becomes

$$p(z) = p_0 e^{-(z-z_0)/H} \tag{4.94}$$

with *scale height*

$$H \equiv \frac{\tilde{R}T}{g}. \tag{4.95}$$

The pressure decreases with height in an exponential manner with an e-folding length called the *scale height* H. For an isothermal atmosphere, the density ρ is directly proportional to p and also decreases with increasing altitude in an exponential manner.

Example 4.2 (Hydrostatic balance in an isothermal neutral atmosphere) Consider a neutral atmosphere with uniform temperature $T = T_n$ and gas constant

$\tilde{R} = k_B/\bar{m}$, where $\bar{m}$ is the mean molecular mass. For air on Earth, $\tilde{R}$ is the ordinary gas constant and we can write Equation (4.95) to obtain the *neutral scale height*:

$$H_n = \frac{k_B T_n}{\bar{m} g}. \tag{4.96}$$

H_n obviously depends on the parameters T_n, $\bar{m}$, and g. For Earth, $T_n \cong 300$ K, $g = 9.88$ m/s^2, and $\bar{m} = 28.8$ amu (20% O_2 and 80% N_2), in which case

$$H_n = 8.7\,\text{km}.$$

For Jupiter, near the cloudtops, $T_n \cong 300$ K, $\bar{m} \cong 2$ amu (mostly H_2), and $g = 23$ m/s^2. The scale height in the Jovian atmosphere is then

$$H_n \cong 52\,\text{km}.$$

The pressure for a plasma must include both the electron and ion partial pressures: $p = p_e + p_i = n_e k_B(T_e + T_i)$, where we have used separate equations of state for electrons and ions plus we have assumed quasi-neutrality. Equations (4.92)–(4.95) still apply if we identify T as $(T_e + T_i)$ and use $\tilde{R} = 2k_B/(m_e + m_i)$. For isothermal electron and ion temperatures, we can easily show (see Problem 4.7) that for a horizontally stratified plasma near a planet (that is, for an ionosphere) the electron density varies with altitude exponentially:

$$n_e(z) = n_{e0} e^{-(z - z_0)/H_p} \tag{4.97}$$

with *plasma scale height*

$$H_p \cong \frac{k_B(T_e + T_i)}{m_i g}. \tag{4.98}$$

Note that this plasma scale height expression includes both the electron and ion temperatures, unlike the neutral scale height, which only included T_n. The presence of the electron temperature term is really due to the electron pressure gradient term in the generalized Ohm's law, Equation (4.45), that was used to eliminate **E** from the momentum equation.

Figure 4.8 shows an electron density profile measured in the mid-latitude terrestrial ionosphere (see the discussion in Rees, 1989). Hydrostatic equilibrium only applies for altitudes above about 300 km, where a maximum exists in the electron density profile. This ionospheric region above the maximum is called the *topside ionosphere*. In the lower ionosphere, on the other hand, chemistry is the controlling process. Note that the topside electron density has an exponential fall-off. The major ion species in the ionospheric "F-region" is O^+($m = 16$ amu). The temperatures in the ionosphere are $T_e \approx 3{,}500$ K and $T_i \approx 1{,}500$ K; hence we can calculate from

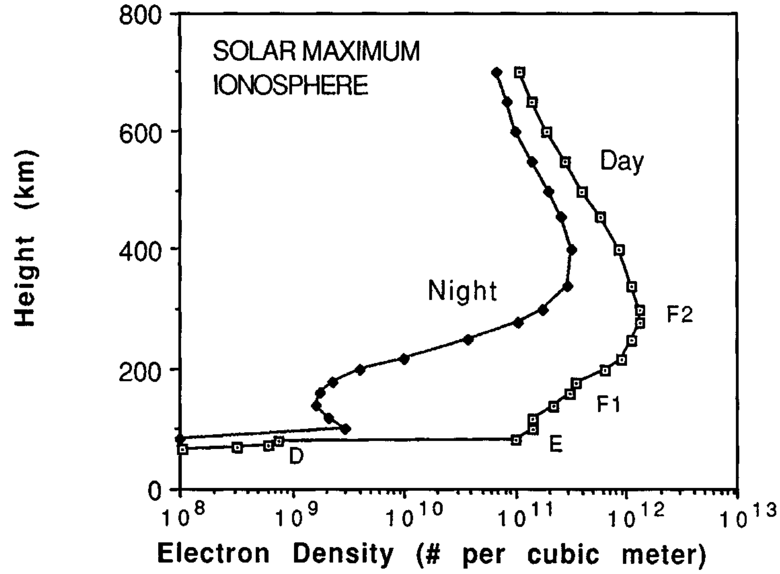

Figure 4.8. Electron density versus altitude in the terrestrial F-region ionosphere versus for solar maximum conditions. Typical day and night profiles are shown. The locations of the D, E, F1, and F2 regions are indicated. Note that the standard convention in the atmospheric sciences is to plot the dependent variable as the abscissa and altitude as the ordinate.

Equation (4.98) that the plasma scale height is

$$H_\mathrm{p} \cong 260\,\mathrm{km}.$$

4.6.2 Static pressure balance in a planar geometry for a magnetized plasma

Now we consider another approximation to the static force balance relation (4.90). We neglect gravity ($g = 0$), so that the equation simply becomes $\mathbf{J} \times \mathbf{B} = \nabla p$. Furthermore, we assume a planar magnetic geometry (i.e., straight magnetic field lines). In this case, we can omit the second term on the right-hand side of expression (4.85) for $\mathbf{J} \times \mathbf{B}$, so that Equation (4.90) becomes

$$\nabla(p + B^2/2\mu_0) = 0, \tag{4.99}$$

which can be integrated to give

$$p + B^2/2\mu_0 = \mathit{constant} \text{ in a direction normal to } \mathbf{B} \tag{4.100}$$

or

$$p_e + p_i + p_B = p_\mathrm{total} = \mathit{constant}.$$

$p_B = B^2/2\mu_0$ is obviously acting like pressure here. Equation (4.100) simply states that the total pressure remains constant; any increase in magnetic pressure must be

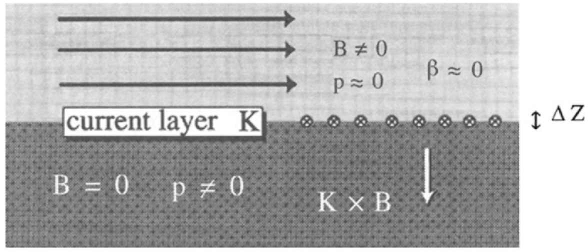

Figure 4.9. Static balance between thermal and magnetic pressure.

compensated for by a decrease in the thermal pressure or vice versa. Consider the following simple illustration of a static pressure balance at an interface between a magnetized, but very low density, plasma and a high-density unmagnetized plasma (Figure 4.9).

To satisfy Equation (4.100), the following condition must apply at the interface between the two media:

$$p = \frac{B^2}{2\mu_0}, \tag{4.101}$$

where p is the thermal pressure in the bottom layer and B is the field strength in the upper layer. However, you should not overlook that what is "really" balancing the thermal pressure at the interface is the $\mathbf{J} \times \mathbf{B}$ force integrated over the narrow extent of the interface. A narrow layer of electrical current, or a *current layer*, is located at the interface. Its current density (per unit length), given by Ampère's law, is

$$K = \int_{\Delta z} J dz = B/\mu_0, \tag{4.102}$$

where Δz is the thickness of the current layer. This current is called the *diamagnetic current*. The "magnetic" (i.e., "electric motor") force per unit area (or pressure) is given by

$$p_B = \int_{\Delta z} (\mathbf{J} \times \mathbf{B}) \cdot \hat{\mathbf{z}}\, dz = \left|\mathbf{K} \times \langle \mathbf{B} \rangle_{\Delta z}\right| = \frac{B}{\mu_0} \cdot \frac{B}{2} = \frac{B^2}{2\mu_0}. \tag{4.103}$$

This magnetic pressure must equal the thermal pressure p in order to keep the interface stationary (or static).

Example 4.3 (The Venus ionopause) The Venus ionopause provides a nice example of a static pressure balance between magnetic and thermal pressures. Figure 4.10 shows the magnetic field strength and the plasma density as functions of height above the surface of Venus for three orbital passes of the *Pioneer Venus Orbiter (PVO)*. *PVO* was launched by NASA and went into orbit around Venus in 1978. The orbit was highly elliptical and so during an orbital pass the ionosphere

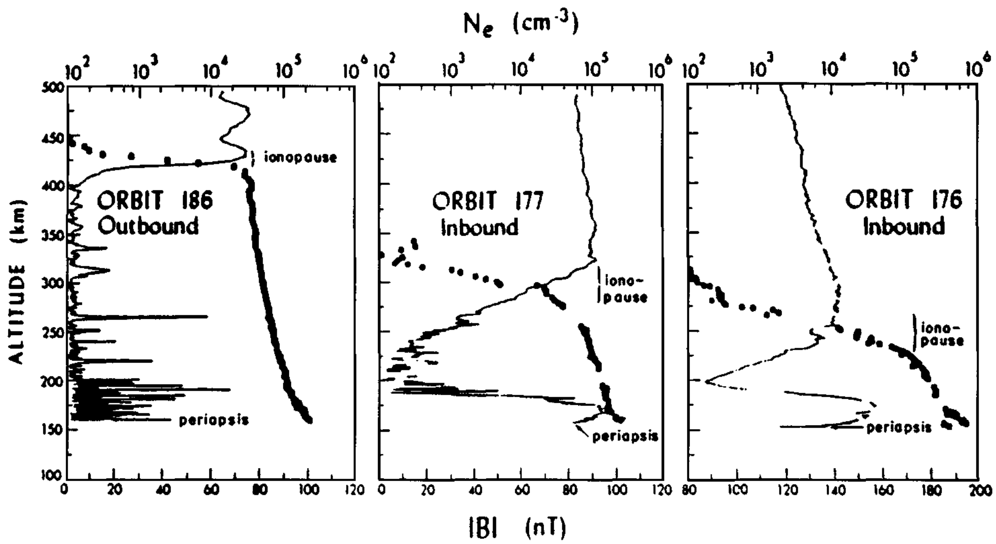

Figure 4.10. Measured magnetic field strength (*Pioneer Venus Orbiter* magnetometer – Russell and Vaisberg, 1983) and measured electron density (Langmuir probe) in the ionosphere of Venus for three orbits. The magnetic field is shown as a solid line and uses the bottom scale. The electron densities are shown as solid dots and use the upper scale. (From Russell and Vaisberg, 1983.)

was sampled above the *periapsis* (i.e., minimum) altitude, which was typically 145 km. Instruments on board measured plasma properties including electron and ion densities and temperatures as well as the magnetic field (Russell and Vaisberg, 1983). The *ionopause* is defined as the region where the ionospheric plasma stops, which is where the solar wind plasma begins.

The plasma just above the ionopause has very low density (and pressure) but is highly magnetized due to the compression of the interplanetary magnetic field. During Orbit 186 the solar wind "pressure" was low so that the magnetic field strength above the ionopause was only about 70 nT. In this case, the magnetic field strength in the ionosphere itself is essentially zero, on the average. However, for Orbits 176 and 177 the solar wind pressure was high, as was the magnetic field strength above the ionopause. Orbits 176 and 177 will not be discussed here, but we will return to them in Chapter 7. The solar wind interaction with Venus is briefly discussed in Chapter 7; what concerns us here is that a pressure balance between thermal and magnetic pressure has been experimentally demonstrated to exist at the Venus ionopause. You can also notice in Figure 4.10 that the ionospheric density (and the pressure, because the electron and ion temperatures are approximately independent of altitude) falls off exponentially as expected from hydrostatic equilibrium, except in the ionopause region.

Now we calculate for Orbit 186 the magnetic pressure just above the ionopause and the thermal pressure just below the ionopause and show that these are equal to each other – that is, we will now demonstrate that Equation (4.101) is satisfied. First, note that above the ionopause $B = 70$ nT so that the magnetic pressure is $B^2/2\mu_0 = 2 \times 10^{-9}$ N/m^2. Next note that the electron density in the ionosphere, just below the ionopause, is $n_e = 3 \times 10^4$ cm^{-3}. The measured electron and ion temperatures are $T_e = 3{,}800$ K and $T_i = 1{,}800$ K, respectively. We find that the thermal pressure is $p = n_e k_B (T_e + T_i) = 2.2 \times 10^{-9}$ N/m^2, which is equal to the magnetic pressure to within 10% (roughly the error of our estimates). Equation (4.101) is indeed satisfied at the Venus ionopause.

4.6.3 Force-free magnetic equilibrium

Now we consider yet another case of a static force balance in which we suppose that $\nabla p \approx 0$ and $g = 0$. Then the static force balance relation, (4.90), is simply

$$\mathbf{J} \times \mathbf{B} = \mathbf{0}. \tag{4.104}$$

This deceptively simple equation can be satisfied in one of two ways: (1) $\mathbf{J} = \mathbf{0}$, but this is not a particularly interesting case, and (2) $\mathbf{J} \parallel \mathbf{B}$; that is, $\mathbf{J}$ is everywhere parallel to $\mathbf{B}$, which results in a *force-free* magnetic structure. A force-free structure exists where the magnetic pressure gradient force part of $\mathbf{J} \times \mathbf{B}$ exactly counterbalances the tension force part of $\mathbf{J} \times \mathbf{B}$ (see Equation (4.85)).

Especially interesting are cylindrically symmetric force-free structures with magnetic field vector $\mathbf{B} = \mathbf{B}(r)$, where r is the radial distance from the axis. For this functional form of the magnetic field, the force relation $\mathbf{J} \times \mathbf{B} = \mathbf{0}$ is satisfied only if $\mathbf{J}$ is proportional to $\mathbf{B}$ multiplied by some scalar function of r; that is, the following relation must be satisfied:

$$\nabla \times \mathbf{B} = \alpha(r)\mathbf{B}, \tag{4.105}$$

where $\alpha(r)$ is some function only of r. In Problem 4.11 you are asked to find $\mathbf{B}(r)$ for the special case of $\alpha = constant$.

Force-free (or almost force-free) magnetic structures are present in many space plasma environments, such as in the solar corona, in the solar wind, and in the ionosphere of Venus. Appearing in the measured magnetic field profile in the ionosphere of Venus for Orbit 186 (Figure 4.10) are narrow spikes that are about 10 km across. A detailed analysis of the magnetic field vector in one of these spikes reveals a ropelike structure, as illustrated in Figure 4.11 (Russell and Elphic, 1979; Elphic et al., 1980). These structures have been given the name *magnetic flux ropes*. Analysis indicates that these ropes are essentially force free.

4.6.4 Stability

We have just finished a discussion of static equilibria. However, not all equilibria are stable. A circus performer balanced on a high wire is in a state of equilibrium,

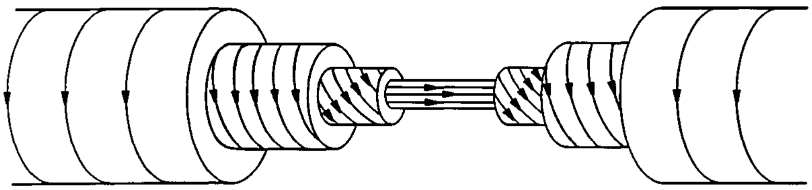

Figure 4.11. Magnetic flux rope structure as deduced from *Pioneer Venus Orbiter* magnetometer measurements. The flux rope is dissected so that its internal structure is visible. Reprinted with permission from *Nature* (Russell and Elphic, *Nature*, **279**, 616, 1979). Copyright (1979) Macmillan Magazines Limited.

albeit an unstable one. A slight unfortunate move to the left or right and (barring other action) the performer is no longer in a "static" equilibrium. Consider a ball in the bottom of a bowl; it is in a *stable* mechanical equilibrium, in that a slight departure from the point of equilibrium results in a restoring force (i.e., gravity in this example) that maintains equilibrium. In contrast, a ball balanced on the top of an overturned bowl is in an *unstable* equilibrium; a slight perturbation from the equilibrium configuration results in an acceleration away from the equilibrium. *Neutral stability* is also possible.

MHD equilibrium states can be stable or unstable. Let us consider the current layer configuration pictured in Figure 4.9. Without gravity, this configuration is neutrally stable. But suppose we introduce gravity. The equilibrium is stable because the low-β and low-density plasma is on top and the high-β, high-density plasma is on the bottom. The stability condition can be verified by displacing a small parcel of plasma from near the interface and determining whether the parcel seeks to return to its equilibrium position (i.e., stable) or not (i.e., unstable). If the gravitational acceleration $\mathbf{g}$ is directed downward in Figure 4.9, a parcel of nonmagnetized dense fluid that is moved up into the low-β, low-density region is denser than its new surroundings and, consequently, has negative buoyancy (see Problem 4.8 for a discussion of buoyancy). The force on this parcel is downward, thus restoring it to its original equilibrium position. Similarly, a parcel of magnetized plasma that is moved downward into the high-β region becomes less dense than the medium surrounding it and buoyancy results in an upward restoring force.

Suppose though that the denser fluid lies on top of the less dense fluid (i.e., invert the two regions in Figure 4.9 but keep $\mathbf{g}$ directed down). Then an upward (downward) displacement of a fluid parcel into the neighboring region results in positive (negative) buoyancy that leads to an acceleration of the parcel away from the equilibrium position (see Figure 4.12). This instability is called the *Rayleigh–Taylor instability* and it occurs whenever a lighter fluid supports a heavier fluid in a gravitational field. An example of a situation that is Rayleigh–Taylor unstable is a layer of oil supported by a layer of water. You can demonstrate this yourself with a

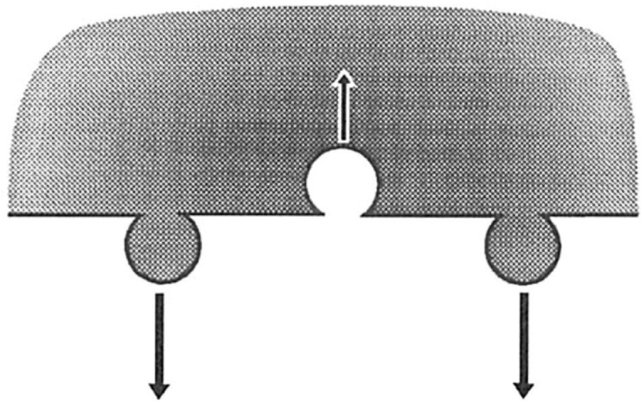

Figure 4.12. Rayleigh–Taylor instability of a heavy fluid supported by a lighter fluid.

container of oil-and-vinegar salad dressing. The Venus ionopause is stable against the Rayleigh–Taylor instability.

Other types of fluid instabilities exist, such as *streaming instabilities*, in whch plasma drifts can upset the equilibrium configuration of a plasma. An important example of a streaming instability is the *Kelvin–Helmholtz instability*. Without gravity, the diamagnetic boundary shown in Figure 4.9 is neutrally stable. But suppose the plasma on either side of the interface is drifting tangentially at different speeds; that is, suppose a *velocity shear* is present at the boundary. In this case, ripples that might be present at the boundary could grow via the Kelvin–Helmholtz instability.

The Rayleigh–Taylor and Kelvin–Helmholtz instabilities are examples of macroscopic, or fluid (or MHD), instabilities, in which the distribution functions f are Maxwellians or drifting Maxwellians. Another category of instabilities comprises *kinetic instabilities*, in which certain kinds of deviations of the distribution function from a Maxwellian distribution result in the growth of plasma perturbations or waves. This topic will not be addressed in this book (although it was briefly alluded to at the end of Chapter 3) and is left to more advanced books on plasma physics and space plasmas.

4.6.5 Diffusion in a partially ionized plasma

For a partially ionized plasma, we can simplify the momentum equation (4.86) (or, equivalently, Equation 4.80) by neglecting the left-hand side (i.e., the inertial terms). This approximation is not as extreme as the static equilibrium approximation made earlier in Section 4.6, because the friction and mass-loading terms, which contain the flow velocity **u**, are still being retained. The now simplified momentum equation can be algebraically manipulated to yield the following explicit expression for the plasma flow velocity:

$$\mathbf{u} = \mathbf{u}_n + \frac{1}{\rho\nu + P_i m_i}(-\nabla p + \mathbf{J} \times \mathbf{B} + \rho\mathbf{g}). \qquad (4.106)$$

Equation (4.106) in effect states that the ion–neutral friction force is in balance with the sum of all other forces on the plasma, including Maxwell stresses, thermal pressure gradient, and gravity.

Obviously, Equation (4.106) is applicable only to plasma environments in which the abundance of neutrals is sufficiently high such that a reasonably large friction term ($\rho\nu$) or mass-loading term ($P_i m_i$) exists. Furthermore, Equation (4.106) is also applicable only if the flow velocity ($\mathbf{u}$) is small enough such that the left-hand side of Equation (4.80) (i.e., the "inertial terms") is much smaller than the largest individual term on the right-hand side. Note that if $\mathbf{u} = \mathbf{u}_n$, then the "static" equilibrium expressed by Equation (4.90) is obtained. Often, $\mathbf{u}_n \approx 0$ is a good assumption because neutral flow velocities are almost always slow in comparison with typical plasma velocities.

Let us further simplify Equation (4.106) by (1) using Equations (4.85)–(4.87b) plus a planar geometry to convert $\mathbf{J} \times \mathbf{B}$ to $-\nabla p_B$, (2) assuming $P_i m_i \ll \rho\nu$ (neglecting mass-loading), and (3) assuming $\mathbf{u}_n = 0$ (stationary neutrals). We obtain

$$\mathbf{u} = -\frac{1}{\rho\nu}(\nabla p_{\mathrm{tot}} - \rho\mathbf{g}), \qquad (4.107)$$

where $p_{\mathrm{tot}} = p_B + p = B^2/2\mu_0 + n_e k_{\mathrm{B}}(T_e + T_i)$. Equation (4.107) is one version of the *plasma diffusion equation*, and it is called the *ambipolar diffusion equation*. This equation states that plasma flows in response to gradients in the total pressure (plus gravity).

The diffusion equation is especially useful for determining the plasma flow in planetary ionospheres. In particular, we can easily convert Equation (4.107) to a form that is useful for plasma transport along a strong external magnetic field, such as is present in the terrestrial ionosphere. First, we assume that $\mathbf{g} = -g\hat{\mathbf{z}}$ (as in Section 4.6.1), where z is altitude. The magnetic field, $\mathbf{B} = B\hat{\mathbf{b}}$, is almost uniform within the relatively narrow ionosphere layer. The angle of inclination of the magnetic field with respect to the horizontal plane, θ_{I}, is given by (see Figure 4.13)

$$\sin\theta_{\mathrm{I}} = +\hat{\mathbf{b}} \cdot \hat{\mathbf{z}}. \qquad (4.108)$$

Starting from Equation (4.107), we can derive an equation for the plasma flow speed (or diffusion velocity) along the magnetic field:

$$u_{\parallel} = -\frac{1}{m_i n_e \nu}\left\{\frac{\partial}{\partial s}[n_e k_{\mathrm{B}}(T_e + T_i)] + n_e g m_i \sin\theta_{\mathrm{I}}\right\}. \qquad (4.109)$$

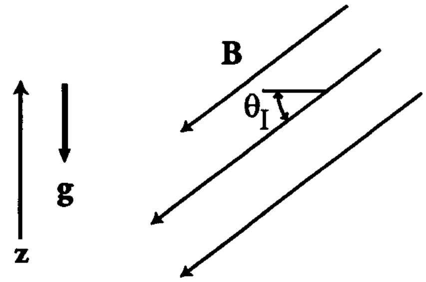

Figure 4.13. Plasma diffusion along a strong magnetic field in a planetary ionosphere. $u_{\|}$ is the plasma flow speed along **B**, and s is the distance along **B**.

A single ion species with mass m_i has been chosen for simplicity. Distance along the magnetic field is denoted s. The component of $\mathbf{u} = u_{\|}\hat{\mathbf{b}}$ in the z direction is the most important flow component for a stratified plasma medium such as a planetary ionosphere. Equation (4.109) can also be expressed (Problem 4.12) as

$$u_z = -D_a \sin^2\theta_I \left[\frac{1}{n_e}\frac{\partial n_e}{\partial z} + \frac{1}{H_p} + \frac{1}{(T_e + T_i)}\frac{\partial}{\partial z}(T_e + T_i)\right], \tag{4.110}$$

where the *ambipolar diffusion coefficient* D_a is given by

$$D_a = \frac{k_B(T_e + T_i)}{m_i \nu} \tag{4.111}$$

and where H_p is the plasma scale height, given by Equation (4.98).

We can combine Equation (4.110) and a one-dimensional version of the continuity Equation (4.79) to find an expression for the electron density n_e:

$$\frac{\partial n_e}{\partial t} + \frac{\partial \phi_i}{\partial z} = S_i. \tag{4.112}$$

The vertical plasma flux is $\phi_i = n_e u_z$ and S_i is the net plasma source.

Equation (4.112) combined with Equation (4.110) has the form of a standard diffusion equation for an isothermal gas if $\mathbf{g} = \mathbf{0}$ (Problem 4.12).

The ion–neutral collision frequency (ν) is proportional to the atmospheric neutral density, which decreases rapidly with increasing altitude. Consequently, D_a, which is inversely proportional to ν, is small at low altitudes (and u_z is small, even for large density gradients) and large at high altitudes. At high altitudes it is very often true that $|u_z/D_a| \ll 1/H_p$, in which case we can set $u_z = 0$ (this approximation is called *diffusive equilibrium*) and reobtain for an isothermal plasma the hydrostatic profile given by Equation (4.97).

The ambipolar diffusion equation will be discussed again in Chapter 7. Note that diffusion equations for individual ion species in a multi-ion species plasma can be also be found by starting with Equations (2.61) and (2.64), instead of the single-fluid momentum equation, by neglecting the inertial terms (i.e., left-hand side) and by solving for $\mathbf{u}_s$.

4.6.6 Dynamic pressure

The inertial terms in the momentum equation are important for fast flows. We now consider a steady-state ($\partial/\partial t = 0$), collisionless ($\nu = 0$), source-free ($S_i = P_i = 0$) plasma. We also consider flow only in one direction with $\mathbf{u} = u(x)\hat{\mathbf{x}}$ and with the magnetic field normal to the flow direction, $\mathbf{B} \perp \mathbf{u}$. With these assumptions, the x component of the momentum Equation (4.86) becomes

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}(p + B^2/2\mu_0). \tag{4.113}$$

This equation can be further transformed (Problem 4.13) into

$$\frac{\partial}{\partial x}(\rho u^2 + p + B^2/2\mu_0) = 0$$

or

$$\rho u^2 + p + B^2/2\mu_0 = C_1, \tag{4.114}$$

where C_1 is a constant independent of x. That is, $\rho u^2 + p + B^2/2\mu_0$ is constant along a *streamline*.

If the total pressure $p + B^2/2\mu_0$ decreases along a streamline, then ρu^2 must increase as prescribed by Equation (4.113) (and vice versa). Thus, ρu^2 acts like pressure and is given the name *dynamic pressure*. Although the simple expression (4.113) is not strictly applicable to more complicated flow patterns, ρu^2 nonetheless provides a good estimate of the relative importance of the $\rho\mathbf{u} \cdot \nabla\mathbf{u}$ term to other terms in the MHD momentum equation. This term is important if the ratio of ρu^2 to the thermal pressure p is large; this ratio is closely related to the sonic Mach number, which we will discuss in Section 4.8.

4.7 MHD energy relations

Both electron and ion pressures appear in the MHD momentum equation (4.80), and thus a complete treatment of a plasma must include energy relations for both plasma species. Often, in place of a complete energy equation, a simple polytropic relation is used to relate the total thermal pressure p to the density ρ (see Equation 4.83) or to relate the partial pressure of an individual species, p_s, to its density ρ_s (see Equation 2.81). However, sometimes a more accurate treatment of the plasma energetics is required.

The energy equations we saw in Section 2.4.5 can be used separately for electrons and ions. The types of approximations that are reasonable can be quite different for electrons than for ions. For ionospheric plasmas, the heat conduction Equation (2.82) is usually appropriate for both species; even the local heat balance relation (2.84) often provides an adequate description of the energetics in the lower ionosphere (altitudes less than 200 km) where the neutral density is high. Local heat balance is often an especially good assumption for the ions if the ionospheric flow is slow (i.e., subsonic, $u \ll u_{i\,\text{therm}}$). However, for fast flows ($u > u_{i\,\text{therm}}$), it becomes necessary to use a more complete energy equation such as those given by Equations (2.73), (2.78), or (2.79).

An equation for the "total" energy (electron thermal energy and bulk kinetic energy, ion thermal energy and bulk kinetic energy, and electromagnetic field energy) is often convenient for MHD problems where we really don't wish to separate out individual plasma species but would rather just deal with the *total* thermal pressure p.

4.7.1 Combined MHD energy relation

We now combine the energy relation (2.73) as applied to both electrons ($s = e$) and ions ($s = i$). Energy density terms for electromagnetic fields (internally or externally generated) must be included. The total energy density (units of J/m^3) of the plasma is then given by

$$W = \sum_{s=e,i} \rho_s \left[U_s + \frac{1}{2}u_s^2 + U_{\text{grav}} \right] + \frac{1}{2}\varepsilon_0 E^2 + B^2/2\mu_0, \tag{4.115}$$

where you recall from Chapter 2 that the internal energy density for species s is $\rho_s U_s = [1/(\gamma_s - 1)]p_s$, U_{grav} is the gravitational potential, $(1/2)\varepsilon_0 E^2$ is the energy density associated with the electric field, and $B^2/2\mu_0$ is the energy density associated with the magnetic field (as well as being the magnetic pressure). Recognizing that the electron mass density, $\rho_e = m_e n_e$, is much less than the ion mass density, $\rho_i = n_i m_i$, we can write the total plasma energy density as

$$\begin{aligned} W &\cong \frac{1}{\gamma_e - 1} p_e + \frac{1}{\gamma_i - 1} p_i + \frac{1}{2}\rho_i u_i^2 + \rho_i U_{\text{grav}} + \frac{1}{2}\varepsilon_0 E^2 + B^2/2\mu_0 \\ &\cong \frac{1}{\gamma_i - 1} p + \frac{1}{2}\rho u^2 + \rho U_{\text{grav}} + \frac{1}{2}\varepsilon_0 E^2 + B^2/2\mu_0, \end{aligned} \tag{4.116}$$

where $\rho \cong \rho_i$, $p = p_e + p_i$, $\gamma_e = \gamma_i$, and $u \cong u_i$ (see Equations (4.51)–(4.54)).

The first term in the combined energy relation should be $\partial W/\partial t$, where W is given by Equation (4.116). The next term is the combined electron and ion divergence term from Equation (2.73):

$$\begin{aligned} &\nabla \cdot \sum_s \left[\rho_s \mathbf{u}_s \left(h_s + \frac{1}{2}u_s^2 + U_{\text{grav}} \right) \right] \\ &\quad \cong \nabla \cdot \left(\frac{\gamma_e}{\gamma_{e-1}} \mathbf{u}_e p_e + \frac{\gamma_i}{\gamma_{i-1}} \mathbf{u}_i p_i + \frac{1}{2}\mathbf{u}_i \rho u_i^2 + \rho \mathbf{u}_i U_{\text{grav}} \right), \end{aligned} \tag{4.117}$$

where γ_e and γ_i are the ratios of specific heats for electrons and ions, respectively. A further simplification of Equation (4.117) is obtained if we take $\gamma_e = \gamma_i$ (calling it simply γ) and if $\mathbf{u}_e = \mathbf{u}_i = \mathbf{u}$:

$$\nabla \cdot \left\{ \rho \mathbf{u} \left[\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2} u^2 + U_{\text{grav}} \right] \right\}. \tag{4.118}$$

The presence of the electromagnetic energy density in Equation (4.116) also necessitates the inclusion of a transport term for electromagnetic energy (i.e., $\nabla \cdot \mathbf{S}$ must be inlcuded, where $\mathbf{S}$ is the Poynting vector) and the inclusion of a source/sink of electromagnetic energy density, $-\mathbf{E} \cdot \mathbf{J}$. The appendix provides a review of this topic. Putting all the parts together, we find the following MHD energy relation:

$$\begin{aligned} &\frac{\partial}{\partial t} \left[\frac{1}{2} \rho u^2 + \frac{p}{\gamma - 1} + \rho U_{\text{grav}} + \frac{1}{2} \varepsilon_0 E^2 + \frac{B^2}{2\mu_0} \right] \\ &\quad + \nabla \cdot \left\{ \rho \mathbf{u} \left[\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2} u^2 + U_{\text{grav}} \right] \right\} + \nabla \cdot \mathbf{S} \\ &\quad + \nabla \cdot (\mathbf{Q}_e + \mathbf{Q}_i) = \left(\frac{\delta E_e}{\delta t} \right)_{\text{coll}} + \left(\frac{\delta E_i}{\delta t} \right)_{\text{coll}} - \mathbf{E} \cdot \mathbf{J}, \end{aligned} \tag{4.119}$$

where $\mathbf{Q}_e$ and $\mathbf{Q}_i$ are conductive heat fluxes for electrons and ions, respectively. Electron and ion collisional energy terms are also included on the right-hand side of (4.119). More details on the ion collision term can be found in Chapter 2.

4.7.2 Bernoulli's equation

A simple energy relation (2.80) was found earlier for the case of steady flow with no collisional sources or sinks of heat and with no heat conduction. An analogous relation can be derived from Equation (4.119) for $\mathbf{u} \perp \mathbf{B}$:

$$\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2} u^2 + U_{\text{grav}} + \frac{B^2}{\mu_0 \rho} = \textit{constant} \text{ along a streamline,} \tag{4.120}$$

where the Poynting vector is approximated with only the motional electric field terms in the generalized Ohm's law. For $\mathbf{u}$ parallel to $\mathbf{B}$ we have the same expression but without the $B^2/\mu_0\rho$ term. Equation (4.120) is *Bernoulli's equation*, which has wide applicability (and is *not* just for one-dimensional flow as is the case for Equation (4.114)). But both Equation (4.114) and Bernoulli's equation (4.120) tell us that as a fluid parcel slows down its pressure (and temperature) increases; or, conversely, as a fluid parcel speeds up, its pressure decreases.

4.8 MHD waves

Earlier in this chapter we considered the topic of Langmuir waves (i.e., plasma oscillations and waves). The frequency for this wave mode was shown to be close to