

Review material for midterm exam (March 18, 2015)

Although I'm not recommending full-on memorization of everything in this document, I do think it's important that you get to a point of *familiarity* with all of the ideas and equations. Nothing in this document should be a surprise to you, is what I'm saying.

### Math & Physics Overview

Various energy quantities associated with individual particles (mass  $m$ , charge  $q$ ):

Kinetic energy	$E = \frac{1}{2}mv^2$	$(v \ll c)$
Thermal energy (per particle)	$E = \frac{3}{2}k_B T$	
Energy of a photon	$E = h\nu = \frac{hc}{\lambda}$	
Gravitational potential energy	$E = -\frac{Gm_1m_2}{r}$	
Electric-field potential energy	$E = \frac{k_e q_1 q_2}{r}$	$k_e = \frac{1}{4\pi\epsilon_0}$

Last two: energy felt by particle 1 in field of particle 2, when they are separated by distance  $r$ .

**Vector calculus:** Be able to apply (i.e., write out in full and manipulate) the various identities and coordinate-specific versions of the vector derivatives (grad, div, curl) in the useful-formula handout.

Electromagnetic **Lorentz force** on a particle embedded in electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$ :

$$\mathbf{F} = m\mathbf{a} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The combined system of E&M fields and charged particles obeys all four **Maxwell's equations**, which relate  $\mathbf{E}$  &  $\mathbf{B}$  to:

- $\rho_c$  (charge density): just what it says: how much electric charge is concentrated into a given volume (Coulombs/m<sup>3</sup>).
- $\mathbf{J}$  (current density): how much charge is in motion in a given volume (units:  $\rho_c \mathbf{v}$ ... also: Amps/m<sup>2</sup>). In a hydrogen plasma,  $\mathbf{J} \approx e(n_p \mathbf{v}_p - n_e \mathbf{v}_e)$ .

I won't list Maxwell's equations here. If you need them, I'll give them in the exam.

To bring this into the context of the Sun and the solar system, you should be familiar with the overall “story” of the changing forms that energy takes on its journey from the Sun to us:

1. Before the Sun was born, it was a giant interstellar gas cloud. It became unstable to *gravitational collapse*, so the formation of the Sun was all about accumulating **gravitational potential energy**.
2. As the new Sun settled into equilibrium, it also converted part of that gravitational energy into **thermal energy** (i.e., its core became hotter and hotter).
3. When  $T_{\text{core}}$  reached a threshold value, the motions of H nuclei became so rapid that they slammed into one another and induced *thermonuclear fusion*. Nucleons arrange themselves into more tightly-bound forms (via the **strong-force potential energy**).
4. Fusion reactions are exothermic, so they give off photons that carry **radiant energy** out through much of the Sun’s interior.
5. Near the solar surface, though, the most efficient way of transporting energy is no longer radiation. *Convection cells* form naturally and transport the energy upwards by a combination of **kinetic & thermal energy** (i.e., bulk flow of hot parcels).
6. At the solar photosphere, **radiant energy** becomes the most efficient way of getting the energy out. However, there is also still some residual **kinetic energy** in the convective “granulation” motions, and **magnetic energy** in the field lines that thread the surface.
7. More than 99.99% of the Sun’s power (generated ultimately by fusion) comes out in the form of radiation. Different parts of the spectrum eventually interact with different layers of the Earth’s atmosphere:
  - Visible/IR: Makes it down to the troposphere (cloud layer) and solid surface. Photons absorbed by solid matter & re-emitted as  $\sim$ blackbody.
  - Near UV: Absorbed in stratosphere (ozone layer); excites electrons & dissociates molecules (endothermic reactions).
  - Far UV/X-ray: Absorbed higher up: ionizes the ionosphere.
8. The outermost layers (chromosphere, corona, solar wind  $\rightarrow$  planetary magnetospheres) undergo additional transformations of energy, but we’ll be covering those in the last part of the course... not on this exam.

(Note: this page is kind of OPTIONAL, in that it helps you maintain the “big picture” in your mind, but it’s not going to correspond to any specific facts or figures on the exam.)

## Plasmas & Fluids

A plasma is an ionized gas... most generally composed of neutral **atoms**, positively-charged **ions**, and free, negatively-charged **electrons**.

In nearly all cases that we care about, the plasma is *quasi-neutral*, i.e., the net charge density  $\rho_c \approx 0$  everywhere. (Charge imbalances get quickly “shorted out” by Coulomb attraction.)

In nearly all cases that we care about, the microscopic velocity distribution function  $f(\mathbf{v})$  is a drifting Maxwell-Boltzmann distribution. It’s parameters are:

$$\text{number density } n = \int d^3\mathbf{v} f(\mathbf{v}) \quad (\text{in units of particles per unit volume})$$

$$\text{bulk/fluid flow speed } \mathbf{u} = \langle \mathbf{v} \rangle = \frac{\int d^3\mathbf{v} f \mathbf{v}}{\int d^3\mathbf{v} f} \quad (\text{in units of m/s})$$

$$\text{gas pressure } P, \text{ related to temperature } T \text{ via the ideal gas law: } P = nk_B T$$

The distribution function  $f(\mathbf{v})$  obeys the Vlasov (or Boltzmann) equation, which is essentially just “no particles are created or destroyed.”

If we multiply each term in the Vlasov equation by various factors of the microscopic/random velocity  $\mathbf{v}$ , and also *integrate* over all  $\mathbf{v}$ , we get the **fluid conservation equations**:

$$\left\{ \begin{array}{l} \text{0th moment: mass conservation} \\ \text{1st moment: momentum conservation} \\ \text{2nd moment: energy conservation} \end{array} \right\}$$

For a mixture of multiple elements (species “ $s$ ”), we define

$$\text{total number density } n_{\text{tot}} = \sum_s n_s \quad \text{mass density } \rho = \sum_s m_s n_s = n_{\text{tot}} \mu m_H$$

$$\text{total gas pressure } P = \sum_s n_s k_B T_s \approx n_{\text{tot}} k_B T = \frac{\rho k_B T}{\mu m_H}$$

where  $\mu$  is the mean mass per particle in units of  $m_H$  (neutral solar mixture:  $\mu \approx 1.26$ , ionized mixture:  $\mu \approx 0.6$ ).

Summing together the fluid conservation equations for each species gives the conservation equations of **magnetohydrodynamics (MHD)**:

$$\text{mass conservation} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\text{momentum conservation} \quad \rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P - \rho \mathbf{g} - \mathbf{J} \times \mathbf{B} = 0$$

and Ampere’s law (with the displacement current term assumed to be zero) can be used to express  $\mathbf{J}$  as proportional to  $\nabla \times \mathbf{B}$ .

In time-steady systems with no motion ( $\mathbf{u} = 0$ ), the momentum equation reduces to the special case of **hydrostatic equilibrium**:

- When  $\mathbf{B} = 0$ ,

$$\nabla P = \rho \mathbf{g}$$

which reduces for an isothermal ( $T = \text{constant}$ ) and thin/Cartesian atmosphere ( $g = \text{constant}$ ) to a simple function of height  $z$ ,

$$\frac{\partial P}{\partial z} = -\rho g \quad \rightsquigarrow \rightsquigarrow \rightsquigarrow \quad \rho(z) = \rho_0 \exp\left(-\frac{z}{H}\right) \quad \text{where} \quad H = \frac{k_B T}{\mu m_H g}.$$

- When  $\mathbf{g}$  is unimportant but there is a magnetic field,

$$\nabla P = \mathbf{J} \times \mathbf{B}$$

and we showed that  $\mathbf{J} \times \mathbf{B}$  can be broken up into two terms: a magnetic pressure and a magnetic tension. **Magnetic pressure** works similarly to gas pressure, so that we often group them together as

$$\nabla P_{\text{tot}} = \nabla (P_{\text{gas}} + P_{\text{mag}}) = \nabla \left( P + \frac{B^2}{2\mu_0} \right)$$

i.e., a bunched-up region of field lines wants to *expand*, just like a region of high  $P_{\text{gas}}$ .

The **thermal energy equation** contains lots of terms, but we often care about simpler cases:

- When there's no external heating or cooling of a "parcel," the gas evolves **adiabatically**:  $P \propto \rho^\gamma$ , where  $\gamma = 5/3$  for an ideal gas.
- When the system is in time-steady equilibrium, it's only the "right-hand side" (sources & sinks) that matters. The total heating rate  $H$  is balanced by the total cooling rate  $C$ . If both rates depend on temperature  $T$ , then we can solve for the equilibrium value of  $T$  at which  $H = C$ .

Lastly, the magnetic field evolves by obeying the **magnetic induction equation**, which comes from Faraday's law and Ohm's law...

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + D_B \nabla^2 \mathbf{B}$$

where  $D_B$  is the magnetic diffusion coefficient (due to friction-like resistivity of the plasma).

When  $D_B = 0$ , the 1st term on the right-hand side tells us the field lines are "frozen-in" to the flow... i.e., the magnetic field is carried along by the plasma velocity vector  $\mathbf{u}$ .

When the 2nd term on the right-hand side dominates, it's a diffusion equation, and the magnetic energy is slowly dissipated away and turned into heat.

## The Solar Interior: Energy Generation

A star releases energy continually with **luminosity** (i.e., power)  $L_*$ . The energy flux (power per unit area) at the surface is given by

$$F = \frac{L_*}{4\pi R_*^2} = \sigma T_{\text{eff}}^4 \quad (\sigma = \text{Stefan-Boltzmann constant})$$

where the effective temperature  $T_{\text{eff}}$  is essentially defined by the above equation. It's also the actual temperature of the *solar photosphere*.

How is the Sun's power generated?

Eddington determined that slow **gravitational contraction** could produce some net energy release. However, we derived the "virial theorem" to show that, at the present-day value of  $L_\odot$ , the Sun would only live for  $\sim 10^7$  years before contracting all the way to nothing.

Thus, there must be some extra source of energy generation inside the Sun to have kept it alive for  $> 4 \times 10^9$  years.

**Nuclear fusion** is of course the answer.

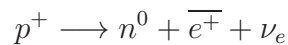
A  ${}^4\text{He}$  nucleus has a lower *mass energy* ( $E = mc^2$ ) than the sum of four separate  ${}^1\text{H}$  nuclei. The difference between these 2 quantities is the relative *binding energy*  $B$  of  ${}^4\text{He}$ .

The nucleus is sitting "deeper" in the potential well (i.e., larger  $B$  means it's more stable).

If you want to break it apart, you'd need to add energy  $B$  to the system. On the other hand, when 4 protons combine into one  ${}^4\text{He}$ , there's extra energy  $B$  "left over" to be released in the form of high-energy photons.

There are **strong** reactions, where two positively-charged nuclei ram into one another (if their kinetic energies are large enough to fight against Coulomb repulsion), form a new, heavier nucleus, and release photons. . . . . Note: forming *heavier* nuclei requires higher  $T$ .

There are also **weak** reactions, where protons and neutrons can turn into one another (if the nucleus has *too many* or *too few* of one type to be stable). One basic form is:

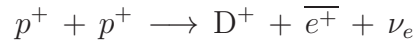


where other particles need to be emitted to conserve charge and other basic laws; i.e.,  $\overline{e^+}$  is a positron (anti-matter! it won't last long before annihilating itself with a corresponding particle of normal matter), and  $\nu_e$  is a neutrino (massless, but measurable).

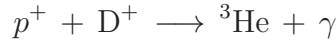
Strong reactions are *much* faster (i.e., they happen much more frequently) than weak reactions.

In the Sun, most of the  ${}^4\text{He}$  is formed by the PP-I chain, which has three steps:

(1) A weak (slow) reaction in which a proton decays into a neutron (only when near another proton) and then they can fuse into deuterium:



(2) Another proton is “captured” by a deuterium to make  ${}^3\text{He}$  in a strong (fast) reaction:



(3) Then, if there are two  ${}^3\text{He}$  nuclei in close proximity, there’s another strong (fast) reaction:



The other two ways of creating  ${}^4\text{He}$  (PP-II and PP-III chains) depend on there already being lots of  ${}^4\text{He}$  around, so they are important for stars with hotter interiors than the Sun.

## The Solar Interior: Energy Transport

How does the energy (mostly generated in the dense core) get out?

$$\text{Most types of } \textit{energy flux} \text{ look like: } \mathbf{F} \propto -\nabla T$$

i.e., a “steeper” temperature gradient (larger  $|\partial T/\partial r|$ ) allows the star to transport energy out more rapidly & efficiently.

Thus, we specify how multiple processes contribute to the total value of  $\partial T/\partial r$ :

- **Radiative diffusion:** photons bounce around randomly, but carry more energy *out* than *in*. Energy is transferred to the plasma via absorption; i.e., nonzero “opacity.”
- **Heat conduction:** similar to above, but with particles bouncing around randomly. Relatively unimportant in the Sun.
- **Convection:** the interior becomes unstable to the production of hot/rising blobs and cool/falling blobs, which carry more energy out than in.

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The radiation field is characterized by **specific intensity**  $I_\nu(\mu)$ , which depends on photon energy ( $E = h\nu$ ) and on photon direction ( $\mu = \cos\theta$ , where  $\theta = 0$  points radially outward).

$I_\nu$  measures the amount of photon energy that is emitted per unit time, per unit frequency, per unit area (passing through), and per unit solid angle (expanding into).

If the interior was completely homogeneous (i.e., nothing changing as a function of  $r$ ), the radiation field would be an isotropic blackbody **Planck function**:

$$I_\nu = B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \quad (\text{doesn't depend on direction angle } \theta)$$

Integrating over all  $\nu$ , this gives  $B = \int_0^\infty d\nu B_\nu(T) = \frac{\sigma T^4}{\pi}$

(but all you'll probably need to remember is that  $B \propto T^4$ ).

However, in the real solar interior,  $I_\nu$  in the outward direction ( $\theta \approx 0$ ) is *slightly larger* than  $I_\nu$  in the inward direction ( $\theta \approx 180^\circ$ ).

This slight imbalance shows up in the radial component of energy flux  $F_r$ , which we related to the radial derivative of the total energy density  $U \propto T^4$ , so that

$$\left(\frac{dT}{dr}\right)_{\text{rad}} \propto \kappa \rho F_r$$

where  $\kappa$  is the **absorption coefficient** of the plasma (a function of  $\rho$  &  $T$ ).

A *higher luminosity* makes a stronger temperature gradient, because more flux = more anisotropy in  $I(\mu)$ , and thus more of a radial change in  $U \propto T^4$ .

Also, a *large opacity* makes a stronger temperature gradient, too... because large opacity = “good insulation” that allows the core to retain its heat while being surrounded by the coldness of space.

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The other major transport mechanism is **convection**. There will always be small “blobs” that are displaced from their initial radii.

But what happens to them once they are displaced? If the initial displacement was “up,” then will they keep rising, or will they fall back down? If it keeps rising, it's *convective instability* (i.e., the blobs transport their energy over long distances).

We started with 3 assumptions:

1. The bubble starts with equal density  $\rho$  as the surroundings.
2. The bubble's evolution is slow enough to keep it in **pressure equilibrium** with its surroundings.
3. The slowness of the process also lets us assume the bubble's evolution in  $\rho$  and  $T$  is **adiabatic** – i.e., the bubble doesn't gain or lose heat to its surroundings.

Putting those together, we estimated the bubble's density at its “new” height. If it's **stable**, then it falls back down, and thus it must be denser than its surroundings (less buoyant).

We also used pressure equilibrium to rewrite everything in terms of the blob temperature: if  $P = \text{constant}$ , then a denser blob will be cooler (and a less dense blob will be hotter). We derived the Schwarzschild convective stability criterion,

$$\left|\frac{\partial T}{\partial r}\right| \leq \nabla_{\text{ad}} \frac{T}{P} \left|\frac{\partial P}{\partial r}\right| \quad \text{for stability.}$$

Note that **shallow** (flat) temperature gradients are stable, and **steep** temperature gradients are the most **unstable** to convection.

$$\text{For an ideal gas, } \gamma = \frac{5}{3} \quad \nabla_{\text{ad}} = \frac{\gamma - 1}{\gamma} = \frac{2}{5}.$$

In the Sun’s deep interior ( $0 < r \lesssim 0.7 R_{\odot}$ ), radiation is the main way of transporting the flux generated by fusion. It’s convectively *stable*.

Above  $\sim 0.7 R_{\odot}$ , convective instability takes over ( $|\partial T/\partial r|_{\text{rad}}$  “wants to” exceed the adiabatic gradient... so the blobs form and keep it close to adiabatic.)

Just below the photosphere ( $r \approx 0.99 R_{\odot}$ ) convection stops (i.e.,  $|\partial T/\partial r|$  gets shallow again), but we can still see the tops of convective cells: **solar granulation**.

## Helioseismology

The Sun isn’t completely hydrostatic and spherical. It oscillates with millions of small-amplitude “pulsation modes.”

At any fixed location ( $r, \theta, \phi$ ), one sees sinusoidal oscillations in  $\rho$ ,  $P$ ,  $T$  (oscillating around a mean value), and  $\mathbf{u}$  (oscillating around zero).

The oscillations have discrete frequencies because they’re in a bound “cavity.” (Think of “nodes” of vibration in a string when held fixed at both ends.)

We can observe brightness variations & Doppler-shift variations (i.e., line-of-sight projected velocity). The ones that we observe on the Sun are all **p-modes**, where the restoring force is the pressure-gradient force. The oscillations behave like trapped acoustic waves, which propagate at a phase speed given by the local **sound speed**,

$$\frac{\omega}{k} = c_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k_B T}{\mu m_H}}$$

where the quantities  $\rho$ ,  $P$ , and  $T$  are the time-averaged background (“zero-order”) quantities, not the fluctuating ones.

Recall that wavelength  $\lambda = 2\pi/k$  and the frequency  $f = \omega/2\pi$ . Also, a wave mode with  $n$  wavelengths from core to surface has  $\lambda \approx R_{\odot}/n$ . Thus,

$$f = \frac{n}{R_{\odot}/c_s} \quad (\text{if } c_s = \text{const}), \quad \text{but in general, } f \sim \frac{n}{\int_0^{R_{\odot}} dr/c_s(r)}$$

and by measuring  $f$  of “neighboring” modes ( $n, n + 1, n + 2$ , etc.) it’s possible to solve for what  $c_s(r)$  must be in the solar interior.

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## The Solar Dynamo

If magnetic diffusion was the only effect governing how the Sun's magnetic field evolves,  $\mathbf{B}$  would have decayed away to  $\sim$ zero. It needs the “flux-freezing” term to generate new field. How is this done? In general, by stretching, twisting, and folding the field lines.

Although we still don't have a complete understanding of the solar dynamo, we believe we know the basic order of steps: **the  $\alpha\Omega$  dynamo**.

Start with a *poloidal* (e.g., dipole) field with only  $B_r$  &  $B_\theta$  at solar minimum.

(1) The  $\Omega$  effect: the equator rotates faster than the poles (“differential rotation”), so the field lines get wrapped around the equator.

The field ends up stretched into the *toroidal* ( $B_\phi$ ) direction.

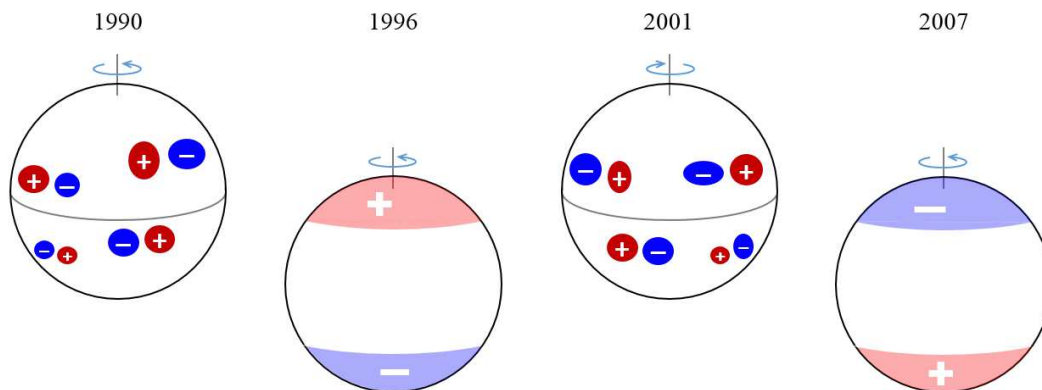
Also, some small regions (convective blobs? buoyant knots of strong  $\mathbf{B}$ ?) rise up and carry bits of toroidal field with it, forming east-to-west **sunspot pairs**.

In general, the differential rotation produces  $u_\phi > 0$  east-to-west flows near the equator, and the **Coriolis effect** deflects them toward the equator (in north, deflection is to the right; in south, deflection is to the left). This reproduces “Joy's law” (observed sunspots tilt toward the equator, too).

(2) The  $\alpha$  effect: There are two proposed general ideas:

- Babcock & Leighton realized sunspots decay, via magnetic diffusion. The near-equator leading spots cancel with one another, but the trailing spots diffuse their  $\mathbf{B}$ -field toward the poles, slowly building up poloidal field there.
- Parker noticed that sunspots act as sites of converging flow. In that case, the Coriolis force produces “cyclonic” twist (counterclockwise in north; clockwise in south). This twists the rising fields in a sunspot pair, from toroidal back to poloidal (north-to-south) direction. The sense of the “new” poloidal field is opposite to the previous poloidal field.

We see this in action, in the form of Hale's polarity laws:



One min  $\rightarrow$  max  $\rightarrow$  min cycle is  $\sim$ 11 years, but to get back to the same polarity, it takes double that:  $\sim$ 22 years. We still don't know why it's 11, and not 1 or 100.