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Physics of Solar System Plasmas

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Chapter

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Introduction to kinetic theory

A gas consisting of charged particles is called a *plasma*, although the use of the term is often restricted to charged particle gases in which collective phenomena, such as plasma oscillations, are more important than collisional phenomena. Collisions generally involve the short-range interactions of discrete particles, whereas collective phenomena involve large numbers of particles working in unison. The charged particle species in most plasmas are positive ions and negative electrons, although negative ions are also present in the D-region of the terrestrial ionosphere. *Fully ionized plasmas* contain only charged particles, whereas *partially ionized plasmas* also contain neutral gas. The solar wind plasma – that is, the interplanetary medium – is a fully ionized plasma; the ionosphere is a partially ionized plasma. A variety of methods have been developed to describe plasmas. *Kinetic theory* uses particle distribution functions to describe plasmas, whereas *fluid theory* (which includes *magnetohydrodynamics* or MHD) only uses a few macroscopic quantities derived from the full particle distribution functions. Because the subject of kinetic theory is largely outside the scope of an introductory book on space physics, this book will primarily use fluid theory to explain plasma phenomena in the solar system. However, a short introduction to kinetic theory and the derivation from kinetic theory of the fluid equations is provided in this chapter. More detailed treatments of kinetic theory can be found in the references listed at the end of the chapter.

2.1 The Boltzmann and Vlasov equations

Classically, a gas can be described by specifying the position and velocity vectors of each particle in that gas (i.e., all the electrons and ions). Let us denote a particular particle by the index $\alpha = 1, \ldots, N$, where N is the total number of particles in the system (or gas). Typically, $N/2$ of the particles are electrons and $N/2$ are ions. The position vector of particle α is $\mathbf{x}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ and the velocity vector is $\mathbf{v}_\alpha = (v_{x\alpha}, v_{y\alpha}, v_{z\alpha})$, where both $\mathbf{x}_\alpha(t)$ and $\mathbf{v}_\alpha(t)$ are functions of time. Each

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particle α obeys Newton's second law for a classical (nonquantum) system:

$$m_\alpha \frac{d\mathbf{v}_\alpha}{dt} = \mathbf{F}_\alpha \quad \text{and} \quad \frac{d\mathbf{x}_\alpha}{dt} = \mathbf{v}_\alpha, \tag{2.1}$$

where m_α is the mass of particle α and $\mathbf{F}_\alpha$ is the total force on particle α at the location of the particle at time t. The force $\mathbf{F}_\alpha$ includes contributions from gravity as well as electric and magnetic forces. In a plasma, the most important force is the *Lorentz force*,

$$\mathbf{F}_\alpha = q_\alpha(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}), \tag{2.2}$$

where q_α is the charge of particle α, and where the electric field intensity, $\mathbf{E}$, and the magnetic flux density, $\mathbf{B}$, must be evaluated at the time and location of the particle.

Vectors $\mathbf{E}$ and $\mathbf{B}$ in general include both contributions from sources external to the plasma and from sources associated with *all* other particles within the plasma. Let us consider a plasma in which only the internal electrostatic force is important and the force on the particle, α, is just the Coulomb force, $\mathbf{F}_\alpha = q_\alpha \mathbf{E}$. The electric field is evaluated at $\mathbf{x}_\alpha(t)$ and is given by

$$\mathbf{E}(\mathbf{x}_\alpha(t)) = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{q_\beta(\mathbf{x}_\alpha - \mathbf{x}_\beta)}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}. \tag{2.3}$$

The sum in this equation is over all particles other than α itself.

The full particle description requires far too much information ($6N$ numbers at each time t) to be practical for realistic-sized systems where $N > 10^{19}$, even for a small-sized plasma. The chief goal of the field of kinetic theory is to reduce this information to manageable proportions using statistical methods, while preserving the essential information associated with macroscopically observed quantities. A key product of kinetic theory is the Boltzmann equation, which will be discussed below, but which will not be derived here – this being beyond the scope of this book.

2.1.1 Single-particle distribution function

Most of the essential information about a plasma is contained in the *single-particle distribution function*, $f_s(\mathbf{x}, \mathbf{v}, t)$, where the position vector $\mathbf{x}$, the velocity vector $\mathbf{v}$, and time t are all independent variables. A separate distribution function is required for each species of plasma particle, s, such as electrons or ions of a particular mass. Phase space encompasses both ordinary space ($\mathbf{x}$) and velocity space ($\mathbf{v}$) and has six dimensions. Thus, to specify a point in phase space, $(\mathbf{x}, \mathbf{v})$, one requires six quantities (x, y, z, v_x, v_y, v_z). The single-particle distribution function (or just distribution function), $f_s(\mathbf{x}, \mathbf{v}, t)$, is defined as the number of particles per unit volume (of phase space) that are present, at time t, in an infinitesimally small
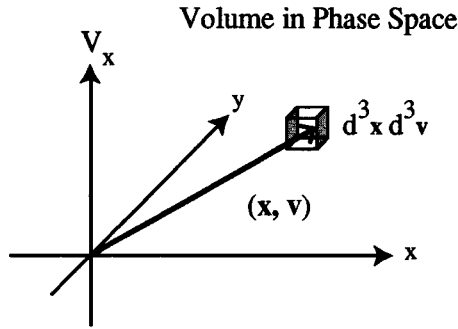
Figure 2.1. Schematic showing part of phase space and a small volume centered at the point $(\mathbf{x}, \mathbf{v}) = (x, y, z, v_x, v_y, v_z)$.

volume of phase space ($\Delta V = \Delta x \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z$), centered at the point in phase space, $(\mathbf{x}, \mathbf{v})$:

$$f_s(\mathbf{x}, \mathbf{v}, t) = \lim_{\Delta V \to 0} \frac{\text{\# particles of type } s \text{ in } \Delta V}{\Delta V}. \tag{2.4}$$

In other words, f_s is the number of type-s particles situated between x and $x + \Delta x$, and also between y and $y + \Delta y$, and z and $z + \Delta z$, and between v_x and $v_x + \Delta v_x$, etc. We are dealing with a statistical distribution here – it does not matter which particles of type s (i.e., which electrons) are in this volume because all these particles are identical. The schematic in Figure 2.1 illustrates the concept of this phase space density. The units of f_s are s³/m⁶.

The integral of $f_s(\mathbf{x}, \mathbf{v}, t)$ over all velocity space yields the number density of particles of type s. The density n_s is a function of position and time:

$$n_s(\mathbf{x}, t) = \int f_s(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{v}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_s(x, y, z, v_x, v_y, v_z, t)\, dv_x\, dv_y\, dv_z. \tag{2.5}$$

The single particle distribution function can be found by solving *the Boltzmann equation*, which can itself be derived from the individual particle picture using the techniques of kinetic theory.

2.1.2 The Boltzmann equation

The Boltzmann equation can be written

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \mathbf{a} \cdot \nabla_{\mathbf{v}} f_s = \left(\frac{\delta f_s}{\delta t}\right)_{\text{collision}}, \tag{2.6}$$

where $\mathbf{a}$ is the acceleration of a particle of type s, located at position $\mathbf{x}$ and possessing velocity $\mathbf{v}$. Note that $\mathbf{a}$ includes the effects of all noncollisional forces on the particles

including gravity. For a plasma, the most important force is the Lorentz force; the acceleration of a particle of species s with charge q_s is given by

$$\mathbf{a}(\mathbf{x}, \mathbf{v}, t) = (q_s/m_s)[\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)]. \qquad (2.7)$$

The electric and magnetic fields in expression (2.7) are *not* the complete fields, such as those given by Equation (2.3), but are *macroscopic*, or average, fields that do not include the microscopic fields associated with discrete particle collisions. These macroscopic fields include long-range average contributions from the plasma particles in a statistical sense. $\mathbf{E}$ and $\mathbf{B}$ can be found using Maxwell's equations; the charge density and current density in these equations are also macroscopic quantities, which can be specified in terms of the single-particle distribution function f_s, as will be discussed later. $\mathbf{E}$ and $\mathbf{B}$ can also include contributions from external sources.

Collisional effects (and the microscopic details of the electromagnetic field) are included in the collision term on the right-hand side of the Boltzmann equation. Coulomb collisions between charged particles involve the long-range electrostatic force, which varies as the inverse square of the particle separation (Equation (2.3)), and this type of interaction requires careful treatment. Formally, the collision term includes the two-particle correlation function, which is proportional to the probability of two particles strongly interacting with each other (Nicholson, 1983; Krall and Trivelpiece, 1973). We will only use simplified versions of the collision term in this book; the reader is referred to the references listed at the end of the chapter for a more complete treatment of the Boltzmann equation and the collision term. An especially simple form of the collision term is the *Krook collision term* (Bhatnagar, Gross, and Krook, 1954), or BGK collision term:

$$\left(\frac{\delta f_s}{\delta t}\right)_{\text{collision}} = -\frac{f_s - f_{s\text{M}}}{\tau_{\text{coll}}}. \qquad (2.8)$$

Here $f_{s\text{M}}$ is the *Maxwell–Boltzmann distribution function* (or *Maxwellian*) for particle species s. τ_{coll} is a collision time – that is, the average time between collisions for a given particle. The BGK collision term is inaccurate but does have the desirable property that a distribution function that is non-Maxwellian at some initial time evolves into a Maxwellian in a period of time of the order of τ_{coll}. The Maxwellian distribution will be described mathematically in a later section. Collisions do indeed have this effect on the distribution function, although the details of this evolution are more complex than is suggested by expression (2.8).

Now let us consider a few aspects of mathematical notation. $\nabla = \hat{\mathbf{x}}\,\partial/\partial x + \hat{\mathbf{y}}\,\partial/\partial y + \hat{\mathbf{z}}\,\partial/\partial z$ is the gradient operator, and $\nabla_{\mathbf{v}} = \hat{\mathbf{x}}\,\partial/\partial v_x + \hat{\mathbf{y}}\,\partial/\partial v_y + \hat{\mathbf{z}}\,\partial/\partial v_z$ is the gradient operator in velocity space. The product $\mathbf{v} \cdot \nabla$ can be also be expressed in component notation as

$$\mathbf{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} = v_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}. \qquad (2.9)$$

All the expressions in Equation (2.9) are equivalent. Note that the index j runs from 1 to 3. For example, $v_1 = v_x$, $v_2 = v_y$, and $v_3 = v_z$. The second to last form uses the summation convention in which the presence of repeated indices (e.g., j) is interpreted as a summation over that index from 1 to 3.

The left-hand side of Equation (2.6) is just equal to the total derivative of f_s in phase space:

$$\frac{D_{\text{ps}} f_s}{D_{\text{ps}} t} = \left(\frac{\delta f_s}{\delta t}\right)_{\text{collision}} \tag{2.10}$$

The Boltzmann equation simply states that the total time derivative of the single-particle distribution function equals the time rate of change of the distribution function due to collisions. If the plasma, or gas, is collisionless, then the right-hand side of Equation (2.10) is zero and the total derivative in phase space is zero. In this case, the phase space density of particles of type s in a small volume of phase space remains constant as this volume moves through phase space on a trajectory specified by the 6-dimensional "vector in phase space" $(\mathbf{v}, \mathbf{a})$. The Boltzmann equation with the collisional term equal to zero is called the *collisionless Boltzmann equation*, or also the *Vlasov equation*:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = 0. \tag{2.11}$$

The Vlasov equation provides the starting point for much of plasma physics. It is a deceptively simple-appearing equation that proves surprisingly difficult to solve for most situations.

2.1.3 The convective derivative

Some insight into the total derivative in phase space can be obtained by studying the total derivative (or *convective derivative*) in ordinary space. Consider a fluid (such as air, water, or a plasma) for which the flow velocity is given by $\mathbf{u}(\mathbf{x})$. $\mathbf{u}(\mathbf{x})$ is the bulk, or average, velocity of a small volume of the fluid (i.e., the "wind" velocity) located at position $\mathbf{x}$. The total derivative of some quantity Q (e.g., density, temperature, etc.) is the time rate of change of Q in a *frame of reference moving with the fluid* (with velocity $\mathbf{u}$). The convective derivative is equal to

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \mathbf{u} \cdot \nabla Q, \tag{2.12}$$

where $Q = Q(\mathbf{x}, t)$ is in general a function of both $\mathbf{x}$ and t.

In one dimension (e.g., $\mathbf{u} = u_x \hat{\mathbf{x}}$) and for a steady-state situation (in which $\partial Q/\partial t = 0$), the total derivative becomes $DQ/Dt = u_x \partial Q/\partial x$. The total time derivative of Q now equals the rate of change of x of a fluid parcel moving with the fluid ($u_x = "dx/dt"$), multiplied by the derivative of Q with respect to x. For example, for a person on a raft floating down a river with speed u_x, the rate of
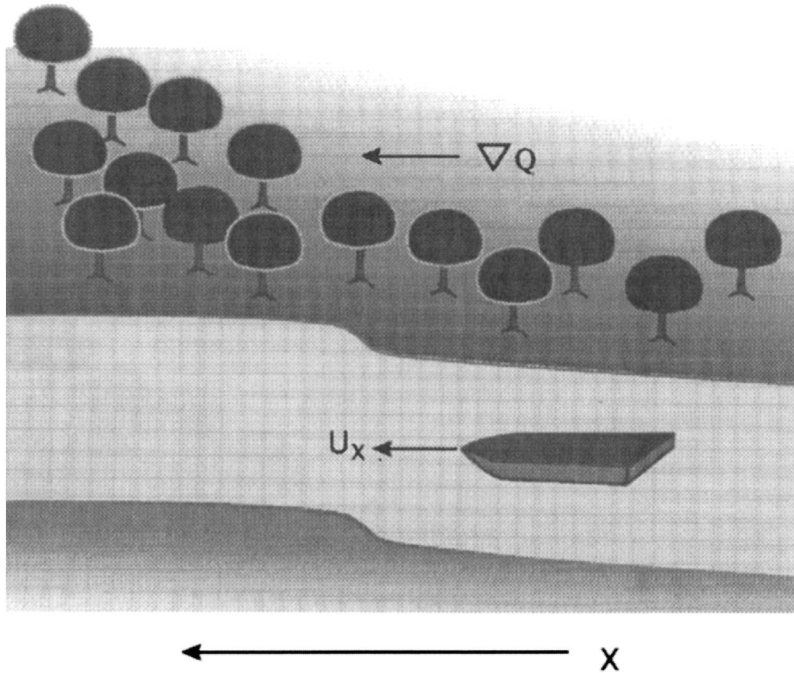
Figure 2.2. A raft floating down a river with speed u_x passes trees on the riverbank. The tree density Q is greater downstream than it is upstream. The raft is carried by the river from a region of low tree density to one of high tree density; that is, the tree density increases with time in the raft's frame of reference.

change – for the rafter – of the density of trees, Q, on the river bank adjacent to the raft is just equal to u_x multiplied by $\partial Q/\partial x$. (See Figure 2.2.) Note that $\partial Q/\partial t = 0$, since trees are immobile in the frame of reference of the Earth (although not in the raft's frame of reference).

2.2 The Maxwell–Boltzmann distribution function

2.2.1 Examples of distribution functions

An infinite number of different distribution functions satisfy the Boltzmann equation. For example, a uniform gas of electrons all moving in the x direction with speed u_{x0} is described by the *beamlike* distribution function,

$$f_e(\mathbf{x}, \mathbf{v}, t) = n_0\delta(v_x - u_{x0})\delta(v_y)\delta(v_z), \qquad (2.13)$$

where $\delta(x)$ is the delta function. The constant n_0 is the electron number density as can be seen by substituting the distribution (2.13) into Equation (2.5) (see Problem 2.1).

The *shell* distribution is another example of a distribution function. For example, consider a uniform gas of ions that move in all directions with equal probability

(i.e., an *istotropic* distribution). All ions in a shell distribution have the same speed $v = |\mathbf{v}| = v_0$, which can thus be written as

$$f_i(\mathbf{x}, \mathbf{v}, t) = \left(n_0/4\pi v_0^2 \right) \delta(v - v_0), \tag{2.14}$$

where n_0 is the ion number density. Although beam and shell distributions are solutions of the Boltzmann equation at a given time, they do not remain beam and shell distributions at later times but eventually evolve into Maxwellian distributions in a time period roughly equal to the collision time.

2.2.2 Maxwellian distribution functions

The Maxwellian distribution for particles of type s is given by the expression

$$f_{s\,\mathrm{M}}(\mathbf{x}, \mathbf{v}, t) = n_s(\mathbf{x}, t) \left(\frac{m_s}{2\pi k_{\mathrm{B}} T_s} \right)^{3/2} \exp\left[-\frac{\frac{1}{2} m_s v^2}{k_{\mathrm{B}} T_s} \right], \tag{2.15}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. The density n_s is a function only of position and time and can be found by integrating $f_{s\mathrm{M}}$ over all velocity space (see Problem 2.2). The temperature of species s is also a function of position and time, $T_s = T_s(\mathbf{x}, t)$. The probability of finding a particle of type s decreases exponentially with increasing v^2 (or, equivalently with increasing kinetic energy) for a Maxwellian. The function $f_{s\mathrm{M}}$ falls off more rapidly for a cold gas (low temperature) than for a hot gas (high temperature) according to Equation (2.15). Note that the Maxwellian distribution is isotropic – that is, $f_{s\mathrm{M}}$ depends only on the magnitude of the velocity vector $\mathbf{v}$ and not on its direction.

A distribution that is closely related to the Maxwellian distribution is the *drifting Maxwellian* distribution for which the gas as a whole moves in some direction with a constant velocity. In this case, the distribution function still looks like the Maxwellian specified by Equation (2.15) but in a frame of reference moving with the uniform velocity $\mathbf{v}_0$. The distribution function for a drifting Maxwellian is given by Equation (2.15), but with v^2 replaced by $|\mathbf{v} - \mathbf{v}_0|^2$.

A reduced, one-dimensional Maxwellian distribution can be obtained by integrating $f_{s\mathrm{M}}$ over all y and z velocity components:

$$\begin{aligned} g_{s\mathrm{M}}(\mathbf{x}, v_x, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{s\,\mathrm{M}}(\mathbf{x}, \mathbf{v}, t) \, dv_y dv_z \\ &= n_s(\mathbf{x}, t) \left(\frac{m_s}{2\pi k_{\mathrm{B}} T_s} \right)^{1/2} \exp\left[-\frac{m_s v_x^2}{2 k_{\mathrm{B}} T_s} \right]. \end{aligned} \tag{2.16}$$

The function $g_{s\mathrm{M}}$ represents the probability of finding a particle of type s with x component of the velocity lying between v_x and $v_x + dv_x$. Like the full Maxwellian,
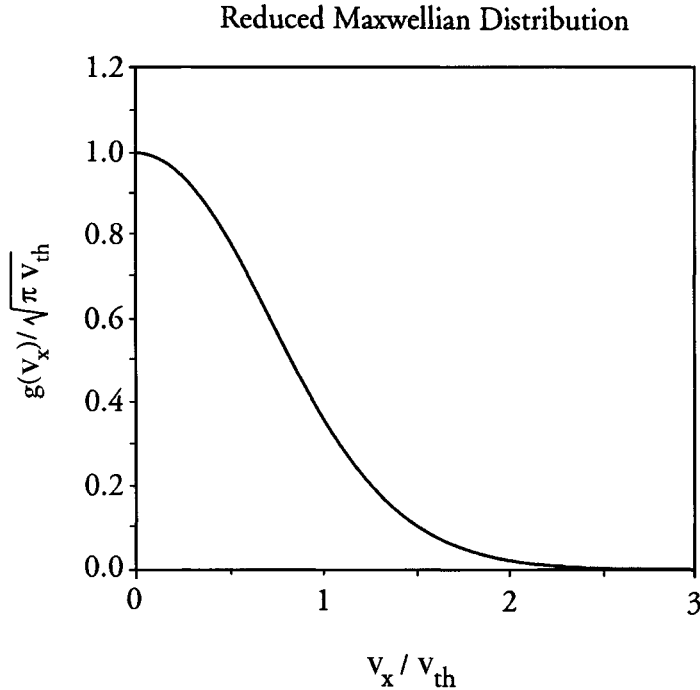
Reduced Maxwellian Distribution

Figure 2.3. Reduced Maxwellian distribution function g_{sM} divided by $n_s[\pi v_{th}]^{1/2}$ plotted versus v_x divided by the thermal speed v_{th}.

g_{sM} is an exponential function (see Figure 2.3). The maximum of g_{sM} is located at $v_x = 0$ and g_{sM} decreases one e-folding as v_x increases from 0 to $v_x = v_{th}$. The thermal speed is defined by

$$v_{th} = [2k_B T_s / m_s]^{1/2}. \tag{2.17}$$

The thermal speed is inversely proportional to the square root of the mass; thus, electrons have a much larger thermal speed than do ions with the same temperature.

2.2.3 Equilibrium Maxwell–Boltzmann distribution

The Maxwellian distribution function given by Equation (2.15), in general, is not an equilibrium distribution function and is a function of time via the time dependence of the density and temperature. In contrast, the Maxwell–Boltzmann distribution for a gas in thermodynamic equilibrium is independent of time and can be expressed in terms of the total energy E of a particle as

$$f_{sM}(\mathbf{x}, \mathbf{v}) = f_{sM}(0, 0) \exp[-E/k_B T], \tag{2.18}$$

where $f_{sM}(0, 0)$ is a reference distribution function. The particle energy, E, for a static electric field is given by

$$E = 1/2 m_s v^2 + q_s V(\mathbf{x}), \tag{2.19}$$

where $V(\mathbf{x})$ is the electrostatic potential at position $\mathbf{x}$. Equation (2.18) can be re-written as

$$f_{sM}(\mathbf{x}, \mathbf{v}) = \left[n_0 e^{-q_s V(x)/K_B T} \right] e^{-m_s v^2 / 2 k_B T}$$

$$n_s(\mathbf{x}) = n_0 e^{-q_s V(\mathbf{x})/k_B T}, \tag{2.20}$$

where n_0 is the density at $\mathbf{x} = 0$. The density is proportional to n_0, as well as to an exponential called the *Boltzmann factor*. We will use this thermal equilibrium distribution near the end of the chapter to derive a shielding length scale in a plasma.

2.3 Macroscopic variables

For many applications we do not need to know the details of the distribution function. It is sufficient to work with a limited number of macroscopic variables. *Velocity moments* of the distribution function have traditionally been used (cf. Burgers, 1969; Gombosi, 1994) to define these basic *macroscopic* (or *fluid*) *variables*. These macroscopic variables are functions only of position (and not velocity) and can be considered to be "measurable" quantities. The nth moment of the single particle distribution function $f_s(\mathbf{x}, \mathbf{v}, t)$ is defined by

$$M^n f_s = \int \mathbf{v}^n f_s(\mathbf{x}, \mathbf{v}, t) \, d^3 \mathbf{v}. \tag{2.21}$$

The zeroeth moment ($n = 0$) is just the number density, $n_s(\mathbf{x}, t)$, as given earlier by Equation (2.5). The first moment ($n = 1$) is just the (net) *particle flux* of species s:

$$\Gamma_s(\mathbf{x}, t) = \int \mathbf{v} f_s(\mathbf{x}, \mathbf{v}, t) \, d^3 \mathbf{v}. \tag{2.22}$$

The particle flux has units of $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$ and is a vector quantity. The integrals in Equations (2.21) and (2.22) are over all of velocity space. In component notation ($j = 1$–3), Equation (2.22) becomes

$$\Gamma_{sj} = \int v_j f_s(\mathbf{x}, \mathbf{v}, t) \, d^3 \mathbf{v}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_j f_s(x, y, z, v_x, v_y, v_z, t) \, dv_x \, dv_y \, dv_z. \tag{2.23}$$

The *bulk flow velocity* of species s is defined as the average velocity and is expressed in terms of the flux as

$$\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{v} \rangle = \Gamma_s(\mathbf{x}, t)/n_s(\mathbf{x}, t). \tag{2.24}$$

The velocity $\mathbf{u}$ will often carry the species subscript s ($\mathbf{u}_s$) if the context is not clear. Note that the average of a quantity Q is defined as

$$\langle Q \rangle = \frac{1}{n_s} \int Q(\mathbf{x}, \mathbf{v}, t) f_s(\mathbf{x}, \mathbf{v}, t) \, d^3\mathbf{v}. \tag{2.25}$$

The second moment of the distribution function is related to the *pressure tensor* of species s. The pressure tensor is best found by using the *peculiar – or random – velocity*, which is defined in terms of the total individual particle velocity and the bulk flow speed:

$$\mathbf{c} = \mathbf{v} - \mathbf{u}. \tag{2.26}$$

The peculiar velocity and the total velocity are the same if the gas as a whole is stationary. The pressure tensor is proportional to the second moment of f_s and can be calculated using $\mathbf{c}$ rather than $\mathbf{v}$:

$$\tilde{\mathbf{P}}_s(\mathbf{x}, t) = m_s n_s \langle \mathbf{c}\,\mathbf{c} \rangle = m_s \int \mathbf{c}\,\mathbf{c} f_s(\mathbf{x}, \mathbf{v}, t) \, d^3\mathbf{v}. \tag{2.27}$$

P_{sij} constitutes a 3×3 matrix. In component notation, the (i, j) element of the pressure tensor $\tilde{\mathbf{P}}_s(\mathbf{x}, t)$ is given as

$$P_{sij}(\mathbf{x}, t) = m_s \int c_i c_j f_s(\mathbf{x}, \mathbf{v}, t) \, d^3\mathbf{v}. \tag{2.28}$$

In Equation (2.28) the substitution $\mathbf{v} = \mathbf{c} + \mathbf{u}$ can be made, and the integral can be evaluated using the peculiar velocity – $d^3\mathbf{c}$ – instead of $d^3\mathbf{v}$. A scalar pressure p_s is defined as one third of the trace of P_{sij}:

$$p_s = \frac{1}{3} \operatorname{Tr} \tilde{\mathbf{P}}_s = \frac{1}{3} \sum_{j=1}^{3} P_{sjj} = \frac{1}{3} P_{sjj}. \tag{2.29}$$

The pressure tensor for an isotropic distribution contains only one independent quantity (i.e., the scalar pressure) and can be expressed in terms of the following matrix:

$$\tilde{\mathbf{P}}_s = \begin{pmatrix} p_s & 0 & 0 \\ 0 & p_s & 0 \\ 0 & 0 & p_s \end{pmatrix}, \tag{2.30}$$

where the scalar pressure $p_{xx} = p_{yy} = p_{zz} = p_s$ is given by

$$p_s(\mathbf{x}, t) = \frac{1}{3} m_s \int_0^\infty c^2 f_s(\mathbf{x}, c, t) 4\pi c^2 dc. \tag{2.31}$$

The distribution function in Equation (2.31) is isotropic and is solely a function of the magnitude of the peculiar velocity. A physical interpretation of pressure will be given later. The pressure p_s has units of N/m^2, which is the same as J/m^3. Equations

(2.30) and (2.31) can be applied to the shell distribution given by Equation (2.14) as well as to the Maxwellian distribution. Using expression (2.31), the scalar pressure for the Maxwellian distribution can be shown (Problem 2.3) to be equal to $p_s = n_s k_B T_s$; this is the *equation of state* of an ideal gas, which relates the pressure of species s to the density n_s and the *temperature T_s*.

The heat flux vector for species s is closely related to the third moment of the distribution function and is given by

$$\tilde{\mathbf{Q}}_s(\mathbf{x}, t) = \frac{1}{2} m_s \int \mathbf{c}\, c^2 f_s(\mathbf{x}, \mathbf{v}, t)\, d^3 \mathbf{v}. \tag{2.32}$$

The SI unit for the heat flux is W/m^2. Most fluid theories in practical use do not go beyond five moments. Rather than carry out the integral in Equation (2.32), it is usually easier to represent the heat flux with a simple phenomenological expression, as will be done later in the chapter.

2.4 The fluid conservation equations

Now that macroscopic variables have been defined, we need some prescription for determining them, without having to undertake the very difficult task of solving the Boltzmann equation. Equations that are easier to solve than the Boltzmann equation can be found by taking moments of the Boltzmann equation. The nth moment of the Boltzmann equation can be represented by

$$\int \mathbf{v}^n \left[\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \mathbf{a} \cdot \nabla_{\mathbf{v}} f_s = \left(\frac{\delta f_s}{\delta t} \right)_{\text{collision}} \right] d^3 \mathbf{v}. \tag{2.33}$$

We will only consider the zeroeth, first, and second moments here. Furthermore, little attention will be devoted to the collision terms.

2.4.1 Continuity equation

Now we evaluate the zeroeth moment of each term of the Boltzmann equation. The first term of Equation (2.33) with $n = 0$ becomes

$$\int \frac{\partial f_s}{\partial t}\, d^3 \mathbf{v} = \frac{\partial}{\partial t} \int f_s\, d^3 \mathbf{v} = \frac{\partial n_s}{\partial t}. \tag{2.34}$$

Note that the order of the time derivative and the integral over velocity has been reversed – this is allowed because time and velocity are independent variables. The definition of density given by Equation (2.5) was also used.

The zeroeth moment of the second term of Equation (2.33) gives

$$\int \mathbf{v} \cdot \nabla f_s\, d^3 \mathbf{v} = \int \frac{\partial}{\partial x_j} (v_j f_s)\, d^3 \mathbf{v}$$

$$= \frac{\partial}{\partial x_j} \int v_j f_s\, d^3 \mathbf{v}. \tag{2.35}$$

Component notation and the summation convention were used in the first line of Equation (2.35), and in the second line the order of the derivative and integral were switched. The integral on the second line is just the definition of the net particle flux, as given by Equations (2.22) and (2.23). The particle flux is equal to $n_s\mathbf{u}_s$, where $\mathbf{u}_s$ is the average (or bulk) flow velocity of species s, whose jth component is u_{sj}. Equation (2.35) then becomes

$$
\begin{aligned}
&= \frac{\partial}{\partial x_j}[n_s\langle v_j\rangle] = \frac{\partial}{\partial x_j}[n_s u_{sj}] \\
&= \nabla \cdot (n_s\mathbf{u}_s).
\end{aligned}
\tag{2.36}
$$

Vector notation is again used in the second line of Equation (2.36). This equation indicates that the second term of the zeroeth moment of Boltzmann's equation is equal to the divergence of the net particle flux of species s.

We can write the acceleration term in Equation (2.33) as

$$
\begin{aligned}
\int \mathbf{a} \cdot \nabla_{\mathbf{v}} f_s \, d^3\mathbf{v} &= \int a_j \frac{\partial f_s}{\partial v_j} \, d^3\mathbf{v} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(a_x \frac{\partial f_s}{\partial v_x} + a_y \frac{\partial f_s}{\partial v_y} + a_z \frac{\partial f_s}{\partial v_z} \right) dv_x dv_y dv_z.
\end{aligned}
\tag{2.37}
$$

It is sufficient for us to evaluate the first term of Equation (2.37) (the other two terms give identical results):

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} a_x \frac{\partial f_s}{\partial v_x} dv_x \right] dv_y \, dv_z.
\tag{2.38}
$$

The integral over v_x (i.e., the term in brackets in this expression) is easy to evaluate if the x component of the acceleration, a_x, does not depend on v_x, as is indeed true for the Lorentz force. Then

$$
\left[\int_{-\infty}^{\infty} a_x \frac{\partial f_s}{\partial v_x} dv_x \right] = \left[a_x \int_{-\infty}^{\infty} \frac{\partial f_s}{\partial v_x} dv_x \right] = a_x[f_s]_{-\infty}^{\infty} = 0,
\tag{2.39}
$$

where we have used the fact that the distribution function is zero at $v_x = \pm\infty$. Thus, the acceleration term of the zeroeth moment of the Boltzmann equation equals zero.

We are left with the task of determining the zeroeth moment of the collision term, which we call S_s. It will be demonstrated below that S_s is the *net source per unit volume* of particles of type s (regardless of velocity) due to collisions. Putting together Equations (2.34) through (2.39), we obtain the familiar *continuity equation*,

$$
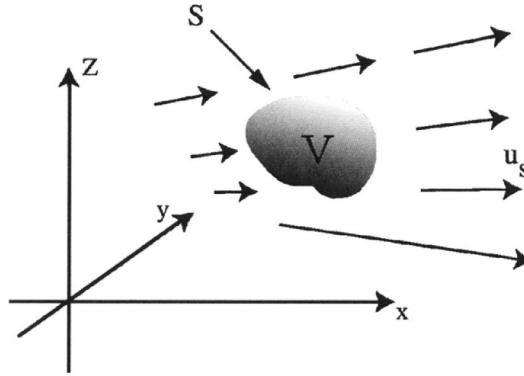\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s\mathbf{u}_s) = S_s.
\tag{2.40}
$$

Figure 2.4. Volume of space V in a fluid of species s, with flow speed $\mathbf{u}_s$. The surface of the volume is designated S. The total number of particles of species s in the volume is N_s

The integral form of the continuity equation is useful for physical interpretation. We integrate both sides of Equation (2.40) over some fixed volume V for a fluid moving with bulk flow speed $\mathbf{u}_s$ (Figure 2.4). The volume integral of the time derivative of the number density, $\partial n_s(\mathbf{x}, t)/\partial t$, is just the time derivative of the total number of particles of species s in the volume, N_s. The volume integral of the divergence term can be converted to a closed surface integral using Gauss's integral theorem, which gives

$$\int_V \nabla \cdot (n_s \mathbf{u}_s)\, d^3\mathbf{x} = \oint_S n_s \mathbf{u}_s \cdot d\mathbf{S},\tag{2.41}$$

where S designates the closed surface and the vector $d\mathbf{S}$ is a differential surface element oriented normal to the surface. Recall that $n_s \mathbf{u}_s$ is the particle flux Γ_s, and thus Equation (2.41) gives the total net flux of particles (units of s^{-1}) through the closed surface S – this is positive for a net flux of particles of type s *out* of the volume V and is negative for a net flux into the volume V.

Now we consider the right-hand side of the continuity equation. The volume integral of the net production rate per unit volume, S_s, gives the total net production rate *inside* the volume:

$$\tilde{S}_s = \int_V S_s\, d^3\mathbf{x}.\tag{2.42}$$

The integral form of the continuity equation becomes

$$\frac{\partial N_s}{\partial t} = \tilde{S}_s - \int_S \Gamma_s \cdot d\mathbf{S}.\tag{2.43}$$

Equation (2.43) states that the time rate of change of the total number of particles (of type s) in the volume is equal to the change due to the flux of particles across the surface plus the total net production of particles within the volume ($\tilde{S}_s$). If the fluid is stationary and $\mathbf{u}_s = 0$, then N_s changes only if there is a net collisional creation

or destruction of particles of species s locally within the volume. For example, in a partially ionized plasma such as that found in the terrestrial ionosphere, ions and electrons can be created by the photoionization of neutrals by solar extreme ultraviolet radiation (i.e., collisions with photons). And ions of one species (e.g., O^+) can be converted to ions of another species (e.g., NO^+) by chemical reactions (e.g., ion–neutral collisions) such as

$$O^+ + N_2 \longrightarrow NO^+ + N. \tag{2.44}$$
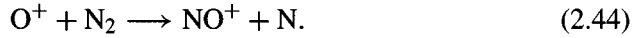
Electrons in the ionosphere can be removed by recombination with ions. All these "collisional/chemical" production and loss terms enter the continuity equation via the net production terms S_s. The net production rate can be expressed as the difference between the production rate, P_s, and the loss rate, L_s; that is, $S_s = P_s - L_s$.

The density is constant in time and space (i.e., $n_s = constant$) for the important special case of an *incompressible* fluid. Furthermore, if there is no production or loss of particles ($S_s = 0$), the continuity equation simply becomes

$$\nabla \cdot \mathbf{u}_s = 0. \tag{2.45}$$

This simple incompressible continuity equation is generally applicable to air in the lower atmosphere or to liquids, but most plasmas are *compressible* and the full continuity equation (2.40) must be used.

2.4.2 Momentum equation

Now let us determine the first moment of the Boltzmann equation and obtain an equation for the flow speed $\mathbf{u}_s$. This is the momentum equation, which is a vector equation with three components. The jth component of the momentum equation is given by

$$\int v_j \{\text{Boltzmann equation}\} \, d^3\mathbf{v}, \tag{2.46}$$

where j runs from 1 to 3 (for the x, y, and z components, respectively).

The first moment of the time-derivative term in Equation (2.33) (or Equation (2.46)) is

$$\boxed{1} = \int v_j \frac{\partial f_s}{\partial t} \, d^3\mathbf{v} = \frac{\partial}{\partial t} \int v_j f_s \, d^3\mathbf{v} = \frac{\partial}{\partial t}(n_s u_{sj}), \tag{2.47}$$

where the order of the time derivative and the integral over velocity were switched. The definition of the flow velocity, Equation (2.24), was employed for the last step. The vector version of (2.47) is given by

$$\frac{\partial}{\partial t}(n_s \mathbf{u}_s). \tag{2.48}$$

The jth component of the second term (the advection term) is

$$\boxed{2} = \int v_j \mathbf{v} \cdot \nabla f_s \, d^3\mathbf{v} = \int v_j v_k \frac{\partial f_s}{\partial x_k} \, d^3\mathbf{v} = \frac{\partial}{\partial x_k} \int v_j v_k f_s \, d^3\mathbf{v}, \qquad (2.49)$$

where the removal of the derivative to outside the integral in the last step in Equation (2.49) was made possible because x_k, v_k, and v_j are all independent variables.

Next, we use the definition of peculiar velocity given in Equation (2.26) to write the jth component of the velocity as $v_j = c_j + u_j$. Putting this into Equation (2.49), we find that

$$\boxed{2} = \frac{\partial}{\partial x_k} \int v_j v_k f_s \, d^3\mathbf{v}$$

$$= \frac{\partial}{\partial x_k} \left\{ \underbrace{\int c_j c_k f_s \, d^3\mathbf{v}}_{(i)} + \underbrace{\int (c_j u_k + c_k u_j) f_s \, d^3\mathbf{v}}_{(ii)} + \underbrace{\int u_j u_k f_s \, d^3\mathbf{v}}_{(iii)} \right\}.$$

$$(2.50)$$

The first integral, (i), can be evaluated by using the definition of the pressure tensor, as given by Equation (2.28):

$$(i) = \int c_j c_k f_s \, d^3\mathbf{v} = n_s \langle c_j c_k \rangle = (1/m_s) P_{s,jk}, \qquad (2.51)$$

where $P_{s,jk}$ is the j, k component of the pressure tensor for species s. Integral (ii) is zero because u_k (or u_j) is independent of $\mathbf{v}$, and the average of c_j or c_k is zero, due to the definition of peculiar velocity. The product $u_j u_k$ can be removed outside the integral (iii), in which case we find

$$(iii) = \int u_j u_k f_s \, d^3\mathbf{v} = u_j u_k \int f_s \, d^3\mathbf{v} = n_s u_j u_k. \qquad (2.52)$$

When we put all the parts of Equation (2.50) back together, we find

$$\boxed{2} = \frac{\partial}{\partial x_k} \left(\frac{1}{m_s} P_{s,jk} + n_s u_j u_k \right). \qquad (2.53)$$

Expression (2.53) can be rearranged in several ways, including

$$\boxed{2} = \frac{1}{m_s} \frac{\partial P_{s,jk}}{\partial x_k} + u_j \frac{\partial}{\partial x_k} (n_s u_k) + n_s u_k \frac{\partial u_j}{\partial x_k}. \qquad (2.54)$$

Let us convert Equation (2.54) to vector form. The first two derivatives in this equation turn into divergence operators and the last derivative becomes the gradient operator:

$$\boxed{2} = \frac{1}{m_s} \nabla \cdot \tilde{\mathbf{P}}_s + \mathbf{u}_s \nabla \cdot (n_s \mathbf{u}_s) + n_s (\mathbf{u}_s \cdot \nabla) \mathbf{u}_s. \qquad (2.55)$$

We now evaluate the first velocity moment of the third term on the left-hand side of the Boltzmann equation – the acceleration term. The jth component of this moment is

$$\boxed{3} = \int v_j a_k \frac{\partial f_s}{\partial v_k} d^3\mathbf{v}$$

$$= \int v_j a_x \frac{\partial f_s}{\partial v_x} d^3\mathbf{v} + \int v_j a_y \frac{\partial f_s}{\partial v_y} d^3\mathbf{v} + \int v_j a_z \frac{\partial f_s}{\partial v_z} d^3\mathbf{v}, \qquad (2.56)$$

where the summation over k was carried out explicitly. Two of the three integrals in expression (2.56) have indices $j \neq k$, and these two integrals are zero (see Problem 2.5). The x component ($j = 1$) of the other integral can be integrated by parts over v_x and is equal to

$$\int v_x a_x \frac{\partial f_s}{\partial v_x} d^3\mathbf{v} = -n_s \langle a_x \rangle, \qquad (2.57)$$

where $\langle a_x \rangle$ is the average of the x component of the acceleration of a particle of species s. The y and z components can similarly be determined. Equation (2.56) can now be written

$$\boxed{3} = -n_s \langle a_j \rangle. \qquad (2.58)$$

We have finished taking the first moment of the left-hand side of the Boltzmann equation. We should now evaluate the velocity moment of the right-hand side of the Boltzmann equation (i.e., the collision term). This represents the change in momentum per unit volume of the fluid species s due to collisions. However, we skip over this very difficult task and simply designate the jth component of the collisional change in momentum as $\delta M_{s,j}/\delta t$. A simple heuristic expression for this collision term will be given later.

The velocity moment equation (i.e., the momentum equation) can now be written in vector form by combining $\boxed{1} + \boxed{2} + \boxed{3}$:

$$\frac{\partial}{\partial t}(\rho_s \mathbf{u}_s) + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s) + \nabla \cdot \tilde{\mathbf{P}}_s - \rho_s \langle \mathbf{a}_s \rangle = \left(\frac{\delta \mathbf{M}_s}{\delta t} \right)_{\text{Collision}}, \qquad (2.59)$$

where the mass density is related to the number density by $\rho_s = n_s m_s$. The average acceleration of a plasma fluid parcel is related to the average force by $\langle \mathbf{a}_s \rangle = \langle \mathbf{F}_s \rangle / m_s$. This average force should include the Lorentz force associated with internal electric and magnetic fields and should include any external forces on the fluid:

$$\langle \mathbf{a}_s \rangle = \frac{q_s}{m_s}[\mathbf{E}(\mathbf{x}, t) + \mathbf{u}_s \times \mathbf{B}(\mathbf{x}, t)] + \mathbf{a}_{\text{external}}. \qquad (2.60)$$

The electric and magnetic fields are understood to be average, or "macroscopic," fields that can be found from Maxwell's equations using macroscopically defined sources (see Section 2.3); all microscopic fields are incorporated into the collision

term. An example of an external acceleration is gravity, $\mathbf{a}_{\text{ext}} = \mathbf{g}$. The acceleration due to gravity near the surface of the Earth is $\mathbf{g} = -9.88\,\hat{\mathbf{z}}\,\text{m/s}^2$.

The momentum equation can be written in several forms, including the following form, which can be derived using the continuity equation plus Equation (2.59) (Problem 2.6):

$$\rho_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right] + \nabla \cdot \tilde{\mathbf{P}}_s = \rho_s \langle \mathbf{a}_s \rangle + \left(\frac{\delta \mathbf{M}_s}{\delta t} \right)_{\text{Collision}} - m_s \mathbf{u}_s S_s. \quad (2.61)$$

Recall that S_s is the net source of particles of species s that appears on the right-hand side of the continuity equation (2.40). The last term in equation (2.61) represents the effect of adding mass to the fluid (this is the "mass-loading" term). The term in brackets is the convective derivative of the flow velocity, which is just the acceleration of a parcel of fluid, $D\mathbf{u}_s/Dt$. Rearranging Equation (2.61) so that only $D\mathbf{u}_s/Dt$ appears on the left-hand side, the right-hand side then specifies this acceleration.

2.4.3 The pressure gradient force

The fluid acceleration not only includes $\langle \mathbf{a}_s \rangle$ and collisional effects but also a contribution due to any nonuniform pressure distribution in the fluid – this is, the *pressure gradient force*. The pressure gradient force per unit mass on a fluid parcel is $-(1/\rho_s)\nabla \cdot \tilde{\mathbf{P}}_s$. For isotropic pressure, this becomes $-(1/\rho_s)\nabla p_s$, where p_s is the scalar pressure. The physical meaning of the pressure gradient force can be illustrated by considering the pressure force on a finite volume of fluid (Figure 2.5). We find the change in the momentum in the volume by taking the volume integral of the divergence of the pressure tensor. This volume integral can be transformed by means of the divergence theorem (or Gauss's theorem) into a surface integral over the surface of the volume:

$$\left(\frac{D(\text{momentum in } V)}{Dt} \right)_{Pressure} = -\int_v \nabla \cdot \tilde{\mathbf{P}}_s d^3 \mathbf{x} = -\int_s \tilde{\mathbf{P}}_s \cdot d\mathbf{S}. \quad (2.62)$$
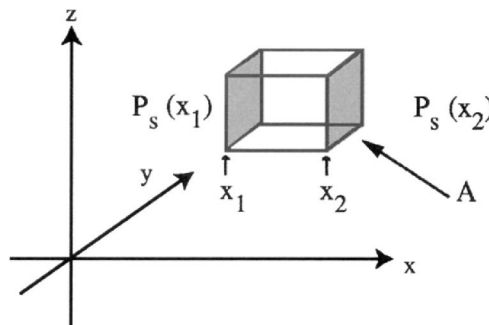

Figure 2.5. A cube-shaped volume (i.e., fluid parcel) is shown, each surface of which has area A. A pressure gradient across this volume in the x direction results in a force on the parcel.

Equation (2.62) represents the force on the fluid parcel due to the pressure gradient force. Consider a simple scalar pressure that is only a function of x: $p_s = p_s(x)$. The only nonzero elements of the pressure tensor are the diagonal terms, which are all equal to p_s. The force on the fluid volume is then solely in the x direction and is equal to $-[p_s(x_2) - p_s(x_1)]A$, where x_1 and x_2 define the x extent of the volume and A is the surface area. Simply put, a net force on a volume results when the pressure is greater on one side of the volume than on the other side. Pressure is due to the random motion of the particles; hence, the pressure gradient force is associated with an excess of collisions on one surface of a volume in comparison with the collisions on the opposite surface.

2.4.4 The collision term

The first moment of the Boltzmann collision term can formally be expressed as

$$\left(\frac{\delta \mathbf{M}_s}{\delta t}\right)_{\text{collision}} = \int \mathbf{v}\, d^3\mathbf{v} \left(\frac{\delta f_s}{\delta t}\right)_{\text{collision}} \tag{2.63}$$

In general, $(\delta f_s / \delta t)_{\text{collision}}$ includes contributions from creation and loss of particles of type s, Coulomb collisions with other charged particles (including species s), and collisions with neutral atoms and molecules.

The term $(\delta f_s / \delta t)_{\text{collision}}$ has the form of a *Fokker–Planck equation* for the case of a fully ionized plasma, because for the long-range Coulomb force, significant changes in f_s can result from a very large number of small-angle "collisions" (see references at the end of this chapter including Spitzer, 1962, and Bittencourt, 1986). However, "collisions" still implies "discrete" interactions of individual particles even if these interactions are long range. For most fully ionized space plasmas, such as the solar wind plasma and plasma in the solar corona, the collision term in the momentum equation can be entirely neglected with little loss of accuracy. However, both Coulomb and ion–neutral collisions are important for the colder and denser plasmas found in planetary ionospheres.

The change in momentum due to collisions can be written (without derivation) as

$$\left(\frac{\delta \mathbf{M}_s}{\delta t}\right)_{\text{collision}} = -\sum_{t \neq s} \nu_{st} \rho_s (\mathbf{u}_s - \mathbf{u}_t) + \rho_s \eta_s \nabla^2 \mathbf{u}_s + P_s m_n \mathbf{u}_n - L_s m_s \mathbf{u}_s. \tag{2.64}$$

The first term on the right-hand side of Equation (2.64) represents the change in fluid momentum due to collisions of species s with all other species (hence, the summation over index t) including charged particle species and neutral species; this is the friction term. The second term is the viscosity term (only an approximate version has been included here), which can usually be neglected for space plasmas; this term handles the change in momentum due to velocity shears in the presence of collisions. The third term accounts for the creation of fluid momentum of species

s via ionization of a neutral species n, which has flow velocity $\mathbf{u}_n$ and mass m_n. The last term in Equation (2.64) accounts for the loss of momentum associated with the chemical loss of species s. Note that the last two terms of Equation (2.64), when combined with the last term of the momentum equation (2.61), yield the following "mass-addition" or "mass-loading" term appropriate for Equation (2.61): $(m_n\mathbf{u}_n - m_s\mathbf{u}_s)P_s$.

A cross section represents the probability of some type of collisional process taking place. For example, suppose that a broad "beam" of projectiles (e.g., pellets fired from a gun) are fired at a target (e.g., a basketball). Then the probability that a pellet will hit the basketball is proportional to the cross-sectional area (or *cross section*) of the basketball, as well as being proportional to the flux of projectiles. The *collision frequency* for some process is the rate at which this process takes place (with units of s^{-1}) and is proportional to the product of the cross section, the relative speed of the projectile and target, and the number density of the targets. For Equation (2.64), the relevant process is the transfer of momentum between colliding species. For the continuity equation, (2.40), the relevant process is that of creation or destruction of species s.

The term ν_{st} is the *momentum transfer collision frequency* between species s and species t and has units of inverse seconds. It is thus apparent that Equation (2.64) must have units of $[\text{N/m}^3]$. The momentum transfer collision frequency can be found by means of a suitable averaging of the momentum transfer cross section over the distribution functions of the two colliding species:

$$\nu_{st} = \frac{m_t}{m_t + m_s} \langle g\sigma_{\mathrm{m}}(g)\rangle_{\mathrm{cm}} n_t. \tag{2.65}$$

Here n_t is the number density of the target species; m_s and m_t are the masses of the "projectile" and "target" species, respectively; $g = |\mathbf{v} - \mathbf{v}'|$ is the relative velocity between particles of type s and t; and σ_{m} is the momentum transfer cross section as a function of g. $\langle \rangle_{\mathrm{cm}}$ is an average in the center of mass reference frame:

$$\langle g\sigma_{\mathrm{m}}(g)\rangle_{\mathrm{cm}} \equiv \frac{1}{n_s n_t} \int d^3\mathbf{v} \int d^3\mathbf{v}' |\mathbf{v} - \mathbf{v}'| \sigma_{\mathrm{m}}(g) f_s(\mathbf{v}) f_t(\mathbf{v}'). \tag{2.66}$$

The cross section depends on the nature of the interaction between the two species of particles. And the distribution functions in Equation (2.66) are generally assumed to be Maxwellians. For electron–neutral collisions, one can write the collision frequency as $\nu_{en} = k_{en}n_n$, where n_n is the neutral density and where the collision coefficient k_{en} depends on the particular neutral species and on the electron energy. Typically, $k_{en} \approx 10^{-8}$ cm^3 s^{-1} for electrons with energies of a few electron volts or less. Similarly, for ion–neutral collisions, the collision frequency can be written as $\nu_{in} = k_{in}n_n$ with $k_{in} \approx 10^{-9}$ cm^3 s^{-1} for most neutral species and for ion energies of the order of several eV or less. Charge transfer is one of the most important types of ion–neutral interactions for space plasmas and can be

represented by the reaction

$$A^+ + B \longrightarrow A + B^+, \tag{2.67}$$

where A and B represent different atomic or molecular species. For example, A might be atomic hydrogen (H) and B might be atomic oxygen (O), in which case reaction (2.67) is just the accidentally resonant charge transfer reaction, $H^+ + O \rightarrow O^+ + H$. For this process, ion A^+ is neutralized and replaced by ion B^+. However, the momentum is largely retained by the neutral particle A. If the neutral particle B is slow, then the final ion B^+ is also slow regardless of the speed of the ion A^+. If the species A and B are the same, then the net result of the reaction (2.67) is a large loss of momentum for ion species A^+. Charge transfer cross sections are typically about 10^{-15} cm^2 for most species. For a more detailed discussion of collisional processes and their consequences you can consult a couple of the references cited at the end of the chapter (Banks and Kockarts, 1973; Rees, 1989).

Now we very briefly consider the collision frequency for interactions between two different charged species; again, for a detailed treatment of this topic consult one of the references listed at the end of the chapter. The momentum transfer cross section for Coulomb collisions varies as the inverse power of g^4, and the collision frequency is (in SI units)

$$\nu_{st} = \frac{e^4 n_t \ln \Lambda}{4\pi \varepsilon_0^2 m_s \mu_{st} \langle g_{st}^3 \rangle}, \tag{2.68}$$

where $\mu_{st} = m_s m_t / (m_s + m_t)$ is the reduced mass for species s and t, and $\ln \Lambda$ is the *Coulomb logarithm*, which has a value of roughly $\ln \Lambda \approx 20$ for space plasmas. Note that ν_{st} is proportional to the average of the product of the speed g and a cross section that varies as g^{-4}. Hence, as the particle speed increases, the Coulomb collision frequency decreases. The average $\langle \rangle$ was defined above and can be evaluated for a Maxwellian gas, in which case Equation (2.68) gives the following collision frequency formula as a function of temperature:

$$\nu_{st} = \frac{n_t e^4 \ln \Lambda \sqrt{\mu_{st}}}{\sqrt{2\pi} \, 32 \varepsilon_0^2 m_s (k_B T_s)^{3/2}} \quad \text{[SI units]}. \tag{2.69}$$

Using Equation (2.69), the following useful expressions for electron–electron (ν_{ee}), electron–ion (ν_{ei}), and ion–ion (ν_{ii}) momentum transfer collision frequencies can be obtained (Banks and Kockarts, 1973):

$$\begin{aligned}
\nu_{ee} &= 54 \, n_e / T_e^{3/2} \\
\nu_{ei} &= \nu_{ee} \\
\nu_{ie} &= (m_e/m_i)\nu_{ei} \\
\nu_{ii} &= (m_e/m_i)^{1/2}\nu_{ee}.
\end{aligned} \tag{2.70}$$

The units of density in expression (2.70) are cm^{-3}. The electron–ion and electron–electron collision frequency formulae are essentially the same (very small differences have been neglected), but the ion–electron and ion–ion collision frequencies are much smaller than the electron–electron or electron–ion collision frequencies.

2.4.5 Energy equation

The continuity and momentum equations can be used to determine the density ρ_s and flow speed $\mathbf{u}_s$ for species s, respectively. The momentum equation, (2.61), contains the pressure p_s, which cannot be simply expressed in terms of density or flow speed from what we have done up to now. Another equation – the *energy equation* – is required to describe how the pressure (or thermal energy density) of a fluid should behave. Pressure has units of force per unit area $[N/m^2]$, but the units $[N/m^2]$ are the same as the units of energy density $[J/m^3]$. In fact, pressure is essentially thermal energy density.

The energy equation for species s is obtained by taking the second moment of the Boltzmann equation. The energy equation takes many forms, several of which will be shown in this section. We will not derive the energy equation; the method is essentially the same as was used to derive the continuity and momentum equations. To give an idea of how this might work, the second moment of the first term of the Boltzmann equation yields

$$\left\langle v_i v_j \frac{\partial f_s}{\partial t} \right\rangle = \frac{1}{m_s} \frac{\partial \tilde{\mathbf{P}}_{s,ij}}{\partial t} + \frac{\partial}{\partial t}(n_s u_i u_j). \tag{2.71}$$

Note that both the pressure tensor and bulk flow velocity appear in this expression.

For an isotropic Maxwellian distribution, the pressure of species s can be related to the density and temperature of species s by means of the *equation of state* for an ideal gas:

$$p_s = n_s k_B T_s, \tag{2.72}$$

where T_s is the temperature of species s, which appears in the Boltzmann factor discussed earlier. For a non-Maxwellian gas, expression (2.72) can be used to define an "effective" temperature using the pressure and the density of the gas. Equation (2.72) can also be written as $p_s = \rho_s R T_s$, where R is the gas constant for species s and $\rho_s = m_s n_s$ is the mass density of species s.

The *conservative* form of the energy equation has the appearance of a continuity equation for the total energy density of the fluid $[J/m^3]$. The total energy density includes: (1) *internal energy* density $(\rho_s U_s)$, where U_s is *specific internal energy*

(units of J/kg), (2) bulk kinetic energy density $(\rho_s u_s^2/2)$, and (3) potential/field energy density $(\rho_s U_{\mathrm{pot}})$. The energy equation is

$$\frac{\partial}{\partial t}\left[\rho_s\left(U_s + \frac{1}{2}u_s^2 + U_{\mathrm{pot}}\right)\right]$$
$$+ \nabla \cdot \left[\rho_s \mathbf{u}_s\left(h_s + \frac{1}{2}u_s^2 + U_{\mathrm{pot}}\right)\right] + \nabla \cdot \mathbf{Q}_s = \left(\frac{\delta E_s}{\delta t}\right)_{\mathrm{collision}}.$$

(2.73)

The first term is the rate of change of the total energy density of species s. The second term takes into account the change of this energy density due to bulk transport of energy in or out of a volume; it also includes, by means of the *specific enthalpy* h_s, the energy gain or loss due to mechanical work associated with changes in the volume. The $\nabla \cdot \mathbf{Q}_s$ term is the heat conduction term that accounts for changes in the energy density associated with microscopic heat transport in or out of a volume. The right-hand side handles the local time rate of change of the energy density associated with collisional processes.

We can write the specific enthalpy, or enthalpy per unit mass, h_s, and the specific internal energy U_s as

$$h_s = U_s + p_s/\rho_s = \frac{\gamma_s}{\gamma_s - 1}\frac{p_s}{\rho_s} \qquad U_s = \frac{f_s}{2}\frac{p_s}{\rho_s} = \frac{1}{\gamma_s - 1}\frac{p_s}{\rho_s},$$

(2.74)

where f_s is the number of degrees of freedom of the gas species s. $f_s = 3$ for an ideal gas whose particles have no internal structure. $f_s = 5$ for ordinary air at room temperature (3 for translational degrees of freedom plus 2 for rotation). The ratio of specific heats for species s is equal to $\gamma_s = (f_s + 2)/f_s$. For an ideal monatomic gas $\gamma_s = 5/3$. $\gamma_s = 7/5$ for air and $\gamma_s = 1$ for a gas with only one degree of translational energy. The internal energy per particle is equal to the number of degrees of freedom multiplied by $k_B T_s/2$. The enthalpy is equal to the internal energy plus an extra term that accounts for mechanical energy gained or lost due to changes in the volume of a fluid parcel.

The potential energy term is often just the gravitational potential: $U_{\mathrm{pot}} = U_{\mathrm{gravity}}$. However, an energy relation for the plasma as a whole can be found by adding the energy equation (2.73) for each of the plasma species (i.e., the index s must include the electrons and all ion species) plus a relation for the electromagnetic energy. In this case, the potential energy term must also include the magnetic energy density, $B^2/2\mu_0$, and the electric field energy density $\varepsilon_0 E^2/2$. Other "electromagnetic terms" also appear throughout the combined plasma energy equation. These terms are explained in Section 7 of the Appendix, and the combined plasma energy equation in its conservative form will be given in Chapter 4.

$(\delta E_s/\delta t)_{\mathrm{coll}}$ is the local change in energy density per unit time due to collisional processes, and it essentially represents the heating or cooling of species s due to collisions with other (colder or warmer) species. For instance, an ion gas can be

heated by Coulomb collisions with hotter electrons, or the ion gas can be cooled by collisions with colder neutrals. Sometimes, for electrons, cooling due to inelastic collisions with neutral species must also be included. Banks and Kockarts (1973) have an extensive discussion of various heating and cooling terms. A relatively simple collisional energy term including only elastic collisions between species s and neutrals (subscript n) can be written as

$$\left(\frac{\delta E_s}{\delta t}\right)_{\text{coll}} = \mathbf{u}_s \cdot \left(\frac{\delta \mathbf{M}_s}{\delta t}\right)_{\text{coll}} + \mu_{sn} \nu_{sn} n_s [(3k_{\text{B}}/m_s)(T_n - T_s) + |\mathbf{u}_s - \mathbf{u}_n|^2]$$
$$- \left(\frac{1}{2} m_s u_s^2 - \frac{3}{2} k_{\text{B}} T_s\right) S_s, \tag{2.75}$$

where μ_{sn} is the reduced mass between species s and the neutral species, T_n is the temperature of the neutrals, $\mathbf{u}_n$ is the neutral flow velocity, and ν_{sn} is the collision frequency for species s and neutrals. Note that the momentum change due to collisions appears in this expression as does the net production rate of species $s (S_s)$.

The vector $\mathbf{Q}_s$ is the heat flux of species s and represents the transport of heat from one location to another by "microscopic" processes (rather than by bulk flow, which was taken care of by other terms on the left-hand side of Equation (2.73)). Although the heat flux vector is quite complicated in general, if the collision frequency is sufficiently large and if the temperature gradient is small enough, the heat conduction expression can be simply approximated by

$$\mathbf{Q}_s = -K_s \nabla T_s. \tag{2.76}$$

Equation (2.76) states that heat flows in response to temperature gradients – heat flows from hot regions to cold regions. K_s is the conductivity coefficient, which is proportional to the collision mean free path λ_{mfp}. The conductivity coefficient can be written, to within a factor of order unity, as $K_s \approx (n_s k_{\text{B}} v_{s,\text{th}}) \lambda_{\text{mfp}}$, where $v_{s,\text{th}}$ is the thermal speed. The conductivity for Coulomb collisions in a fully ionized Maxwellian plasma (i.e., the *Spitzer conductivity* – see Banks and Kockarts again and Spitzer, 1962) is given by

$$K_s = C T^{5/2} \quad [\text{eV/m/s/K}] \tag{2.77}$$

with $C \approx 7.7 \times 10^7$ for most space plasmas.

Two other very useful forms of the energy equation can be derived from the energy conservation form by using the continuity and momentum equations (see Problems 2.10 and 2.11). One form of the energy equation is written in terms of the convective derivative of the pressure (i.e., thermal energy density) of species s:

$$\frac{D}{Dt}\left(\frac{1}{\gamma_s - 1} p_s\right) + \frac{1}{\gamma_s - 1} p_s \nabla \cdot \mathbf{u}_s + (\tilde{\mathbf{P}}_s \cdot \nabla) \cdot \mathbf{u}_s + \nabla \cdot \mathbf{Q}_s$$
$$= \left(\frac{\delta E_s}{\delta t}\right)_{\text{coll}} - \mathbf{u}_s \cdot \left(\frac{\delta \mathbf{M}_s}{\delta t}\right)_{\text{coll}} + \frac{1}{2} m_s u_s^2 S_s. \tag{2.78}$$

If species s were an ideal monatomic gas then the polytropic index would be $\gamma_s = 5/3$. The second and third terms account for the thermal energy changes due to compression or expansion of a fluid volume. For isotropic pressure, the third term is equal to $(p_s \nabla \cdot \mathbf{u}_s)$ and the second and third terms together become $[\gamma_s/(\gamma_s - 1)]p_s \nabla \cdot \mathbf{u}_s = h_s \nabla \cdot \mathbf{u}_s$. Yet another form of the energy equation can be written in terms of the convective derivative of the temperature:

$$\frac{3}{2} n_s k_B \frac{DT_s}{Dt} + p_s \nabla \cdot \mathbf{u}_s + \nabla \cdot \mathbf{Q}_s$$
$$= \left(\frac{\delta E_s}{\delta t} \right)_{coll} - \mathbf{u}_s \cdot \left(\frac{\delta \mathbf{M}_s}{\delta t} \right)_{coll} + \left(\frac{1}{2} m_s u_s^2 - \frac{3}{2} k_B T_s \right) S_s. \quad (2.79)$$

Isotropic pressure has been assumed here.

The above forms of the energy equation can often be simplified. Let us consider four possible simplifications that are often useful in space physics applications.

2.4.5.1 Steady flow without heat sources or sinks: Bernoulli's equation

In addition to assuming that the flow is steady ($\partial/\partial t = 0$) and that there are no collisional sources or sinks of heat, let us assume that heat conduction is unimportant and that we can neglect the potential energy. The following expression, which is one form of *Bernoulli's equation*, can then be derived from Equations (2.73) and (2.74) plus the continuity equation for steady ($\partial/\partial t = 0$) flow without sources or sinks (see Problem 2.12):

$$\frac{\gamma_s}{\gamma_s - 1} \frac{p_s}{\rho_s} + \frac{u_s^2}{2} = constant. \quad (2.80)$$

Equation (2.80) states that $h_s + u_s^2/2$ is constant along a streamline. For an ideal monatomic gas, Equation (2.80) states that $(5/2)k_B T_s + u_s^2/2$ is a constant. For example, as a parcel of fluid moves from a region of fast flow to a region of slower flow, it heats up; kinetic energy of the bulk flow is converted into thermal kinetic energy. The volume of a fluid parcel decreases as it slows down, and from elementary thermodynamics we know that a gas that is *adiabatically* compressed (i.e., no heat transfer occurs into or out of the volume although mechanical work can be done on the gas) has its internal energy increased.

2.4.5.2 Polytropic energy relation

An even simpler energy relation can be found when the conductive heat transport and the collisional terms are unimportant:

$$p/\rho^\gamma = constant. \quad (2.81)$$

Relation (2.81) even applies to time-dependent situations but does not apply across discontinuities in the flow. This equation is applicable when *specific entropy* (i.e., entropy per unit mass) is conserved by the flow. This is called *isentropic* flow.

$\gamma = 1$ is appropriate for an isothermal gas and $\gamma = 5/3$ for an ideal monatomic gas (see Problem 2.13). $\gamma = 2$ is appropriate for a strongly magnetized plasma and for plasma motions perpendicular to the magnetic field.

2.4.5.3 Heat conduction equation

When the thermal speed is much greater than the fluid speed ($v_{\text{th},s} \gg u_s$), the "dynamical" terms and mass-loading terms can be neglected. The dominant processes are then heat conduction and local collisional heating and cooling. The energy equation (2.79) can then be reduced to the *heat conduction equation* for species s:

$$\frac{3}{2} n_s k_{\text{B}} \frac{\partial T_s}{\partial t} = -\nabla \cdot \mathbf{Q}_s + H_s - L_s(T_s). \tag{2.82}$$

The conductive heat flux was given by Equation (2.76). The collisional terms in Equation (2.79) have been reorganized into local heating and cooling rate terms, H_s and $L_s(T_s)$, respectively. The heating and cooling terms can be neglected for some situations, in which case the one-dimensional heat conduction equation simply becomes

$$\frac{\partial T_s}{\partial t} = \frac{2}{3 n_s k_{\text{B}}} \frac{\partial}{\partial z} \left(K_s \frac{\partial T_s}{\partial z} \right), \tag{2.83}$$

where z is the relevant spatial coordinate (e.g., distance along the magnetic field line). In Problem 2.14, Equation (2.83) is solved for an ionospheric electron gas.

2.4.5.4 Local collisional energy balance

At low altitudes in a planetary ionosphere where the neutral density (and thus the electron–neutral and ion–neutral collision frequencies) is high, local collisional energy transfer becomes more important in the energy balance relation than either convective or conductive heat transport. In this case, the simple energy equation (2.82) can be approximated by the following local heat balance equation:

$$H_s = L_s(T_s). \tag{2.84}$$

Equation (2.84) can sometimes be solved to give an analytic expression for T_s. Banks and Kockarts (1972) discussed the heating and cooling terms (H_s and L_s) appropriate for the ionosphere of Earth. The electron–neutral cooling rate can usually be expressed in the form $L_{e,en}(T_e) = b_{en} n_n n_e (T_e - T_n)$, where the parameter b_{en} is roughly independent of altitude z, although it does depend on the electron temperature T_e and on the neutral composition. For the terrestrial ionosphere between an altitude of about 120 km and 200 km, taking into account the neutral composition of this atmospheric region (N_2, O_2, and O), one finds that $b_{en} \approx 5 \times 10^{-13}$ [eV cm^3/(Ks)] if the electron and neutral densities (n_e and n_n, respectively) are given in units of cm^{-3}. Expressing the electron heating rate (due to collisions with suprathermal electrons – photoelectrons or auroral precipitating electrons) as

$H_e(z) = a_{ee}(z)n_e$, we can see that the electron temperature as a function of altitude is given, in this approximation, by

$$T_e = T_n + a_{ee}(z)/(b_{en}n_n). \tag{2.85}$$

For the terrestrial daytime ionosphere, the function $a_{ee}(z)$ has a maximum value of about 0.05 eV/s at an altitude of about 160 km. Clearly, according to equation (2.85), the electron temperature is simply equal to the neutral temperature $(T_e = T_n)$ at the lowest altitudes $(z < 130$ km$)$ where the neutral density n_n is very large. However, at higher altitudes the electron temperature increases with altitude since n_n decreases with increasing altitude. Electron temperatures measured in the terrestrial ionosphere for altitudes above 200 km or so typically exceed the neutral temperature by several thousand degrees. However, at these higher altitudes vertical heat conduction becomes an important part of the electron energetics and Equation (2.85) is no longer valid.

2.5 Macroscopic sources for Maxwell's equations

The Vlasov equation, and the moment/conservation equations derived from it, include the electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, respectively, via the Lorentz force contribution to the average acceleration $\langle \mathbf{a} \rangle$. These fields are macroscopically averaged fields and do not include the very small-scale, microscopic fields associated with collisions. The fields can be found from Maxwell's equations (see the appendix) where the source terms are macroscopic (or average) quantities:

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} \tag{2.86}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.87}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.88}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.89}$$

The macroscopic source terms – charge density $\rho_c(\mathbf{x}, t)$ and the current density $\mathbf{J}(\mathbf{x}, t)$ – are expressed in terms of the densities and bulk flow velocities of all the charged particle species in the plasma:

$$\rho_c(\mathbf{x}, t) = \sum_s q_s n_s(\mathbf{x}, t), \qquad \mathbf{J}(\mathbf{x}, t) = \sum_s q_s n_s(\mathbf{x}, t)\mathbf{u}_s(\mathbf{x}, t). \tag{2.90}$$

The sum over s must include both electrons and ions.

The continuity and momentum equations provide prescriptions for finding n_s and $\mathbf{u}_s$, respectively, and these equations in turn require $\mathbf{E}$ and $\mathbf{B}$, which are determined by Maxwell's equations using the source functions ρ_c and $\mathbf{J}$. In order to complete

the set of self-consistent fluid equations, ρ_c and $\mathbf{J}$ are specified in terms of n_s and $\mathbf{u}_s$ by means of Equation (2.90).

The fluid equations that we have presented in this chapter will be used in Chapter 4 to develop a fluid theory useful for describing space plasmas. In the next section, we will use the material in this chapter to discuss a basic property of plasmas – Debye shielding.

2.6 Debye shielding and the plasma parameter

Plasmas are different from neutral gases in that they are composed of charged particles that can exert forces on each other through the electric and magnetic fields they create. In space plasmas, the number densities of electrons and ions are equal, on the average, and the plasma is said to be *quasi-neutral*. Slight departures from charge neutrality ($\rho_c = e(n_i - n_e) = 0$) can occur on large spatial scales for nonequilibrium plasmas but not for most equilibrium plasmas. However, significant departures from neutrality can exist on short spatial scales even for an equilibrium plasma. What we mean by "short" spatial scale will become clear from the following discussion of *Debye shielding* of a test charge in an equilibrium Maxwellian plasma.

2.6.1 Electrostatic potential of a test charge in a plasma

First let us consider the electric potential, $V(r)$, of a point test charge, q_T, located at the origin in a vacuum. $r = |\mathbf{x}|$ is the radial distance from the charge. Combining Gauss's law – Equation (2.86) – and the electrostatic relation, $\mathbf{E} = -\nabla V$, we obtain Poisson's equation,

$$\nabla^2 V = -\rho_c/\varepsilon_0. \tag{2.91}$$

For a point charge in a vacuum, the charge density is $\rho_c = 0$ for $r > 0$. The solution of Equation (2.91) for a point charge in a vacuum is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_T}{r}. \tag{2.92}$$

Now we suppose that at time $t = 0$, the test charge is immersed within an initially uniform plasma in which the electron and ion densities are equal to each other as well as to a reference density, $n_i = n_e = n_0$. The initial charge density is zero and the electric potential is still given by Equation (2.92). For positive values of q_T, the ions in the plasma are repelled by the test charge and the electrons are attracted; for negative q_T, the electrons are repelled and the ions attracted by the test charge. After a sufficiently long time, the electrons and ions rearrange themselves in response to the electrostatic forces on them, and the plasma eventually reaches a new equilibrium configuration that takes into account the existence of the test charge. The ions move much more slowly than the electrons, so that for an intermediate
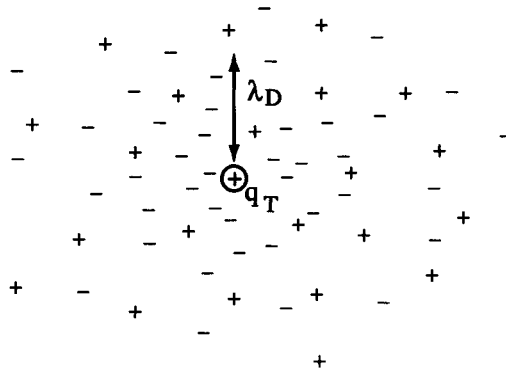
Figure 2.6. Shielding cloud surrounding a test charge.

time scale we assume that they remain motionless. The ion density then remains uniform: $n_i = n_0$. However, the density of electrons near the test charge increases so that $n_e > n_0$ (for $q_T > 0$) or $n_e < n_0$ (for $q_T < 0$). The charge density is no longer zero near the test charge because $n_i \neq n_e$ (Figure 2.6), and Poisson's equation becomes

$$\nabla^2 V = -e[n_i - n_e]/\varepsilon_0. \tag{2.93}$$

For a Maxwell–Boltzmann distribution of electrons in an electrostatic field, we can use expression (2.20),

$$n_e(r) = n_0 \exp(+eV(r)/k_B T_e). \tag{2.94}$$

This expression tells us that, in thermal equilibrium, the electron density is greatest at those locations where the electric potential V is the most positive – that is, n_e is higher in the vicinity of the test charge (for positive values of q_T). Here T_e is the electron temperature. The density variation is greater when the electron gas is cold than when the gas is hot. Substituting expression (2.94) into the spherical coordinate version of Equation (2.91), we find

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -\frac{n_0 e}{\varepsilon_0}\left[1 - \exp\left(\frac{eV(r)}{k_B T_e}\right)\right], \quad r > 0, \tag{2.95}$$

where we used the assumption that the ion density was uniform. For longer time scales, for which both the ions and electrons are in thermal equilibrium, another exponential term with the ion temperature would appear in the brackets on the right-hand side of (2.95) in place of the 1. Equation (2.95) is a complete equation for the potential V as function of r, subject to the condition that as $r \to 0$, the potential should look like that of a point charge, q_T.

We can approximate Equation (2.95) by restricting ourselves to radial distances that are large enough so that $|eV| \ll k_B T_e$. This condition means that a typical

electron's kinetic energy is much greater than its potential energy. We can now expand the exponential function in Equation (2.95) using

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots, \quad x \ll 1, \tag{2.96}$$

where $x = eV(r)/k_B T_e$. Poisson's equation then becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) \cong \left[\frac{n_0 e^2}{\varepsilon_0 k_B T_e}\right]V(r) = \frac{1}{\lambda_D^2}V(r). \tag{2.97}$$

The constants contained within the brackets have been collected to form the parameter λ_D, which is called the *Debye length* or *Debye shielding length*. In SI units

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{n_0 e^2}} \quad [\text{m}], \tag{2.98a}$$

and in cgs units

$$\lambda_D = \sqrt{\frac{k_B T_e}{4\pi n_0 e^2}} \quad [\text{cm}]. \tag{2.98b}$$

The solution of Equation (2.97) is given by Problem (2.15):

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_T}{r}\exp\left(-\frac{r}{\lambda_D}\right). \tag{2.99}$$

Clearly, as $r \to 0$, the potential is essentially that of a point charge in a vacuum, whereas for $r \gg \lambda_D$, Equation (2.99) demonstrates that $V(r) \to 0$ (as does the electric field) much faster than it does for a point charge. The vacuum Coulomb force is long range, but now in a plasma this force only extends a Debye length or so from the source, as a consequence of the Debye shielding cloud. For a positive test charge ($q_T > 0$) and positive potential, the shielding cloud contains an excess of electrons, whereas for a negative potential, the cloud has a deficit of electrons. It can easily be shown, using Gauss's law, that the total net charge contained in the shielding cloud, q_c, is equal and opposite to the test charge: $q_c = -q_T$. Another way of interpreting the Debye shielding phenomenon is that, although on small scales ($L \approx \lambda_D$) a plasma in thermal equilibrium can have significant departures from charge neutrality ($n_e \neq n_i$), for long spatial scales ($L \gg \lambda_D$) an equilibrium plasma must maintain charge neutrality. This property is called *quasi-neutrality*.

The size of the shielding cloud (λ_D) increases as the electron temperature increases because electrons with greater kinetic energy are better able to overcome the Coulomb attraction associated with the potential. And λ_D is smaller for a denser plasma because more electrons are available to populate the shielding cloud. We now consider the value of λ_D for two typical space plasmas.

Example 2.1 (Debye length in the ionosphere and in the solar wind) A numerical expression for the Debye length [m] is given by $\lambda_D = 69 [T_e/n_e]^{1/2}$ with electron temperature T_e in units of K and n_e in units of m^{-3}.

Typical temperature and density values in the topside terrestrial ionosphere are $T_e \approx 1000\,\text{K}$ and $n_e \approx 10^{11}\,\text{m}^{-3}$, respectively, giving the following value for the Debye length:

$$\lambda_D = .007\,\text{m}\ (\approx 1\ \text{cm}).$$

This Debye length is much less than either the vertical ($L \approx 300\,\text{km}$) or horizontal ($L \approx 3{,}000\,\text{km}$) extent of the ionosphere: That is, $\lambda_D \ll L$. Hence, the ionosphere can be considered to be quasi-neutral.

Typical parameters in the solar wind near 1 AU are $T_e \approx 10^5\,\text{K}$ and $n_e \approx 10^7\,\text{m}^{-3}$, giving

$$\lambda_D = 7\,\text{m}.$$

Seven meters is much less than the macroscopic spatial scale of the solar wind ($L \approx 1\ \text{AU} \approx 10^8\,\text{km}$). Hence the solar wind can also be considered to be quasi-neutral. However, note that λ_D is greater than, or comparable to, the size of most spacecraft that have traversed the interplanetary medium. This must be taken into account when designing instruments to measure solar wind plasma properties.

2.6.2 The plasma parameter

Each particle in a plasma – be it an electron or an ion – can act as a "test charge" and carry its own shielding cloud. The concept of Debye shielding as it has just been developed requires the presence of a sufficiently large number of electrons and ions so that "density" can be defined in a statistically meaningful way. A useful parameter in this regard is the *number of particles in a Debye sphere*,

$$N_D = n_0 \left[4\pi \lambda_D^3 / 3 \right]. \tag{2.100}$$

N_D is approximately equal to the Λ parameter that appears in the Coulomb logarithm. The *plasma parameter* is defined by

$$g_{\text{plasma}} = 1/N_D. \tag{2.101}$$

A useful expression for N_D in SI units is $N_D = 138\,T^{3/2}/n^{1/2}$. For example, in the solar wind and in the ionosphere, we have

$$\text{solar wind:}\quad N_D \approx 10^{10}, \qquad g_{\text{plasma}} \approx 10^{-10}$$
$$\text{ionosphere:}\quad N_D \approx 10^5, \qquad g_{\text{plasma}} \approx 10^{-5}.$$

In both of these plasmas, the number of particles in a Debye sphere is extremely large and the plasma parameter is very small, which indicates that Debye shielding is a statistically valid concept.

Another interpretation of the plasma parameter is that for $g_{plasma} \ll 1$, large-scale plasma phenomena are much more important than short-range Coulomb collisions (Nicholson, 1983). This is equivalent to the statement that for an equilibrium plasma for which $g_{plasma} \ll 1$, the average kinetic energy of a plasma particle ($\langle KE \rangle$) is much larger than its average potential energy ($\langle PE \rangle$). One can show (see Problem 2.17) that

$$g_{plasma} \approx \langle PE \rangle / \langle KE \rangle. \qquad (2.102)$$

$\langle PE \rangle$ for an equilibrium plasma indicates the importance of the Coulomb collisions.

Three criteria are commonly used (see references listed at the end of this chapter) for determining whether or not a charged particle gas is a "good" plasma. "Good" is not meant to be an ethical judgment, but indicates that the plasma is quasi-neutral on important length scales and that collective, collisionless, and long-range phenomena such as plasma oscillations (discussed in Chapter 4) are more important than short-range collisional phenomena. The three criteria are:

(1) $\lambda_D \ll L$ (quasi-neutrality on length scales of interest),
(2) $g_{plasma} \ll 1$ (Coulomb collisions are not important and λ_D is defined),
(3) $\omega \tau_n \gg 1$ (other collisional processes are not important).

In criterion (3), ($\omega = 2\pi f$) denotes the angular frequency of the relevant plasma process (such as a wave mode) and τ_n is a collision time for electron or ion collisions with neutrals. The last criterion states that collisions with neutrals do not constitute an important process on time scales of interest. For example, for a plasma to sustain a large-scale oscillation (or waves) with wave frequency ω, then ω must be much greater than the collision frequency for ion–neutral or electron–neutral collisions ($\tau_n^{-1} \approx \nu_{in}$ or ν_{en}). This last criterion is not always met in an ionospheric plasma or in the solar atmosphere, although it usually is met in the solar wind or magnetosphere; in fact in the ionosphere, some of the most interesting phenomena are associated with charged particle collisions with neutrals.

In this chapter a fluid theory has been introduced that can be used to describe the statistical behavior of a collection of charged particles. This theory will be used in Chapter 4 to derive a more refined set of equations (e.g., the magnetohydrodynamic equations) that can be directly applied to space plasma problems. But first, in Chapter 3, we will study the behavior of individual (or single) particles in specified electric and magnetic fields. The study of single particle motion is often useful for understanding very low density plasmas, but it can also be useful for obtaining a physical understanding of the effects of fields on charged particles in fluids.

Problems

2.1   Find the number density associated with the "beam" distribution function given by Equation (2.13).

2.2   Demonstrate by substitution that the Maxwellian distribution, given by Equation (2.15), is a solution of the collisionless Boltzmann equation (2.11).

2.3   The equation of state for an ideal gas relates pressure to density and temperature: $p_s = n_s k_B T_s$.

   Show that for a Maxwellian gas the average kinetic energy of a particle is given by

$$\langle KE \rangle = (3/2) k_B T_s$$

   and that T_s is indeed the temperature appearing in the Boltzmann factor. Then show that for a Maxwellian distribution, Equation (2.31) yields the equation of state.

2.4   Find the scalar pressure p_s associated with the shell distribution function given by Equation (2.14).

2.5   In the derivation of the momentum equation from the Boltzmann equation several mathematical manipulations were undertaken. Derive the following expressions:

   (a) $\int v_i a_x \frac{\partial f_s}{\partial v_x} d^3 \mathbf{v} = 0$ for $i = y, z$,
        where the acceleration $\mathbf{a}$ includes the Lorentz force.
   (b) $\int v_x a_x \frac{\partial f_s}{\partial v_x} d^3 \mathbf{v} = -n_s \langle a_x \rangle$.

2.6   Start with the following "conservative" form of the momentum equation for species s:

$$\frac{\partial}{\partial t}(\rho_s \mathbf{u}_s) + \nabla \cdot [\rho_s \mathbf{u}_s \mathbf{u}_s] + \nabla p_s = \rho_s \langle \mathbf{a}_s \rangle + \left(\frac{\delta \mathbf{M}_s}{\delta t} \right)_{\text{collisions}}$$

   Use the continuity equation for species s to transform this form of the momentum equation into the following form:

$$\rho_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right] + \nabla p_s = \rho_s \langle \mathbf{a}_s \rangle - m_s \mathbf{u}_s S_s + \left(\frac{\delta \mathbf{M}_s}{\delta t} \right)_{\text{collisions}},$$

   where m_s is the mass of a species s particle and where S_s is the net production rate of species s.

   Note that in component notation

$$\{ \nabla \cdot [\rho_s \mathbf{u}_s \mathbf{u}_s] \}_i = \nabla \cdot [(\rho_s u_{s,i}) \mathbf{u}_s] = \frac{\partial}{\partial x_j}(\rho_s u_{s,i} u_{s,j}).$$

2.7   Show that the units of the following collision term in the momentum equation are $[\text{N m}^{-3}]$:

$$\left(\frac{\delta \mathbf{M}_s}{\delta t} \right)_{\text{collisions}} = -\rho_s \sum_{t \neq s} \nu_{st} (\mathbf{u}_s - \mathbf{u}_t).$$

Also show that the units of the collision frequency ν_{st} must be $[s^{-1}]$, starting from an expression for this collision frequency that includes the momentum transfer cross section.

2.8 For a slow-moving, uniform, charged particle gas (i.e., species s) with zero electric and magnetic field, moving through a stationary background neutral gas, the momentum equation simply becomes

$$\partial \mathbf{u}_s / \partial t = -\nu_{sn} \mathbf{u}_s.$$

Find the flow velocity as a function of time given an initial flow velocity of $\mathbf{u}_{s0}(t = 0)$.

2.9 Derive the Spitzer (i.e., Coulomb) heat conduction expression as given by Equation (2.77) (to within a factor of order unity) by using an expression for the Coulomb collision mean free path,

$$\lambda_{\mathrm{mfp}} = \upsilon_{\mathrm{th}} / \nu,$$

where υ_{th} is the thermal speed and ν is the Coulomb collision frequency.

2.10 Derive the form of the energy equation given by Equation (2.78) from the "conservative" form of the energy equation (Equation (2.73)). Use the continuity and momentum equations. Assume that the electric and magnetic fields are zero.

2.11 Derive the form of the energy equation (2.79) from the form given by Equation (2.78), assuming an isotropic pressure.

2.12 Derive the simple energy relation, Equation (2.80), from the "conservative" form of the energy equation (Equation (2.73)) using Equation (2.74). Carefully consider the assumptions needed for this simple equation that were discussed in the text.

2.13 Derive the polytropic energy relation (2.81) for an ideal gas, starting from the energy equation (2.78) by neglecting the collision terms and heat conduction.
Hint: You will also need the continuity equation without sources or sinks.

2.14 The electron temperature as a function of altitude z in a planetary ionosphere can frequently be described using a one-dimensional, steady-state, heat conduction equation without local heating and cooling:

$$K_e(\partial T_e / \partial z) = Q_{e0},$$

where Q_{e0} is the downward electron heat flux at the top of the ionosphere $(z = z_{\mathrm{top}})$. The constant Q_{e0} can often be equated to the integrated heating rate at higher altitudes associated with such processes as magnetospheric heating. The neutral density, and therefore electron–neutral collisional cooling rate, decreases sharply with increasing altitude in an ionosphere; hence the assumption of no heating and cooling is not unreasonable. The

lower boundary can be placed at an altitude z_{bottom} where cooling first begins to be important. At this lower boundary, assume that the temperature is specified: $T_e(z = z_{bottom}) = T_{e0}$. It is usually appropriate to use the Coulomb heat conductivity coefficient.

(a) Derive the steady-state heat conduction equation given in this problem from Equation (2.83).

(b) Show that the solution, $T_e(z)$, of the steady-state heat conduction equation is given by

$$T_e^{7/2} = T_{e0}^{7/2} + (7/2)(Q_{e0}/C)(z - z_{bottom})$$

where C is the constant in Equation (2.77).

(c) Let $T_{e0} = 300$ K at $z_{bottom} = 100$ km and let $z_{top} = 1000$ km. Also let the heat flux into the top of the ionosphere have a (typical) value of $Q_{e0} = 10^{13}$ eV/m²/s. Plot the electron temperature as a function of altitude for these conditions. What is T_e at the top of the ionosphere?

2.15   Demonstrate by substitution that the electrostatic potential given by Equation (2.99) is the solution of the differential Equation (2.97).

2.16   The Debye shielding length was found in Chapter 2 with the assumption that the ions were stationary. Repeat the analysis given in the chapter, but for mobile ions (which are in thermal equilibrium like the electrons) with temperature T_i, and find a new expression for the Debye length λ_D. How does this expression for λ_D compare with the expression given by Equation (2.98)?

2.17   Demonstrate that the plasma parameter is approximately (to within a factor of order unity) given by the ratio of the average potential energy of an electron in the plasma to the average kinetic energy:

$$g_{plasma} \approx \langle PE \rangle / \langle KE \rangle.$$

Assume that the relevant value of the test charge is $q_T = e$.

Bibliography

Banks, P. G. and G. Kockarts, *Aeronomy*, Academic Press, New York, 1973.

Bhatnagar, P. L., E. P. Gross, and M. Krook, A model for collision processes in gases, I, *Phys. Rev.*, **94**, 511, 1954.

Bittencourt, J. A., *Fundamentals of Plasma Physics*, Pergamon Press, New York, 1986.

Burgers, J. M., *Flow Equations for Composite Gases*, Academic Press, New York, 1969.

Gombosi, T. I., *Gaskinetic Theory*, Cambridge Univ. Press, Cambridge, UK, 1994.

Krall, N. A. and A. W. Trivelpiece, *Principles of Plasma Physics*, McGraw-Hill, New York, 1973.

Nicholson, D. R., *Introduction to Plasma Theory*, Wiley, New York, 1983.

Rees, M. H., *Physics and Chemistry of the Upper Atmosphere*, Cambridge Univ. Press, Cambridge, UK, 1989.

Spitzer, L., Jr., *Physics of Fully Ionized Gases*, Interscience Publishers (J. Wiley and Sons), New York, 1962.