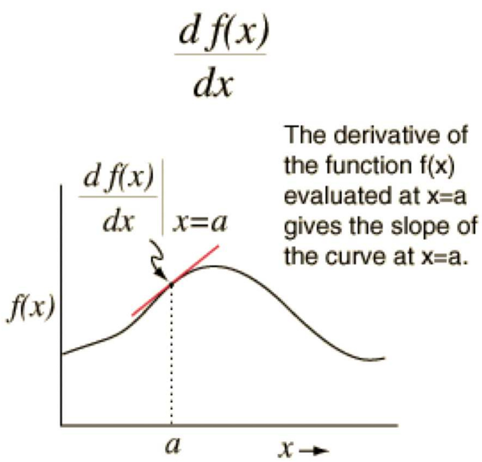
Partial Derivatives, Multiple Integrals, & Vector Calculus

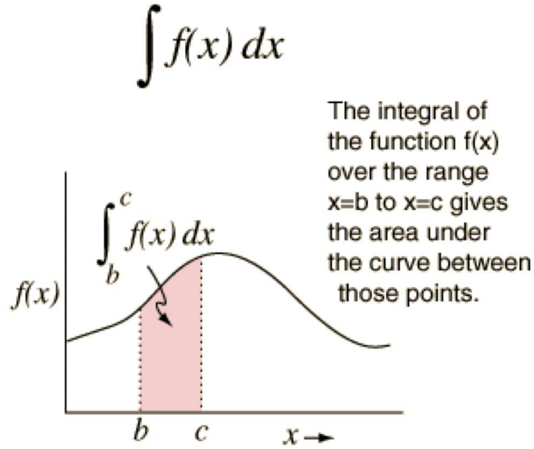
This is essentially part two of our introduction to vectors. We've learned how to add, subtract, and multiply them, and to take derivatives of vectors, but only with respect to scalars like **time.**

However, that doesn't really explore the full variety of derivatives and integrals that one can do... **in 3D...**

Topics: (1) Partial Derivatives (∂) and the Gradient (∇)
(2) Multiple Integrals
(3) Other Kinds of Vector Derivative (Div & Curl)
(4) Partial/Vector Differential Equations (a few examples)

Previously in calculus, taking derivatives and integrals assumed that the functions depend on just one variable, typically called x,





However, we've seen in physics that we can have functions of two, three, or four variables:

$$f(x,y) \qquad\qquad T(x,y,z) \qquad\qquad \mathbf{v}(x,y,z,t) \ .$$

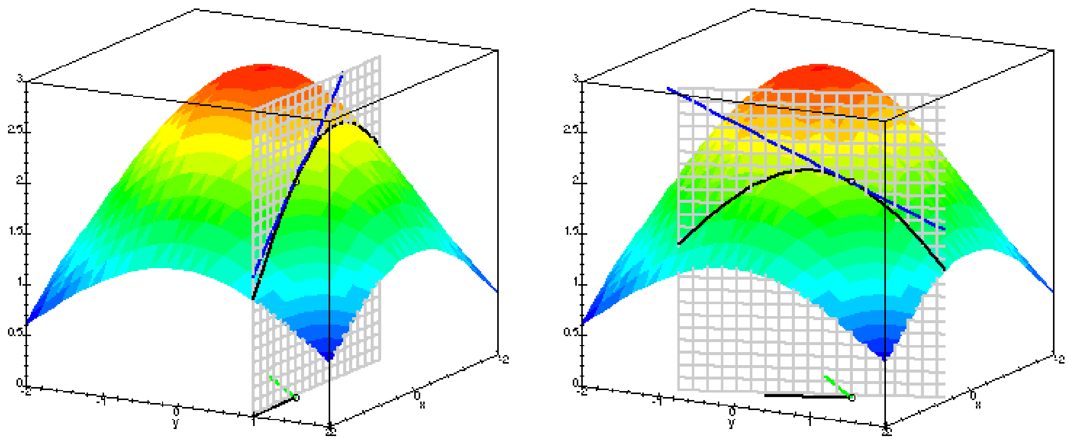
The more variables, the more difficult it tends to be to make plots & graphs of these functions.

Thus, many of our examples now will just be two-variable functions, like $f(x,y)$. Everything generalizes to more dimensions.

Think of f as mountain-range "elevation," and (x,y) as position along the ground (north–south, east–west).

We can now introduce **PARTIAL DERIVATIVES.**

Just like normal derivatives, they are still interpreted as slopes. But if it's a multidimensional function, there are multiple ways to "cut through the mountain" to define the slopes:



Remember our old definition of the derivative with respect to one variable:

$$f(t): \qquad \frac{df}{dt} = \lim_{\epsilon \to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon}$$

can be modified for functions of multiple variables, as long as we vary only **one variable at a time:**

$$f(x,y): \qquad \frac{\partial f}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon, y) - f(x,y)}{\epsilon}$$

$$\text{and} \qquad \frac{\partial f}{\partial y} = \lim_{\epsilon \to 0} \frac{f(x, y+\epsilon) - f(x,y)}{\epsilon} .$$

The rounded ∂ symbol is often just called *partial,* as in $\partial f/\partial x$ is spoken as "partial of f with respect to x."

The way we actual **do** the derivatives is just to assume the "other" variables are all held constant.

Example:

$$f(x,y,z) = e^{xy} + \frac{x}{y}$$

Determine $\partial f/\partial x$ and $\partial f/\partial y$ and $\partial f/\partial z$.

For $\partial f/\partial x$, just pretend y & z are constants:

$$\frac{\partial f}{\partial x} \;=\; ye^{xy} + \frac{1}{y}\;.$$

For $\partial f/\partial y$, just pretend x & z are constants:

$$\frac{\partial f}{\partial y} \;=\; xe^{xy} - \frac{x}{y^2}\;.$$

For $\partial f/\partial z$, just pretend x & y are constants:

$$\frac{\partial f}{\partial z} \;=\; 0\;.$$

. . . .

We can define **higher-order** partial derivatives, and it's interesting that the "types" proliferate...

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad\qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} \qquad\qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}\;.$$

If you work through "well-behaved" examples, you'll find that it's *nearly always* true that

$$\frac{\partial^2 f}{\partial x\,\partial y} \;=\; \frac{\partial^2 f}{\partial y\,\partial x}\;.$$

We won't see these "mixed second partials" very much, though.

There are also new versions of the **chain rule.** Recall the old version. If we know a function $f(x)$, but we also know $x(t)$, then it's possible to figure out that

$$\frac{df}{dt} \;=\; \frac{df}{dx}\,\frac{dx}{dt}\;.$$

There are two new versions:

(a) Multi-then-One:

Start with a function of multiple variables, say, mountain height $h(x, y)$. Then, assume each of those variables depends on one variable (like time t). Think of the ocean, where you can have waves flowing north–south, colliding with waves flowing east–west.

Thus, how does the height change with time, at a given location, when feeling both kinds of "sloshing" simultaneously?

The answer is that we must **sum** over both variations:

$$\text{i.e., for} \quad h(x(t), y(t)) \quad \Longrightarrow \quad \boxed{\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}}$$

and note that ordinary derivatives (d/dt) are taken when there's just one variable (t), and partials are taken when there are two (x, y).

(b) Multi-then-Multi:

Here we're dealing with some function of multiple variables $f(x, y)$ and **each** of the coordinates depends on some set of multiple **other** variables.

Let's use the example of coordinate systems. Think of a 2D plane, which can be described by either Cartesian or polar coordinates. You know the conversions:

$$x = r\cos\phi \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\phi \qquad \phi = \tan^{-1}(y/x)$$

and it's clear you could describe "mountain height" by either $f(x, y)$ or $f(r, \phi)$.

However, each coordinate depends on **all** of the others in the other set, too. Thus, these functions can be written as

$$f(x(r, \phi), y(r, \phi)) \qquad \text{or} \qquad f(r(x, y), \phi(x, y)) \; .$$

If you start by describing slopes in one coordinate system, you can convert to the other by using the multi/multi chain rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \qquad = \cos\phi\,\frac{\partial f}{\partial x} + \sin\phi\,\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} \qquad = -r\sin\phi\,\frac{\partial f}{\partial x} + r\cos\phi\,\frac{\partial f}{\partial y}$$

where the black equations are general, and red apply to this coordinate-system example only.

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We now define a new kind of spatial derivative: **the gradient.**

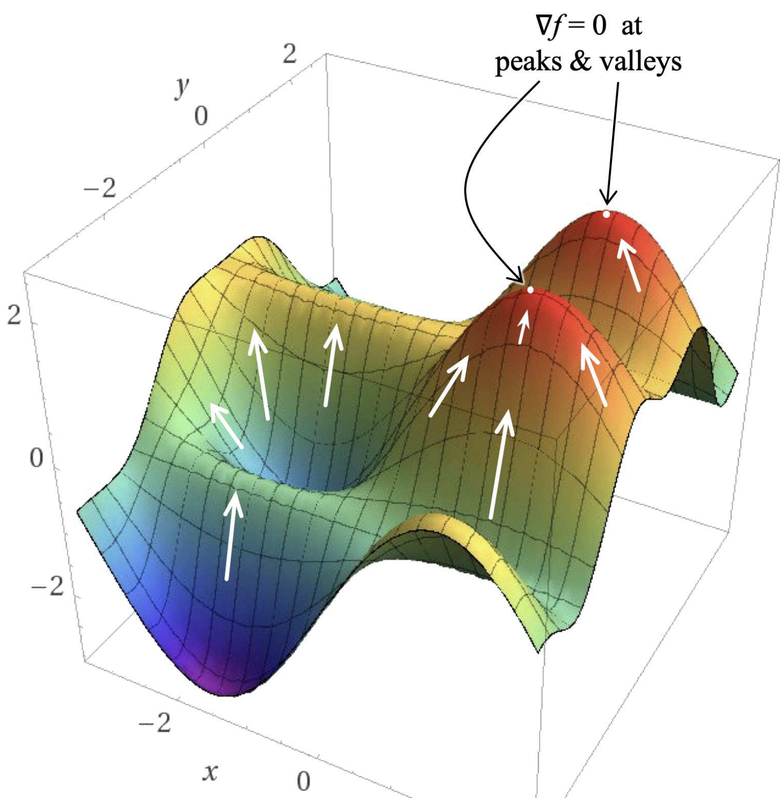
Let's first define it, then discuss what it means. If we have a scalar function of 3D space $f(x, y, z)$, the gradient of f is a vector comprised of the partial derivatives of f in each direction:

$$\mathbf{grad}\, f \;=\; \nabla f \;=\; \hat{\mathbf{e}}_x \frac{\partial f}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial f}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial f}{\partial z}$$

The symbol ∇, sometimes called "del" or "nabla" (latter: Greek word for a type of harp) is a **vector operator.** Some people render it as $\boldsymbol{\nabla}$ or $\vec{\nabla}$ to make sure we remember that.

The direction of ∇f points along the direction of **fastest growth,** or most rapid increase, in f.

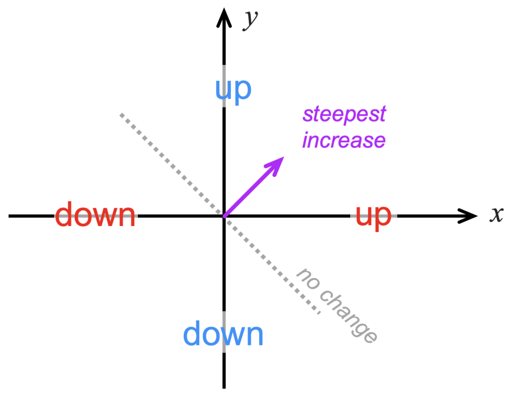
In other words, if $f(x, y)$ is the height of a mountain, ∇f points in the direction of **steepest ascent:**



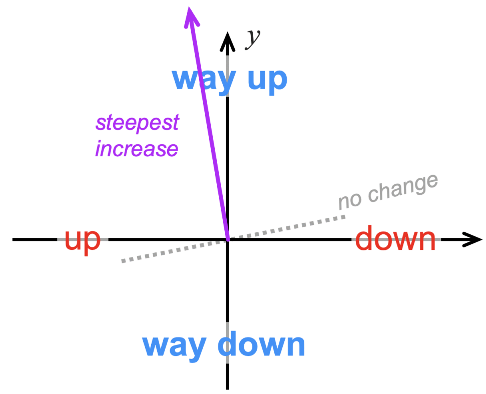
The magnitude of ∇f tells us **how** steep is the steepest slope.

Let's look at some specific examples for $f(x, y)$:

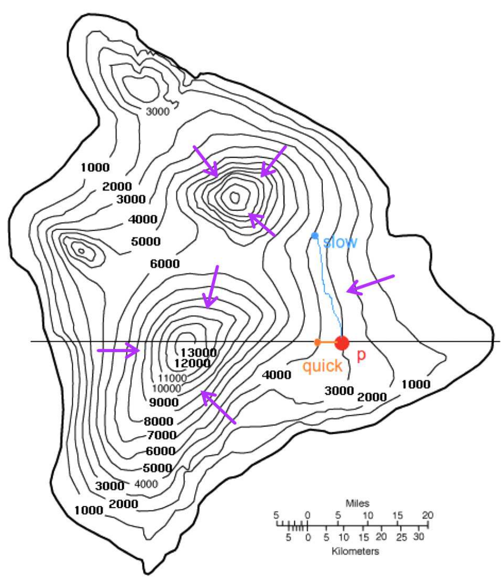
$\dfrac{\partial f}{\partial x} = +1 \qquad \dfrac{\partial f}{\partial y} = +1$



$\dfrac{\partial f}{\partial x} = -1 \qquad \dfrac{\partial f}{\partial y} = +10$



Thus, constructing the vector ∇f is just like constructing the "position vector" $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$. Go east this much, go north that much, then go up this much. But here, we do it with **slopes.**



The directions along which NOTHING changes (i.e., "level surfaces") are always perpendicular to ∇f.

In other words, if we draw "mountain height" as contours, the gradient always points along a 90° direction to the local contour curves.

Do you want to take the **shortest path** from one "level" up to the next? Just follow the gradient.

In physics, most gradients will be 3D: $\nabla f(x, y, z)$. Think of a dog sniffing for a scent... the vector points toward the **most rapid increase** in the concentration of f.

. . . .

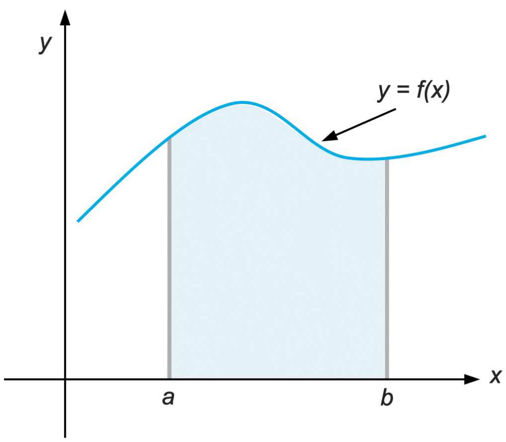
There's one additional piece of information that may be helpful to visualizing what a gradient is...

Remember our coordinate-system **unit vectors?** They point along the direction of **maximum growth** for a given coordinate. Thus, the formal way to define one of them is:

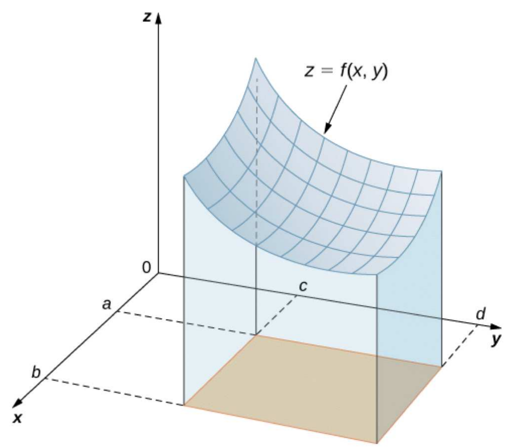
$$\hat{\mathbf{e}}_w = \frac{\nabla w}{|\nabla w|} \qquad \text{where } w \text{ could be } x,\ y,\ r,\ \theta, \text{ etc.; any coordinate.}$$

We've extended the concept of derivatives to multiple dimensions... and we can do that with integrals, too. Thus: **MULTIPLE INTEGRALS.**

If a regular 1D integral gives us the **area** under a curve, then we can define a 2D integral to give us the **volume:**



$$\int_a^b dx\ f(x)$$



$$\iint dA\ f(x,y)$$

How do we carry out this new kind of **double integral?** Similar philosophy as taking partial derivatives... just do one dimension after another, while holding the *other* variable constant.

There are two ways to do it:

$$\iint dA\ f(x,y) \ = \ \int_a^b dx \left[\int_c^d dy\ \boldsymbol{f(x,y)}\right] \ = \ \int_c^d dy \left[\int_a^b dx\ \boldsymbol{f(x,y)}\right]$$

where, in each case, the successive integrals are evaluated "inside-out..." (see next page!)

- In the 1st case, the **red *f*** is evaluated by integrating with respect to y, **holding $\boldsymbol{x}$ constant.** Then the result in square brackets is just a function of x, and the outer integral is done with respect to x as usual.
- In the 2nd case, the **blue *f*** is evaluated by integrating with respect to x, **holding $\boldsymbol{y}$ constant.** Then the result in square brackets is just a function of y, and the outer integral is done with respect to y as usual.

Fubini's theorem: because we're just summing up little pieces of volume, it doesn't matter in which order you do the "partial integrals."

I think nearly all of the multiple integrals in this course will be given as **definite** integrals (i.e., "known" limits in all dimensions).

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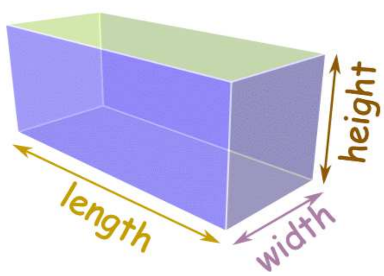
Silly example: what's the volume of a box of length L, width W, height H?



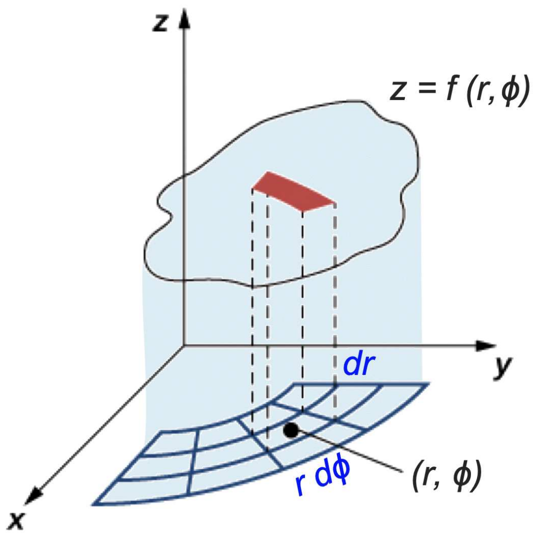
The function to integrate is the height, $z = f(x,y) = H$. The integration limits are straightforward...

$$V \;=\; \iint dA\; f(x,y) \;=\; \left\{ \int_0^L dx \left[\int_0^W dy\; H \right] \right\}$$

$$=\; H \left\{ \int_0^L dx \left[y \right]_0^W \right\} \;=\; WH \left\{ \int_0^L dx \right\} \;=\; WH \Big\{ x \Big\}_0^L$$

$$=\; LWH\,. \qquad \checkmark$$

We can extend the concept of multiple integrals to other coordinate systems, but we have to use the proper volume elements. For example, cylindrical coordinates:



Instead of integrating over a rectangle in the (x, y) plane, it's straightforward to integrate over a **"wedge"** in (r, ϕ).

This way, the limits on the integrals (a, b, c, d) can just be numbers.

However, the "area element for the x, y plane in cylindrical coordaintes is:

$$dA \;=\; (dr)(r\,d\phi) \;=\; (r\,dr)(d\phi)$$

and that rearrangement is useful for figuring out what goes into each partial integral:

$$\iint dA\; f(r,\phi) \;=\; \int_{r=a}^{r=b} dr\; r \int_{\phi=c}^{\phi=d} d\phi\;\; f(r,\phi)$$

Written this way, the inner integral is done with respect to ϕ (holding r constant), then the result is a function of r. The outer integral contains an "extra" factor of r that can't be neglected.

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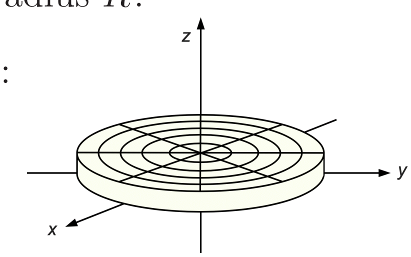
Another silly example: What's the area of a circle of radius R?

This time, z is just a "dummy coordinate" of height 1:



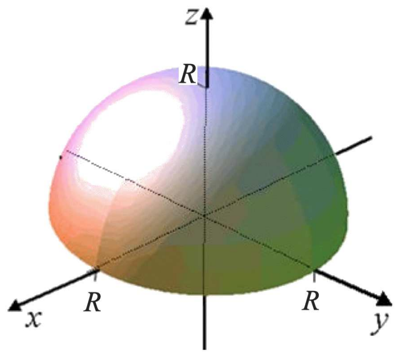
The real action is in the (r, ϕ)-plane **limits:**

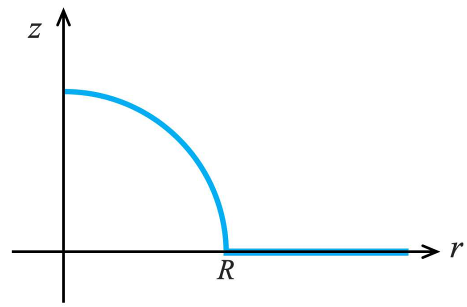
- Limits in ϕ: 0 to 2π ... all the way around.
- Limits in r: 0 to R only.

$$A \;=\; \int_{r=0}^{r=R} dr\; r \int_{\phi=0}^{\phi=2\pi} d\phi\; [1] \;=\; 2\pi \int_{r=0}^{r=R} dr\; r \;=\; 2\pi \left[\frac{r^2}{2}\right]_0^R \;=\; \pi R^2 \quad \checkmark$$

. . . .

Another example: What's the volume of a sphere of radius R? We will start with just half of this problem: a hemisphere:





The "height" of the surface is given by

$$z \;=\; f(r,\phi) \;=\; \begin{cases} \sqrt{R^2 - r^2}\ , & \text{if } r \le R, \\ 0\ , & \text{if } r > R. \end{cases}$$

and note that the function doesn't depend on ϕ. The volume is a double integral with the same limits as the circle case:

$$V \;=\; \int_{r=0}^{r=R} dr\ r \int_{\phi=0}^{\phi=2\pi} d\phi\ f(r,\phi) \;=\; 2\pi \int_{r=0}^{r=R} dr\ r\ \sqrt{R^2 - r^2}$$

I looked up how to do this integral in Wolfram Alpha...

$$V \;=\; -\frac{2\pi}{3}\left[(R^2 - r^2)^{3/2}\right]_0^R \;=\; \frac{2\pi R^3}{3}\ .$$

This is just for the "top half." The full sphere volume is double that, so

$$V_{\text{sphere}} \;=\; \frac{4\pi R^3}{3} \qquad \checkmark$$

. . . .

It's interesting how you can get <u>either</u> areas <u>or</u> volumes out of double integrals.

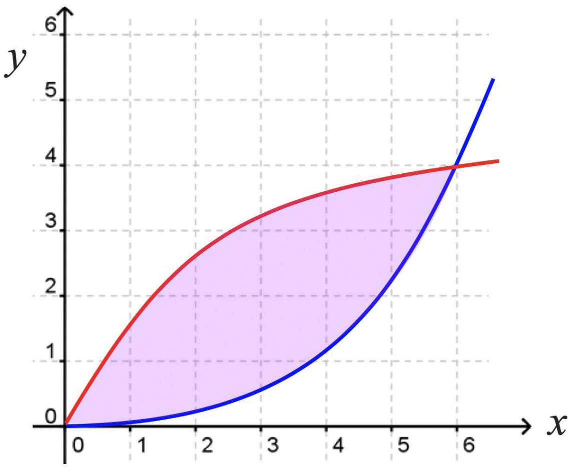
The process of integrating over dA gives you an A for the *base.* Then, if the integrand is a *height,* then you've got $V = \{\textit{area of base}\}\{\textit{height}\}$.

Alas, not all multiple integrals can be evaluated with integration limits that are just pure numbers.

- In 2D Cartesian, "pure numbers" limits us to dA = rectangles.
- In 2D cylindrical, "pure numbers" limits us to dA = circular wedges.

What if the dA shape is more general?

If we can describe the shape with functions of x and y, we can still do these kinds of integrals. But now, **order matters.** Let's see an example:



The bottom and top of the purple region are defined by:

$$\boldsymbol{y = \frac{x^3}{54}} \quad \text{and} \quad \boldsymbol{y = \sqrt{\frac{8x}{3}}}$$

The left and right are defined by:

$$\boldsymbol{x = 0} \quad \text{and} \quad \boldsymbol{x = 6}\ .$$

At any point along the x axis, between 0 and 6, the upper and lower limits for the y integral are **known** (just like if the region was a simple rectangle), it's just that those limits **VARY** as a function of x.

But that's okay! If the inner integral is done over y, those x-dependent limits will just carry over into the outer integral.

Let's figure out the **area** of the purple region. Thus, our integrand is just $f(x, y) = 1$,

$$A \;=\; \int_{x=0}^{x=6} dx \int_{y=x^3/54}^{y=\sqrt{8x/3}} dy\,[1] \;=\; \int_0^6 dx \left[\sqrt{\frac{8x}{3}} - \frac{x^3}{54}\right] \;=\; \rightsquigarrow \rightsquigarrow \;=\; \mathbf{10}\ .$$

So, really, the area is just the area under the red curve, minus the area under the blue curve, as it ought to be.

But we can now put in other integrands, to get volumes!

Let's note one more thing: **we can change the order** of integration, but we have to be careful. If we turn our heads 90°, the new bottom & top of the purple region are described by "inverse functions:"

$$\boldsymbol{x = \frac{3y^2}{8}} \qquad \text{and} \qquad \boldsymbol{x = (54y)^{1/3}}$$

and the "left" and "right" sides are defined by:

$$\boldsymbol{y = 0} \qquad \text{and} \qquad \boldsymbol{y = 4}\ .$$

Thus, we can also compute the area in a new way, and it ought to have the same value...

$$A \ = \int_{y=0}^{y=4} dy \int_{x=3y^2/8}^{x=(54y)^{1/3}} dx\,[1] \ = \int_0^4 dy \left[(54y)^{1/3} - \frac{3y^2}{8}\right] \ = \rightsquigarrow \ = \ \mathbf{10}\ . \quad \checkmark$$

. . . .

All of our previous examples were "double" integrals. In physics, we often see **triple integrals.**

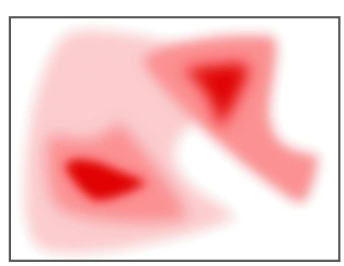
It's difficult to visualize how to integrate "under" a 3D curve $f(x, y, z)$. Our brains don't work in 4D space!

However, we've been lazy. In our 2D drawings, we've wasted one of those precious dimensions to show the **magnitude** of some quantity. (Yes, sometimes that magnitude was *truly* interpreted as another dimension—i.e., integrating the height over dA to get volume.)

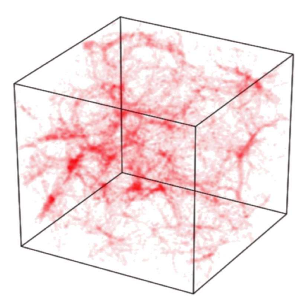
But we didn't need to do that. What we thought of the function f as the concentration of "stuff," say **red ink?**

Thus, $f(x)$ can be shown in 1D, $f(x, y)$ in 2D, and $f(x, y, z)$ in 3D (sort of):

1D: $f(x)$



2D: $f(x,y)$



3D: $f(x,y,z)$

Recall the way we pictured the gradient in 3D. Like a dog sniffing for a scent... the gradient $\nabla f(x, y, z)$ points toward the most rapid increase in the concentration of f.

Thus, if f is the "number density" of scent molecules in the air (# per m^3), the triple integral over $f(x, y, z)$ **sums up** the total number of molecules over a specified region of space:

$$\iiint dV \; f(x, y, z) \;=\; \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz \; f(x, y, z) \; .$$

Of course, if the function $f(x, y, z)$ is just 1, then the triple integral over dV by itself just gives the 3D **volume** of the region being integrated.

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In other coordinate systems, The "volume element" dV is the product of differential lengths in each dimension along the displacement vector $d\mathbf{r}$, with

$$\begin{array}{rlll} \text{Cartesian:} & dV = dx\,dy\,dz & & \\ \text{Cylindrical:} & dV = (dr)\,(r\,d\phi)\,(dz) & = & r\,dr\,d\phi\,dz \\ \text{Spherical:} & dV = (dr)\,(r\,d\theta)\,(r\,\sin\theta\,d\phi) & = & r^2\,\sin\theta\,dr\,d\theta\,d\phi \end{array}$$

As another example, we can **re**-compute the volume of a sphere by using spherical coordinates. With the integrand $f = 1$, it's just the product of 3 separate integrals:

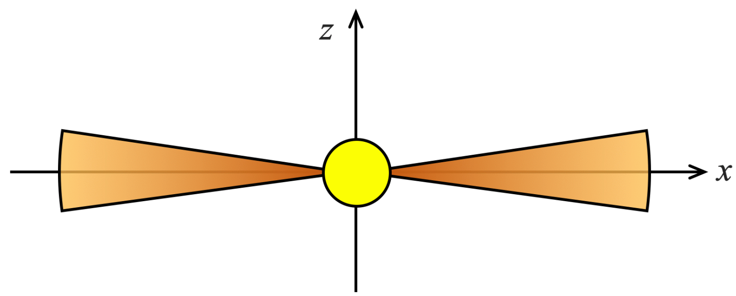
$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \int_0^R dr \; r^2 \;=$$

$$=\; 2\pi \Big[-\cos\theta \Big]_0^{\pi} \left[\frac{r^3}{3} \right]_0^R \;=\; 2\pi\,(2) \left(\frac{R^3}{3} \right) \;=\; \frac{4\pi R^3}{3} \; . \quad \checkmark$$

Another example: consider a **protoplanetary disk** around a newly-formed Sun-like star. It has an "opening angle" of 10° and is observed to be filled with stuff that has a mass density that depends only on radial distance:

$$\rho(r) \;=\; \rho_0 \left(\frac{R_\odot}{r}\right)^{3.5}$$

where $R_\odot$ is the radius of the star (assumed equal to the Sun) and ρ_0 is measured to be 3 kg/m^3.



What is the total mass of the disk? How does it compare to the mass of the central star ($M_\odot$)?

Density = mass/volume, so we need to integrate the density "over" the entire volume of the disk:

$$M_{\rm disk} \;=\; \iiint dV \;\rho(r,\theta,\phi)$$

where the geometry of this problem points us to spherical coordinates as the most natural choice. Thus...

$$M_{\rm disk} \;=\; \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_2} d\theta \,\sin\theta \int_{R_\odot}^{\infty} dr\, r^2\, \rho(r)$$

where $\theta_1 = 85°$ and $\theta_1 = 95°$ describe the 10° opening angle. So,

$$M_{\rm disk} \;=\; 2\pi\rho_0\, R_\odot^{3.5}\Big[-\cos\theta\Big]_{\theta_1}^{\theta_2}\Big[-2r^{-1/2}\Big]_{R_\odot}^{\infty} \;=\; 4\pi\rho_0\, R_\odot^3\Big[\cos\theta_1 - \cos\theta_2\Big] \;,$$

and that last part in square brackets is ≈ 0.174. Thus, $\boxed{M_{\rm disk} \approx 2.2\,\rho_0 R_\odot^3.}$

Plugging in ρ_0 and $R_\odot$, we get $M_{\rm disk}/M_\odot \approx 0.001$, which makes sense since that's similar to the total mass of all the **planets** in our solar system.

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In astronomy, we often work with **solid angles** on the sky. The entire sky is a full sphere; i.e., a "closed surface" that is sometimes denoted by a special integration symbol:

$$\oint d\Omega \;=\; \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \,\sin\theta$$

If the integrand is 1, you can see that the integral is just

$$\oint d\Omega \;=\; 2\pi \Big[-\cos\theta\Big]_0^{\pi} \;=\; 2\pi\,(\cos 0 - \cos\pi) \;=\; \boldsymbol{4\pi}$$

which is the number of steradians in a full sphere.

You may also see the $\oint$ symbol used for integrals like WORK, in cases where the path followed by the particle **closes back on itself.**

Next topic: **OTHER VECTOR DERIVATIVES**

You've seen deriv's of vectors with respect to scalars: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$

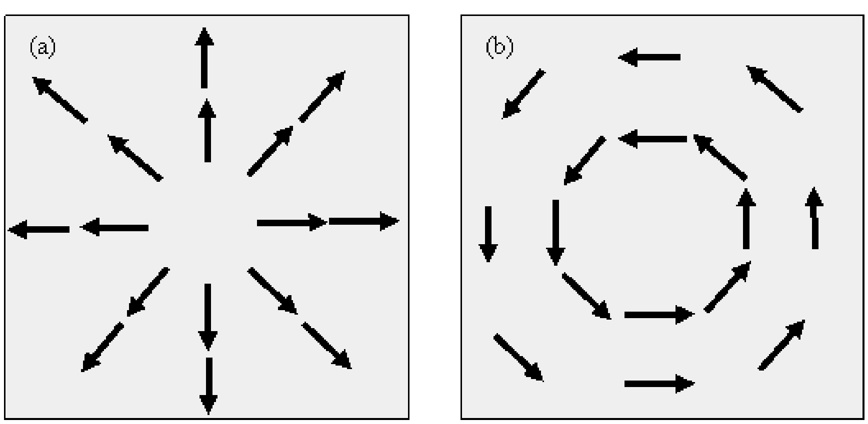
You've seen deriv's of scalars with respect to vectors: $\nabla f =$ "$\dfrac{df}{d\mathbf{r}}$"

and here we need to define 2 new types of derivatives of a **vector field** (like velocity $\mathbf{v}$ or force $\mathbf{F}$) with respect to the vector position $\mathbf{r}$.

Just like we had dot & cross products for vector multiplication, these two new vector derivatives are similar:

Divergence and **Curl:**

$\nabla \cdot \mathbf{F}$



$\nabla \times \mathbf{F}$

Divergence measures how much a field spreads out from a given point.
Curl measures how much a field swirls or torques around a given point.

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad\qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

or, for Cartesian coordinates:

$$\nabla \times \mathbf{F} \;=\; \mathbf{e}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad .$$

In other words, you can think of the ∇ operator as a vector with the components

$$\nabla \;=\; \mathbf{e}_x \left(\frac{\partial}{\partial x} \right) + \mathbf{e}_y \left(\frac{\partial}{\partial y} \right) + \mathbf{e}_z \left(\frac{\partial}{\partial z} \right)$$

and the definitions of gradient, divergence, & curl line up better with concepts of scalar multiplication, dot product, & cross product.

In other coordinate systems, the way to write ∇ is more complicated, just like for the extra terms in $\mathbf{v}$ & $\mathbf{a}$ vectors in dynamics. We can always look them up in the *"useful formula"* document.

Note: ∇ always represents a derivative with respect to **length units.**

We'll talk about each operation in more detail, including another new one called the **Laplacian** ($\nabla^2 f = \nabla \cdot \nabla f$), but for now here's a summary:

Gradient	$\nabla f \;=\;$ vector	Measures the rate & direction of maximum change in a scalar field
Divergence	$\nabla \cdot \mathbf{F} \;=\;$ scalar	Measures the strength of a point-like source or sink at a given location in a vector field
Curl	$\nabla \times \mathbf{F} \;=\;$ vector	Measures the tendency for a vector field to rotate about a given point
Laplacian	$\nabla^2 f \;=\;$ scalar	Measures how "steep" are local changes in a scalar field (i.e., large $\nabla^2 f$ means sharp edges exist in f)

More about Divergence

In general,

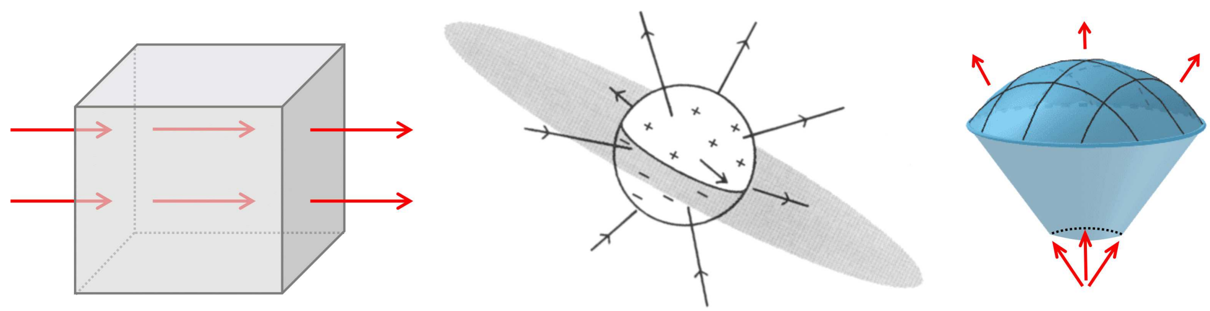
- If $\nabla \cdot \mathbf{F} > 0$, then $\mathbf{F}$ is an **expanding** vector field.
- If $\nabla \cdot \mathbf{F} < 0$, then $\mathbf{F}$ is an **converging** vector field.

If you can draw a little cube around a region, and the divergence is **positive inside,** you can think of that box containing an ever-spewing SOURCE of the stuff that $|\mathbf{F}|$ measures the concentration of.

Similarly, a little cube surrounding a region of **negative divergence** essentially contains a tiny "attractor," or SINK, that is continually allowing stuff to be pulled in (accumulate) over time.

The case of $\nabla \cdot \mathbf{F} = 0$ is special. If there are no sources or sinks, then that means no "particles" are being created or destroyed in the little box.

However, it's possible for there to still be some particles flowing **into** the cube, as long as an equal number are flowing **out.** Examples of **zero divergence:**



The first example is just $\mathbf{F} =$ constant. Any derivative of a constant is zero.

The second one is what we sometimes call a "split monopole." The field points inward in one hemisphere, outward in the other, so the **net flux** of stuff through the sphere is zero.

The third one is deceptive. It sort-of *looks like* a piece of a larger thing with positive divergence. But the actual result (i.e., is $\nabla \cdot \mathbf{F}$ + or − or 0?) depends on how much stuff is streaming through both ends. Does it balance?

In spherical coordinates, we can write it as a radial vector with $\mathbf{F} = (r^n)\hat{\mathbf{e}}_r$. The full definition of the divergence is:

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) \;+\; \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta) \;+\; \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

and thus, for our radial-only vector,

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{2+n}\right) \;=\; \begin{cases} (2+n)r^{n-1}\ , & \text{if } n \neq -2 \\ 0\ , & \text{if } n = -2 \quad ! \end{cases}$$

If you're in a region of space where $\mathbf{F}$ is dropping off as the inverse-square of radial distance, this means there are **NO local sources or sinks** where you are sitting now, at distance r from the origin.

(However, there may be sources or sinks *elsewhere;* say at the origin?!?)

We will see that vector fields due to gravity & electric charges act like this, when there are point-like sources and surrounding "free space."

Other vector fields, like the magnetic field, have zero divergence everywhere... i.e., "there are no magnetic monopoles."

. . . .

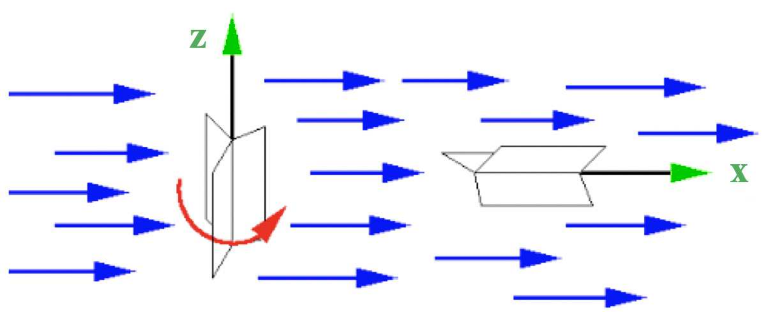
More about Curl

The right-hand rule is helpful for interpreting its meaning:

- If $(\nabla \times \mathbf{F})_x > 0$, then $\mathbf{F}$ swirls **counter-clockwise** around the x-axis.
- If $(\nabla \times \mathbf{F})_x < 0$, then $\mathbf{F}$ swirls **clockwise** around the x-axis.

However, the use of "swirling" may be misleading. A vector field may have $\nabla \times \mathbf{F} \neq 0$ even when NOT appearing to circulate.

Imagine little "paddle-wheels" oriented along each axis:



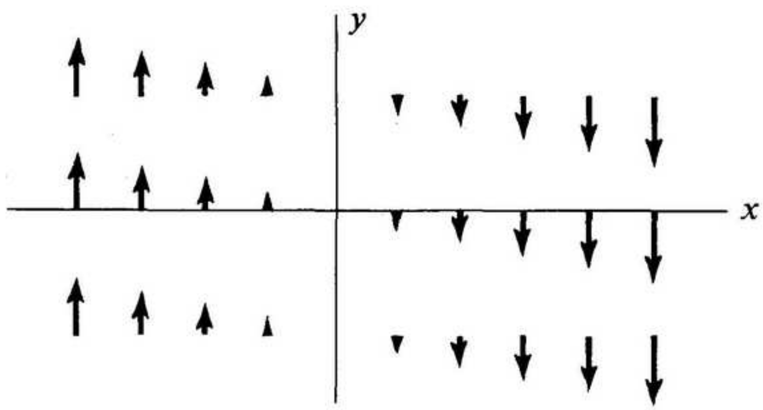
For the one aligned with the z axis, if the vector field $\mathbf{F}$ would push the paddle-wheel in the counter-clockwise direction, then the z-component of $\nabla \times \mathbf{F}$ would be positive.

If it pushes clockwise, then the z-component of $\nabla \times \mathbf{F}$ would be negative.

The magnitude of the curl (in each direction) is proportional to how "fast" the paddle wheel will turn; i.e., the strength of the "torque" around that axis.

Div was a scalar, but curl is a vector. Three components are needed, because 3D flows are complex. The flow *may* turn one of the wheels, while not affecting the others.

Note that you can have a nonzero curl even if the vectors are straight lines! The following **shear flow** has a curl:



In this case, the z-component of the curl vector is $\neq 0$. What's its sign?

. . . .

The handout gives various kinds of identities that boil down to the **chain rule.** Again: you don't have to memorize them, but it's good to know they exist.

Just like how dot & cross products "pick out" completely different (mutually orthogonal) projections of a vector, the div and curl (or the grad and curl) **cancel each other out:**

$$\nabla \times \nabla f = 0 \qquad\qquad \nabla \cdot \nabla \times \mathbf{F} = 0 \ .$$

Div and gradient are kind of like "siblings." Thus, when combined, they form a second derivative:

$$\nabla \cdot \nabla f \;=\; \nabla^2 f \;\neq\; 0 \qquad (\text{“Laplacian operator”})$$

We'll see the Laplacian operator crop up again. For now, let's just have a look at what it looks like in Cartesian coordinates:

$$\nabla^2 f \;=\; \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \;.$$

. . . .

One additional physics application: a call-back to Newtonian dynamics.

Recall the concept of a **conservative force.** The work done as you go along a path from a to b does not depend on which path you choose:

$$W_{ab} \;=\; \int_a^b \mathbf{F} \cdot d\mathbf{r} \;=\; U_a - U_b \;.$$

Conservative forces correspond to specific *kinds* of vectors, and we now have the language to describe them.

We can now say some additional things about conservative vector fields:

(1) Notice that if $a = b$ (i.e., the path is a closed loop), then

$$W_{aa} \;=\; \oint \mathbf{F} \cdot d\mathbf{r} \;=\; 0 \qquad (\text{because } U_a = U_a).$$

(2) All conservative fields can be written as the gradient of a scalar potential function:

$$\mathbf{F} \;=\; -\nabla U$$

where the minus sign is a typical convention in physics.

We can prove it. Let's assume that $\mathbf{F} = -\nabla U$, and show that we get the familiar definition for the work:

$$W_{ab} \;=\; \int_a^b \mathbf{F} \cdot d\mathbf{r} \;=\; -\int_a^b (\nabla U \cdot d\mathbf{r}) \;=\; -\int_a^b \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right)$$

(continued on next page)

The previous step is equivalent to

$$W_{ab} \;=\; -\int_a^b \left(\frac{\partial U}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial U}{\partial y}\frac{dy}{dt} \;+\; \frac{\partial U}{\partial z}\frac{dz}{dt}\right) dt \;=\; -\int_a^b \frac{dU}{dt} dt$$

where the last part is just the total derivative of a function $U(x(t), y(t), z(t))$. Thus,

$$W_{ab} \;=\; -\int_a^b dU \;=\; -(U_b - U_a) \;=\; U_a - U_b$$

which is what we used in Newtonian dynamics.

(3) Since the curl of a gradient is always zero, it's true that all conservative forces must obey:

$$\nabla \times \mathbf{F} \;=\; 0$$

which should make intuitive sense in the case of **gravity,** since there was nothing "swirly" about that force-law: it points radially inward toward the source of mass.

PARTIAL DIFFERENTIAL EQUATIONS

There are dozens of courses at CU Boulder devoted solely to solving these "PDEs." We will just survey a few commonly-seen types of these equations, and take note of **what they mean** in terms of physical applications.

For a function $f(x,t)$ of position and time, it may obey something like the **advection equation:**

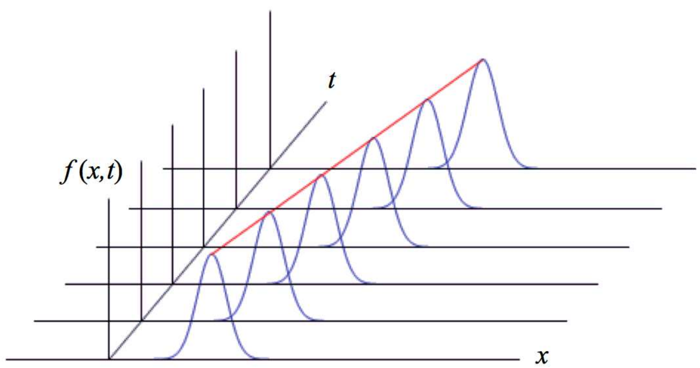
$$\frac{\partial f}{\partial t} + V_x \frac{\partial f}{\partial x} = 0$$

where V_x is a constant.

The solution for $f(x,t)$ is a pattern that drifts along the x-axis, over time, with a speed of V_x:

$$f(x,t) = f_0(x - V_x t) \ .$$

and the "template" of the pattern $f_0(x)$ can be anything.



We won't show how the PDE can be solved to get the solution, but we can show that the solution does satisfy the PDE. Take the 2 partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f_0}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial t} = -V_x \frac{\partial f_0}{\partial x} \ .$$

If the "drift" is happening in 3D space, the advection equation for $f(\mathbf{r},t)$ generalizes to:

$$\frac{\partial f}{\partial t} + V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0$$

but that is just

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla f = 0$$

and the corresponding solution is

$$f(\mathbf{r},t) \;=\; f_0(\mathbf{r}-\mathbf{V}t)\ .$$

In the advection equation, the general "operator" that operates on f arises a lot in physics, and is sometimes called the **advective derivative** (or material derivative):

$$\frac{D}{Dt} \;=\; \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla$$

It's a derivative with respect to time, but it follows around a "parcel" that is moving through space with velocity $\mathbf{V}$

i.e., it's unlike $\partial/\partial t$, which looks at the change in time only at one fixed location in space.

Silly example: You're sitting in a meadow, watching the trees. f is the "density" of leafy green stuff that you see.

If we're just sitting still, the trees are (slooooowly) growing where they stand. Nothing is moving around in space. Thus, $f = f(t)$ only, and the only rate of change you see is $\partial f/\partial t$.

However, what if we're in a boat on a river, drifting along the x direction, and the trees are getting denser/thicker as we go down river. In our reference frame, $f = f(x,t)$, and our position x is a function of time.

$$f = f(x(t),t)\ , \qquad \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{dx}{dt}}_{v_x}\frac{\partial f}{\partial x} \qquad \text{(chain rule!)}$$

In other words, Df/Dt is the same as the "total derivative."

If f increases as x increases, then $\partial f/\partial x > 0$. Df/Dt gives us the summed effect of the two kinds of "increase" in tree density that we see:
in time, and in space.

$$\text{The advection equation PDE is thus:} \qquad \frac{Df}{Dt} \;=\; 0$$

and a quantity that obeys it remains time-steady (i.e., unchanging in time) **within** its moving frame of reference.

. . . .

Another common PDE is the **wave equation.** Again, starting in 1D with $f(x,t)$, the wave equation is

$$\frac{\partial^2 f}{\partial t^2} = V^2 \frac{\partial^2 f}{\partial x^2}$$

and it has oscillating sinusoid type solutions like

$$f(x,t) = A \sin(kx - \omega t + \phi_0) \quad \text{where } V = \omega/k \text{ is the wave } \textit{phase speed.}$$

Actually, $k = \pm\omega/V$, since it supports waves going in both directions along x. You can freely choose either ω or k, but not both.

The amplitude A and phase ϕ_0 are arbitrary; any values satisfy the wave equation. The angular frequency ω and wavenumber k relate to some other commonly-seen quantities:

$$\text{period} = \mathcal{P} = \frac{2\pi}{\omega} \qquad\qquad \text{frequency} = \nu = \frac{\omega}{2\pi} = \frac{1}{\mathcal{P}}$$

$$\text{wavelength} = \lambda = \frac{2\pi}{k} \ .$$

Note that a **sum** of multiple sinusoids, each with arbitrary amplitude and phase, also satisfies the wave equation (as long as they all describes waves that propagate with $\pm V$).

. . . .

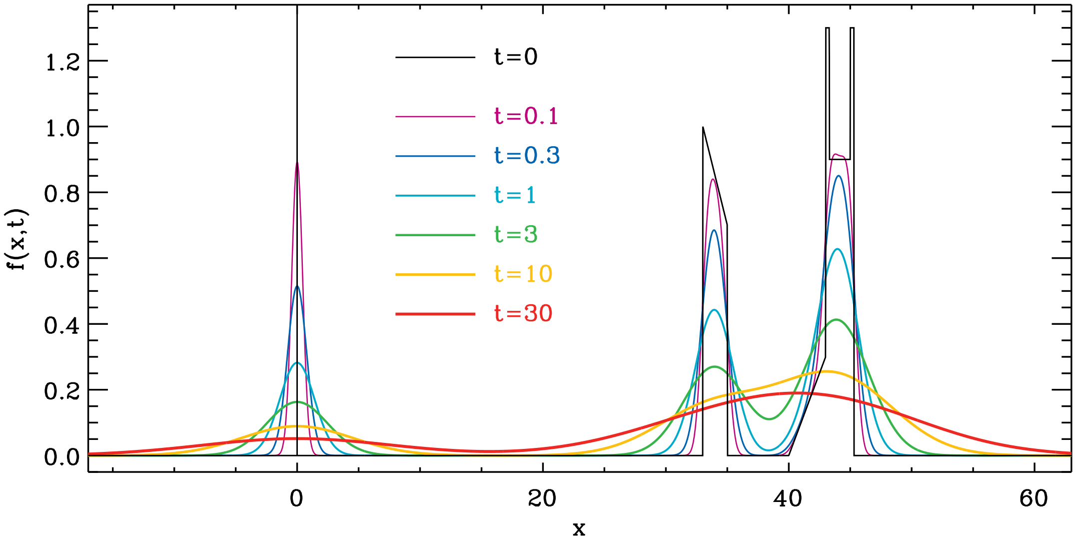
We won't say too much about the **diffusion equation,** but it's good to at least know it exists. It is a hybrid: 1st order in time, 2nd order in space:

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}$$

where the diffusion coefficient D has units of length^2/time.

If the function $f(x,t)$ is "sharply peaked" at $t=0$, the diffusion equation describes a time evolution that continuously *spreads out* the peak.

See example solution plot on next page:



There's a clear **directionality** of time to the diffusion equation. Any initial "structure" is eventually & **irreversibly** smeared out as t marches on.

As $t \to \infty$, f diffuses to a constant value. If x subtends all space, $f \to 0$.

. . . .

For many of the above, in 3D one should replace $\partial f/\partial x$ by ∇f, and $\partial^2 f/\partial x^2$ by $\nabla^2 f$ (the Laplacian).

If there is no change in time, then both the 3D wave equation and the 3D diffusion equation reduce to **Laplace's equation:**

$$\nabla^2 f \ = \ 0 \ .$$

Such a simple-looking little equation can exhibit some complex behavior, as we will see in E&M and quantum mechanics.