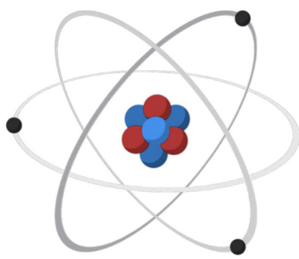
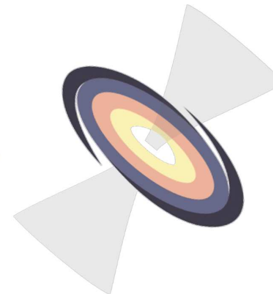
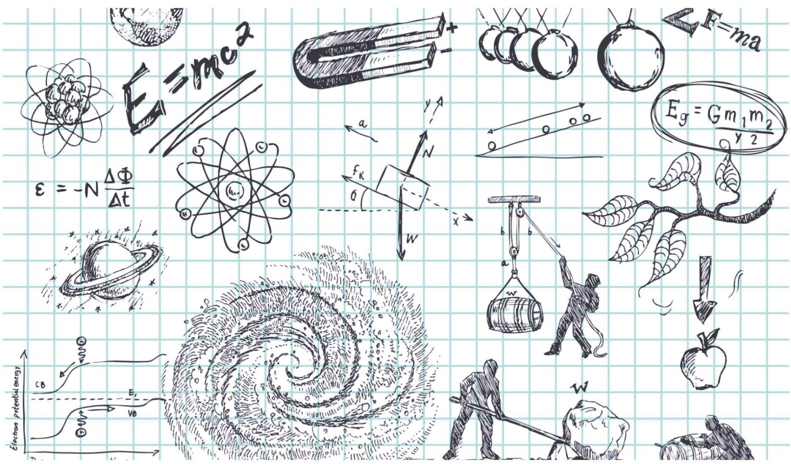
ASTR-2100: Fundamental Concepts in Astrophysics ... Spring 2023



Introduces the **math & modern physics** (quantum mechanics, electromagnetic spectra, atomic & nuclear physics, special relativity, thermodynamics) that prepares you for upper-division ASTR courses





- Previously, some ASTR courses required Calculus III (APPM 2350 or MATH 2400) and Modern Physics (PHYS 2130/2170).
- ASTR-2100 now fulfills these requirements, too.
- All APS students starting at CU in Fall 2019 or afterwards must take ***either*** ASTR-2100 ***or*** Calc III & Modern Physics for graduation, regardless of track.
- Pre-requisites for this course: Calc II (APPM 1360 or MATH 2300) and Physics II (PHYS 1120 or 1125).

This course is meant to provide you with a **full toolbox** for advanced courses in astrophysics, planetary science, and solar/space physics, and the skills to use those tools.

On the first day of class, we'll go through the syllabus in detail and discuss what components go into your total course grade.

These lecture notes are the primary "textbook" for the course (so please read everything, even if we skip some parts in class). Other books and online resources are listed in the syllabus & web page.

Several things about math...

- This course is mostly about physics and its application to stuff "out there" in the universe, but math is an important tool we use to make sense of it all.

- Thus, you'll get a **handout** with useful mathematical formulae. Some of what's in there should be a review of what you've learned in eariler courses, and some may be new. You can use the handout as a resource for homeworks. (For exams, I'll provide the bits & pieces you'll need.)

- We'll also do a lot of **approximating!** Seeing this done may be surprising, if you're used to every problem having an exact solution.

  For scientists doing research, approximation/assumption is something we do all the time...

  $$\text{It's an } \textbf{art} \text{ to figure out} \begin{cases} \text{what to simplify} \\ \text{what to neglect} \\ \text{what to flat-out ignore} \end{cases}$$

  Hopefully, by seeing how it's done (both here and in other courses) you'll start to get a feel for doing it yourself. It takes a while...

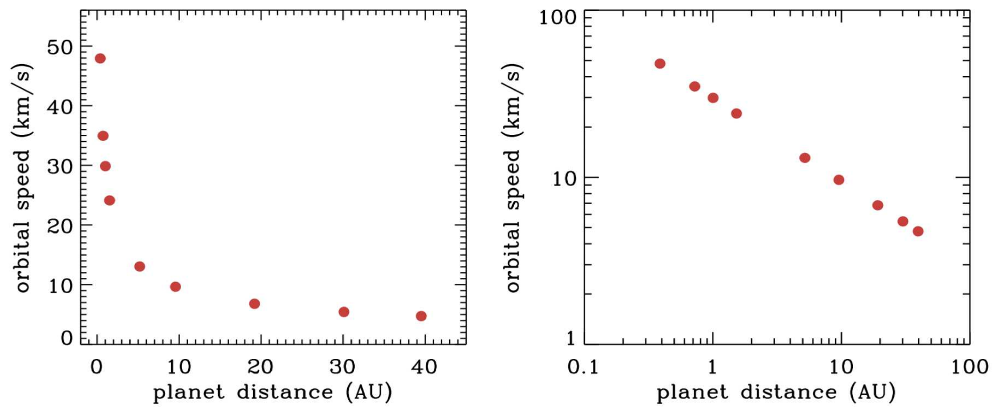
  $=$ the "exact equality" will often give way to
  $\approx$ "is approximately equal to," or sometimes even
  $\sim$ "very roughly equal to" (within an order of magnitude!?)
  $\propto$ and sometimes we just care about which quantities are "proportional to" one another, ignoring normalizing constants.

- You should have a calculator that uses scientific notation.
  (Good free phone-app: *TechCalc*. Good web resource: *Wolfram Alpha.*)
  Googling answers to math problems at this level can be unreliable...

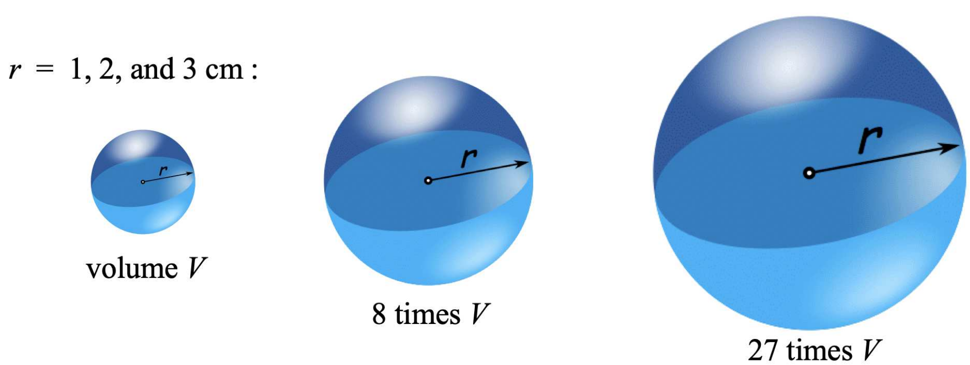
BRIEF MATH/PHYSICS REVIEW

Some topics from earlier calculus & physics classes that will crop up a lot:

- **Scientific notation** is super-useful for writing "astronomically" large or small numbers compactly... $43{,}200{,}000{,}000{,}000 = 4.32 \times 10^{13}$.
- We'll certainly encounter **derivatives & integrals** of basic functions.
- **Logarithmic plots** help us find trends in data that may be hard to see:



- **Metric units & conversions** should be familiar, but please always feel free to use the handout to look up unfamilar prefixes. We'll stick with SI ("mks") as much as possible.
- **Proportional reasoning** is helpful, because we sometimes care much more about **relative relationships** between quantities, and not so much about solving for exact numbers...



$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V \propto r^3 \quad \Rightarrow \quad \frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^3$$

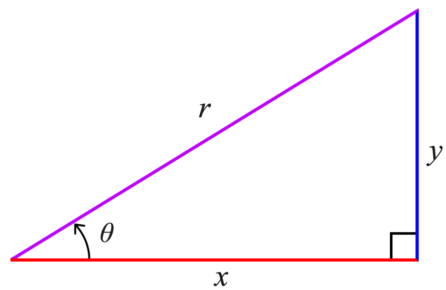
ANGLES & TRIGONOMETRY

We'll be using the classic trigonometric functions a lot, so it's good to recall their definitions, too:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

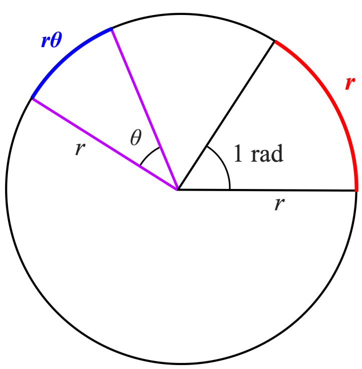


$$x^2 + y^2 = r^2$$

$$\sin^2\theta + \cos^2\theta = 1$$

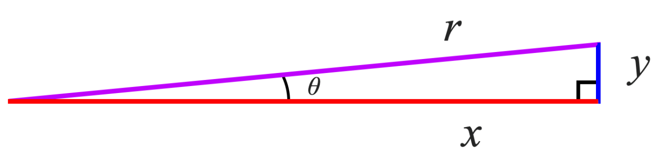
$$1 + \tan^2\theta = \sec^2\theta$$

For angles in physics formulas, we will often use units of **radians** because they are the most natural; i.e., the quantity $r\theta$ is the actual path-length along a wedge (of angle θ) of a circle of radius r.



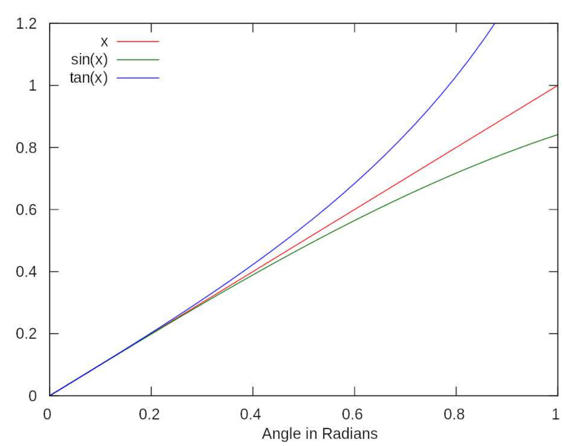
1 circle $=$ 360° $=$ 2π radians (rad)
1 radian $=$ $360°/2\pi$ $=$ $180°/\pi$ $\approx$ 57.296°
1 degree $=$ $60'$ (i.e., 60 arcminutes) $=$ $3600''$
1 arcminute $=$ $60''$ (i.e., 60 arcseconds)
Thus, 1 rad $\approx$ 206,265$''$ & 1 circle $=$ 1,296,000$''$

Radians are also useful in the "small-angle limit." For $\theta \ll 1$,



$$x \approx r\ , \quad \text{so} \quad \sin\theta = \frac{y}{r} \approx \frac{y}{x} = \tan\theta$$

$$y \approx r\theta\ , \quad \text{so} \quad \sin\theta = \frac{y}{r} \approx \frac{r\theta}{r} = \theta$$



What is $\cos\theta$ in the small-angle limit?

Very roughly, we see that $\cos\theta = x/r \approx 1$. But sometimes it's best to be a bit more accurate.

If $\sin\theta \approx \theta$, and $\cos\theta = \sqrt{1-\sin^2\theta}$, then

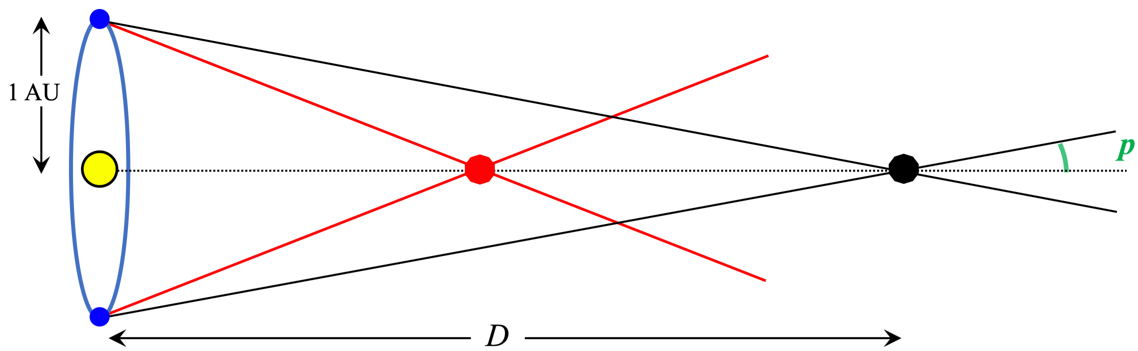
$$\cos\theta \;\approx\; \sqrt{1-\theta^2} \;\approx\; 1 - \frac{\theta^2}{2}$$

where the last approximation comes from the **binomial expansion** for small quantities:

$$\text{For } |x| \ll 1, \qquad (1+x)^n \;=\; 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

. . . .

When looking at distant sources on the sky, very small angles are the norm. For example: **parsecs!**



When $p = 1''$, we define $D = 1$ parsec (**PAR**allax of one arc**SEC**ond).
As D increases, p decreases, and the small-angle approximation gives the usual expression:

$$\tan p \;=\; \frac{1 \text{ AU}}{D \text{ (in AU)}} \;\approx\; p \;\text{(in radians)}$$

$$\text{This is equivalent to} \qquad \frac{1}{D \text{ (in parsecs)}} \;\approx\; p \;\text{(in arcseconds)}\;.$$

. . . .

The last trig topic is to generalize from angles to **solid angles.**

When we use radians for angle θ, the arc-length along a circle of radius r is

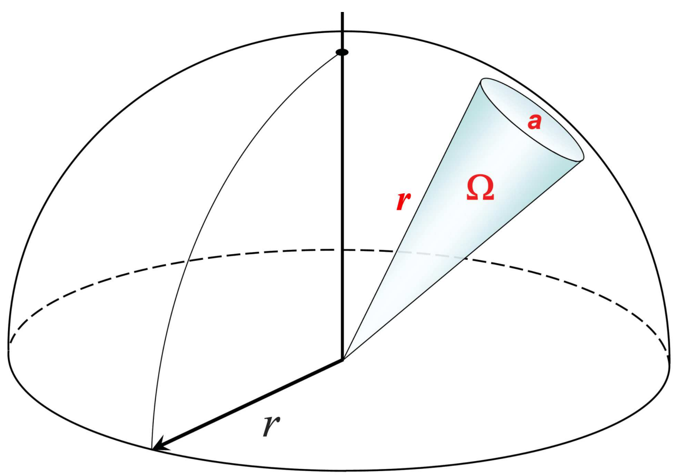
$$\ell = r\theta$$

and the circumference of a *full* circle is defined when we reach $\theta = 2\pi$, because we know that $C = 2\pi r$.   (This is **why** 2π radians = full circle!)

Similarly, for a sphere of radius r, let's define the solid-angle quantity Ω (in units of **steradians**) so that it covers a given area

$$a = r^2\Omega$$

on the sphere's surface. The sphere's *full* area is $A = 4\pi r^2$. Thus, a full sphere is covered by a solid angle of $\Omega = 4\pi$.



For small solid angles (i.e., distant sources), it's easy to make more direct use of the analogy:

$$\frac{\text{angle}}{\text{length}} = \frac{\text{solid angle}}{\text{area}} .$$

For example:

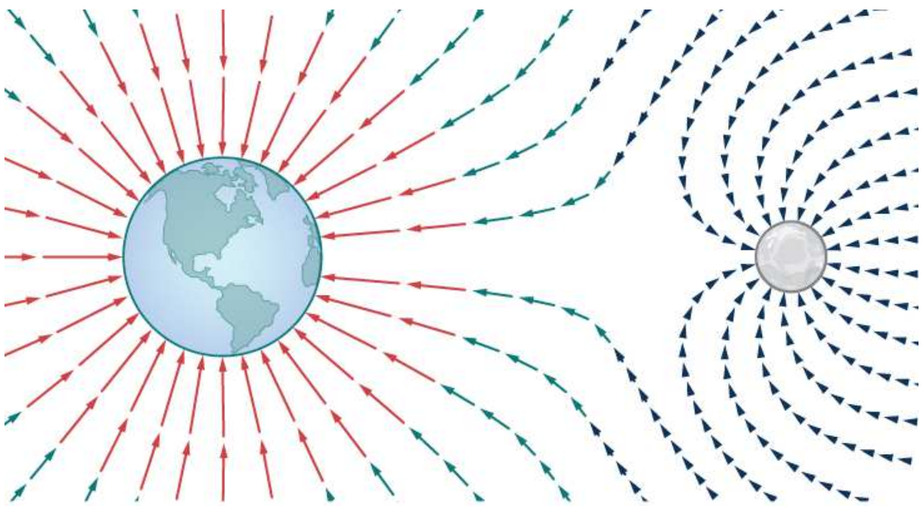
- For a tiny CIRCULAR patch "on the sky" with angular radius θ, $\boxed{\Omega \approx \pi\theta^2 .}$
- For a tiny SQUARE patch "on the sky" subtending angle θ on one side, $\boxed{\Omega \approx \theta^2 .}$

VECTOR ANALYSIS

Scalars are fine if you just want to measure the *magnitude* of a quantity.

Vectors are needed if a quantity has a magnitude and a **direction** (e.g., wind velocity in a weather report). Convention: bold **V** or arrow $\overrightarrow{V}$.

Another example: **forces** in physics, like gravity:



In drawings like this, the "length" of the vector is only a schematic way of visualizing the magnitude. A given amount of force doesn't really correspond to an actual length as shown here.

Note: Some quantities have direction ONLY, without a magnitude. For them, we use **unit vectors,** which always have length = 1.
Convention: "hatted" lower-case $\hat{\mathbf{n}}$.

In Cartesian coordinates, there are three unit vectors that allow us to specify magnitudes along each of the axes:

$\hat{\mathbf{e}}_x$	vector tip coordinates (1,0,0)	sometimes called $\mathbf{i}$ or $\hat{\mathbf{x}}$
$\hat{\mathbf{e}}_y$	vector tip coordinates (0,1,0)	sometimes called $\mathbf{j}$ or $\hat{\mathbf{y}}$
$\hat{\mathbf{e}}_z$	vector tip coordinates (0,0,1)	sometimes called $\mathbf{k}$ or $\hat{\mathbf{z}}$

Thus, any vector can be specified by giving its magnitudes in the three Cartesian directions:

$$\text{Example:} \quad \mathbf{F} = 3\hat{\mathbf{e}}_x + 5\hat{\mathbf{e}}_y + 7\hat{\mathbf{e}}_z \ .$$

The **magnitude** of a vector is its total 3D "length,"

$$|\mathbf{F}| \;=\; F \;=\; \sqrt{F_x^2 + F_y^2 + F_z^2}$$

For the example above, $\quad |\mathbf{F}| \;=\; \sqrt{3^2 + 5^2 + 7^2} \;=\; \sqrt{83} \;\approx\; 9.1$.

(Remember: units of vector magnitude don't have to be *actual* length.)

In physics, both scalars and vectors will tend to be continuous functions of both position and time...

scalar $f(x, y, z, t)$; $\quad$ vector $\mathbf{v}(x, y, z, t)$ is really made up of:

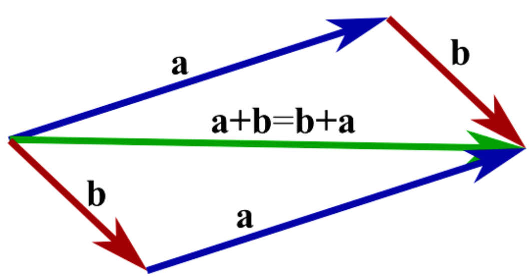
$$v_x(x, y, z, t)\hat{\mathbf{e}}_x \;+\; v_y(x, y, z, t)\hat{\mathbf{e}}_y \;+\; v_z(x, y, z, t)\hat{\mathbf{e}}_z \;\; .$$

. . . .

Vector Addition

The sum of two vectors **A** and **B** can be constructed by placing them end-to-end (i.e., placing the initial point of **B** at the end point of **A**) and constructing a new vector that goes from the initial point of **A** to the end point of **B**.

Thus, $\mathbf{A} + \mathbf{B}$ is the "summed effect" of following both vectors in succession:



Note: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, as it ought to be for anything we call "addition."

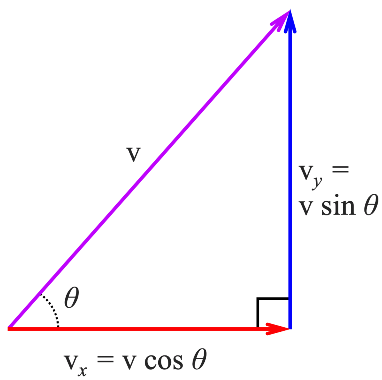
Vector addition is *not* just adding their magnitudes! The diagram above brings to mind the "triangle inequality" from geometry. In general, it's true that

$$|\mathbf{A} + \mathbf{B}| \;\leq\; |\mathbf{A}| + |\mathbf{B}| \;\; .$$

Another way to think about vector addition is that the Cartesian components add together like normal scalars:

$$\left\{ \begin{array}{l} \mathbf{A} = A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z \\ \mathbf{B} = B_x\hat{\mathbf{e}}_x + B_y\hat{\mathbf{e}}_y + B_z\hat{\mathbf{e}}_z \end{array} \right\} \implies \mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{e}}_x + (A_y + B_y)\hat{\mathbf{e}}_y + (A_z + B_z)\hat{\mathbf{e}}_z \ .$$

Thinking of some familiar things as "summed vectors" can be a useful aid to memorization:



You may already know every fact given in the triangle pictured here. However, if the x and y axes are treated like vectors, their sum (the hypotenuse) should be something you can compute without having to know anything about triangles...

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v(\sin^2\theta + \cos^2\theta)} = v$$

The Pythagorean theorem is something else you can just "derive!"

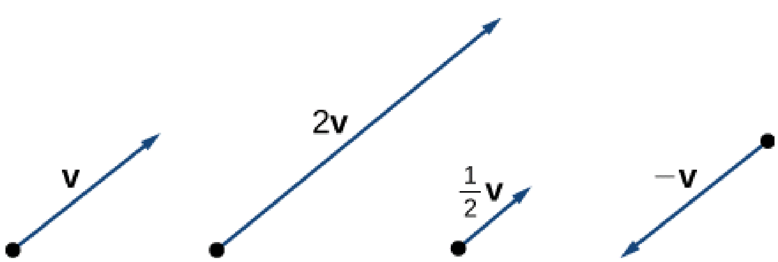
Vector Multiplication

We'll get to vector subtraction in a minute, but it's actually *easier* to start by talking about one of the (several!) types of vector multiplication.

(1) Scalar Multiplication:

If we add $\mathbf{v} + \mathbf{v}$, that's the same as $2\mathbf{v}$. Also, $\mathbf{v} + \mathbf{v} + \mathbf{v} = 3\mathbf{v}$, and so on.

Thus, we can multiply a vector $\mathbf{v}$ by any scalar number k, so that the result is a vector with magnitude $|k|$ times the magnitude of the original $\mathbf{v}$.



Note that if k is negative, the new direction of the vector is **opposite** that of the original $\mathbf{v}$.

The magnitude of the vector $k\mathbf{v}$ is just: $|k|\,|\mathbf{v}|$.

Also, scalar multiplication can be used to **normalize** any vector;
i.e., convert it into a unit-vector of length 1:

$$\hat{\mathbf{n}} = \frac{\mathbf{v}}{|\mathbf{v}|} \ .$$

Here, $\hat{\mathbf{n}}$ points in the same direction as $\mathbf{v}$, but the magnitude information has been "removed."

. . . .

This now lets us talk about...

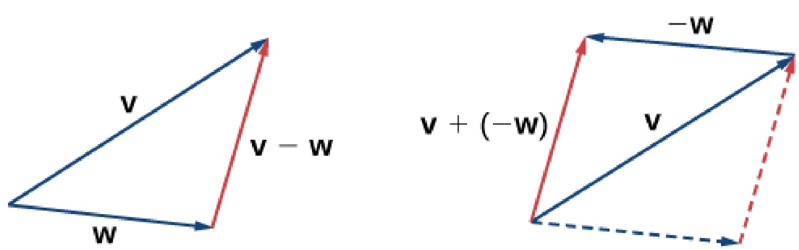
Vector Subtraction

The "difference" between two vectors is the difference between their two end-points (if they have the same initial point).

However, a more natural way of thinking about it is to realize that

$$\mathbf{V} - \mathbf{W} = \mathbf{V} + (-\mathbf{W})$$

where we use the above definition of the negative of a vector:



(It's still the same vector even if it is "translated" in space.)

When broken down into components, the definition of subtraction is straightforward, too:

$$\left\{ \begin{array}{l} \mathbf{A} = A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z \\ \mathbf{B} = B_x\hat{\mathbf{e}}_x + B_y\hat{\mathbf{e}}_y + B_z\hat{\mathbf{e}}_z \end{array} \right\} \implies \mathbf{A} - \mathbf{B} = (A_x - B_x)\hat{\mathbf{e}}_x + (A_y - B_y)\hat{\mathbf{e}}_y + (A_z - B_z)\hat{\mathbf{e}}_z$$

(and some books use *this* as the main definition of vector subtraction, rather than the above visual ideas about arrows).

Example: Using Vectors to Describe Motion

We now know enough to apply vectors to kinematics: the study of motion. We'll often describe the **position vector** of an object as

$$\mathbf{r}(t) \;=\; x(t)\hat{\mathbf{e}}_x \,+\, y(t)\hat{\mathbf{e}}_y \,+\, z(t)\hat{\mathbf{e}}_z \;.$$

As long as we specify a coordinate system (i.e., where is the origin), this vector points to the position of an object that moves over time.

Just like in Physics 1, we can take time derivatives to describe the velocity of the object:

$$\mathbf{v}(t) \;=\; \frac{d\mathbf{r}}{dt} \;=\; \frac{dx}{dt}\hat{\mathbf{e}}_x \,+\, \frac{dy}{dt}\hat{\mathbf{e}}_y \,+\, \frac{dz}{dt}\hat{\mathbf{e}}_z \qquad \left(\text{where we call } v_x = \frac{dx}{dt},\ \text{etc.}\right)$$

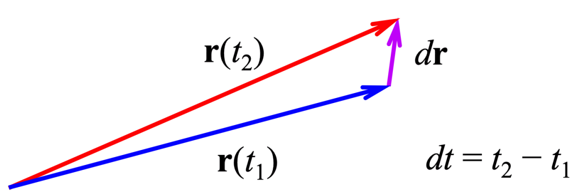
and **"speed"** is a scalar equal to the magnitude of the velocity vector: $|\mathbf{v}|$. Also, we can keep differentiating to get the acceleration:

$$\mathbf{a}(t) \;=\; \frac{d\mathbf{v}}{dt} \;=\; \frac{d^2\mathbf{r}}{dt^2} \;=\; \frac{dv_x}{dt}\hat{\mathbf{e}}_x + \frac{dv_y}{dt}\hat{\mathbf{e}}_y + \frac{dv_z}{dt}\hat{\mathbf{e}}_z \;=\; \frac{d^2x}{dt^2}\hat{\mathbf{e}}_x + \frac{d^2y}{dt^2}\hat{\mathbf{e}}_y + \frac{d^2z}{dt^2}\hat{\mathbf{e}}_z$$

and we can also integrate to go backwards... For example:

$$\mathbf{r}(t) \;=\; \int_a^b dt\ \mathbf{v}(t) \;=\; \left(\int_a^b dt\ v_x(t)\right)\hat{\mathbf{e}}_x + \left(\int_a^b dt\ v_y(t)\right)\hat{\mathbf{e}}_y + \left(\int_a^b dt\ v_z(t)\right)\hat{\mathbf{e}}_z \,.$$

Lastly, it's sometimes useful to consider the numerator of the velocity derivative as a thing unto itself: an infinitesimal **displacement vector:**



$$d\mathbf{r} \;=\; (dx)\hat{\mathbf{e}}_x + (dy)\hat{\mathbf{e}}_y + (dz)\hat{\mathbf{e}}_z$$

If we decided to integrate just this quantity over time, we would obtain the total **path length:**

$$\ell \;=\; \left|\int d\mathbf{r}\right| \;.$$

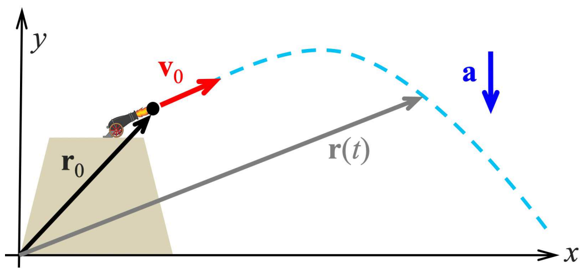
. . . .

What do we do with these quantities? Apply Newton's second law... which can be written as a vector equation (with some scalar multiplication thrown in to make the units work out):

$$\boxed{\mathbf{F} = m\mathbf{a}\ .}$$

Let's work out the classic example of projectile motion.

Assume the flat ground is the x-axis, "up" is the y-axis, and we can ignore z:



There's only one force: gravity, which supplies a constant acceleration in the downward (minus y) direction:

$$\mathbf{a} = -g\,\hat{\mathbf{e}}_y$$

Thus, we can integrate to get the vector velocity. Note the *integration constant* is now a vector $\mathbf{C}$...

$$\mathbf{v}(t) = \int \mathbf{a}\ dt = \int (-g\hat{\mathbf{e}}_y)\,dt = -gt\,\hat{\mathbf{e}}_y + \mathbf{C}$$

However, when $t = 0$, $\mathbf{v} = \mathbf{C}$, so we can call $\mathbf{C} = \mathbf{v}_0$.

Note that the initial velocity $\mathbf{v}_0$ can point in ANY direction. Thus,

$$\mathbf{v}(t) = \mathbf{v}_0 - gt\,\hat{\mathbf{e}}_y$$

and we can integrate again to get the position vector $\mathbf{r}$,

$$\mathbf{r}(t) = \int \mathbf{v}\ dt = \mathbf{r}_0 + \mathbf{v}_0 t + -\frac{1}{2}gt^2\,\hat{\mathbf{e}}_y$$

where we did the same trick with the vector integration constant.

Just like in Physics 1, the answer breaks down into separate components for the horizontal and vertical motion:

$$x(t) = x_0 + v_{x,0}t \qquad\qquad y(t) = y_0 + v_{y,0}t - \frac{1}{2}gt^2$$

but vectors allowed us to "do the physics" all in one piece.

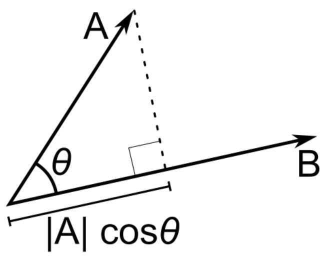
. . . .

(Back to) Vector Multiplication

For scalars, just one "type" of multiplication was all we needed. However, for vectors, we need more...

(2) The Dot Product: $\mathbf{A} \cdot \mathbf{B} = AB \cos\theta$

It's the projection of $\mathbf{A}$ along $\mathbf{B}$, and also the projection of $\mathbf{B}$ along $\mathbf{A}$.



θ is the angle "between" the vectors.

Note: $(A\cos\theta)B \;=\; A(B\cos\theta)$

Overall, it tells you how well 2 vectors line up:

$$
\begin{aligned}
\text{If } \mathbf{A} \text{ is parallel to } \mathbf{B}, &\quad \mathbf{A}\cdot\mathbf{B} = AB \quad \text{(max)} \\
\text{If } \mathbf{A} \text{ is perpendicular to } \mathbf{B}, &\quad \mathbf{A}\cdot\mathbf{B} = 0 \\
\text{If } \mathbf{A} \text{ is anti-parallel to } \mathbf{B}, &\quad \mathbf{A}\cdot\mathbf{B} = -AB \quad \text{(min)} \\
\text{And, of course, } &\quad \mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A^2
\end{aligned}
$$

Note: it only makes sense to talk about θ being between 0 and 180°.

One can also compute the dot product using components:

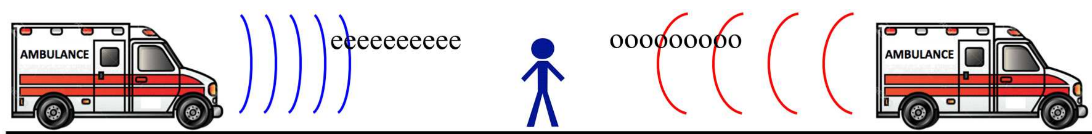
$$\mathbf{A}\cdot\mathbf{B} \;=\; A_x B_x + A_y B_y + A_z B_z \;\; .$$

and it's clear how this also tells us that $A^2 = \mathbf{A}\cdot\mathbf{A}$.

. . . .

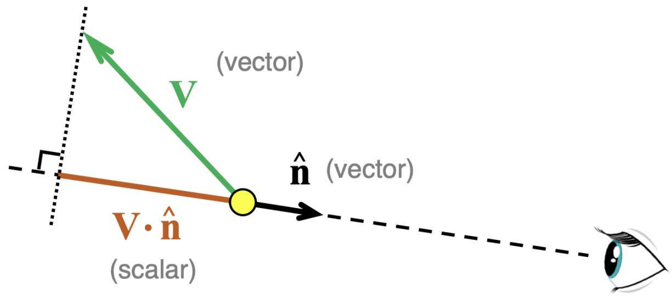
Example: The Doppler Effect

You know it already...



However, for an object moving arbitrarily in 3D, the redshift/blueshift only depends on the component of the motion along the observer's "line-of-sight."

Consider the velocity vector $\mathbf{V}$ as something with real units, and a unit-vector $\hat{\mathbf{n}}$ that points from the moving object to the observer:

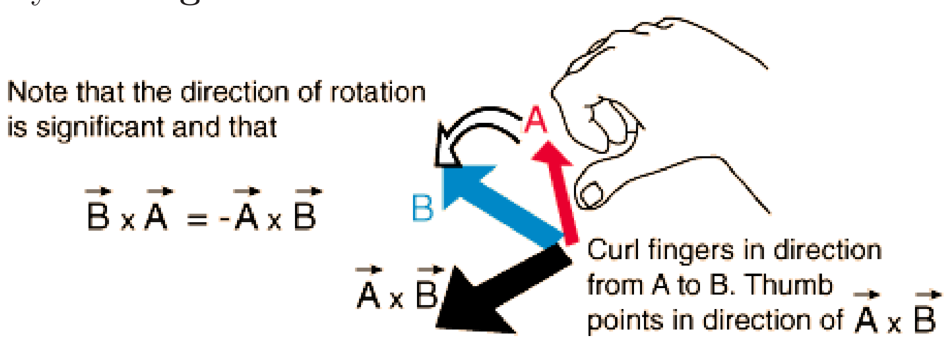


Thus, the scalar $V_{\rm LOS} = \mathbf{V} \cdot \hat{\mathbf{n}}$, and it can be positive, negative, or zero.

. . . .

(3) The Cross Product: $\quad \mathbf{A} \times \mathbf{B} = (AB\sin\theta)\,\hat{\mathbf{n}}$

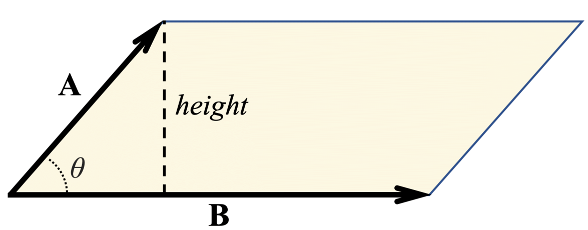
where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, with a direction formed by the **"right-hand rule:"**



The cross product picks out how well 2 vectors are *transverse* to one another:

$$
\begin{array}{rll}
\text{If } \mathbf{A} \text{ is parallel to } \mathbf{B}, & \mathbf{A}\times\mathbf{B} = 0 & \text{(min)} \\
\text{If } \mathbf{A} \text{ is perpendicular to } \mathbf{B}, & |\mathbf{A}\times\mathbf{B}| = AB & \text{(max)} \\
\text{And, of course, } & \mathbf{A}\times\mathbf{A} = 0 &
\end{array}
$$

Interestingly, the magnitude of a cross product is equal to the area of the parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$:



Area = *base* × *height*
= $B \times (A \sin\theta)$

In Cartesian coordinates, it's a matrix determinant,

$$\mathbf{A} \times \mathbf{B} \;=\; \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} \;=\; \hat{\mathbf{e}}_x(A_yB_z - A_zB_y) \,+\, \hat{\mathbf{e}}_y(A_zB_x - A_xB_z) \,+\, \hat{\mathbf{e}}_z(A_xB_y - A_yB_x)$$

Note that for the cross product, order matters: $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$.

Also: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ . . . so we need to be careful.

Here's a quick review of matrix determinants, but you won't be asked to memorize how to do them:

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

i.e., $\searrow$ minus $\swarrow$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3\,(1) - 5\,(-2) = 3 + 10 = 13.$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

In physics, the cross product is useful in cases where quantities have the most effect when they're *NOT* lined up – e.g., **torque, magnetic fields.**

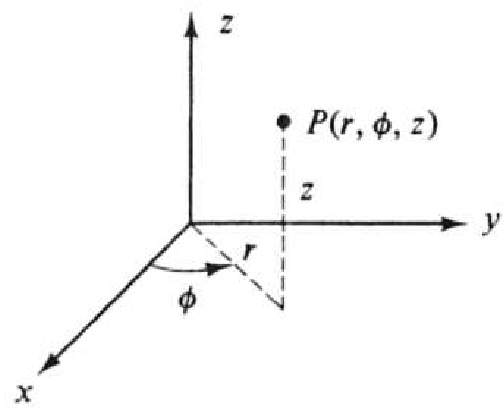
The handout gives various kinds of commutative, distributive properties. You don't have to memorize them either, but it's good to know they exist.

COORDINATE SYSTEMS

Depending on the **symmetries** in a system, we might find it more useful to use other 3D coordinate systems besides good old Cartesian (x, y, z).

Cylindrical coordinates replace x and y by "polar coordinates" r and ϕ (where ϕ is measured counter-clockwise from the $+x$ axis), and they keep z the same:

$$\begin{aligned} x &= r\cos\phi \qquad & r &= \sqrt{x^2+y^2} \\ y &= r\sin\phi \qquad & \phi &= \tan^{-1}(y/x) \\ z &= z \qquad & z &= z \end{aligned}$$



Of course, despite the fact that x, y, and z can take on values from $-\infty$ to $+\infty$, we can see that we must have:

$$0 \leq r < +\infty \qquad \text{and} \qquad 0 \leq \phi \leq 2\pi \ .$$

Note that a "surface" defined by a constant value of r (i.e., the collection of all possible points that have that value of r) is just a **cylinder** that extends up and down in the z direction indefinitely.

In some textbooks, the horizontal distance from the z-axis (which we call r) is called ρ or ϖ or R. The azimuthal angle ϕ is sometimes called θ.

In physics, cylindrical coordinates are useful for objects that exhibit "axisymmetry" (i.e., rotational symmetry around one axis), like accretion disks, or currents & magnetic fields associated with a straight wire.

. . . .

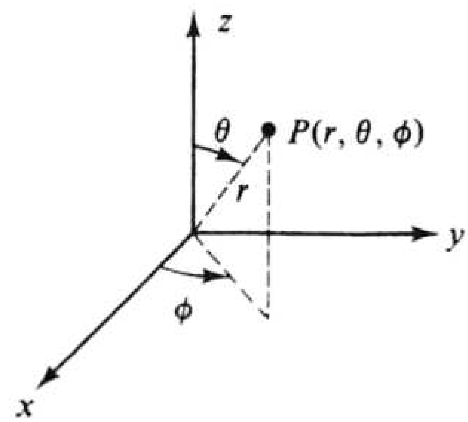
Cartesian coordinates describe a point in 3D space by 3 numbers that each represent distances.

Cylindrical coordinates describe a point using 2 distances and 1 angle.

Can we describe a point with 1 distance and 2 angles?   Yes...

Spherical coordinates are the natural choice for systems in spherical symmetry. There's just one radius coordinate r that describes how far from the origin a point is, and its location on an imaginary sphere is described by two angles θ and ϕ:

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}(z/r) \\ \phi &= \tan^{-1}(y/x) \end{aligned}$$



The azimuthal angle ϕ (which is sort of like **longitude**) is the same as in cylindrical coordinates. It can extend from 0 to 360°.

The polar angle θ (which is sort of like **latitude,** and is sometimes called "co-latitude") only goes from 0 to 180°, with:

- $\theta = 0$ at the north pole,
- $\theta = 90°$ at the equator,
- $\theta = 180°$ at the south pole.

Horrifically, some books switch the meanings of θ and ϕ, but *most* physicists and astrophysicists define them the way we do here.

....

The "useful formulas" handout contains many other conversions and identities for vector operations in these other coordinate systems.

When doing vector addition & subtraction, it's best to stick with Cartesian coordinates (i.e., convert **from** either cylindrical or spherical **to** Cartesian, then add them, then convert back).

When doing vector multiplication, one can just use non-Cartesian components:

Cylindrical:

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} \;=\; (A_\phi B_z - A_z B_\phi)\hat{\mathbf{e}}_r \;+\; (A_z B_r - A_r B_z)\hat{\mathbf{e}}_\phi \;+\; (A_r B_\phi - A_\phi B_r)\hat{\mathbf{e}}_z$$

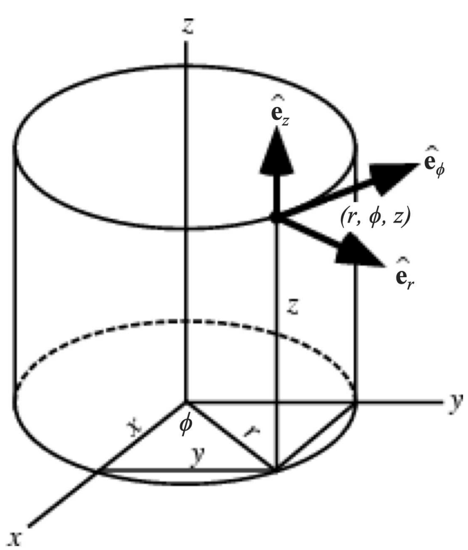
Spherical:

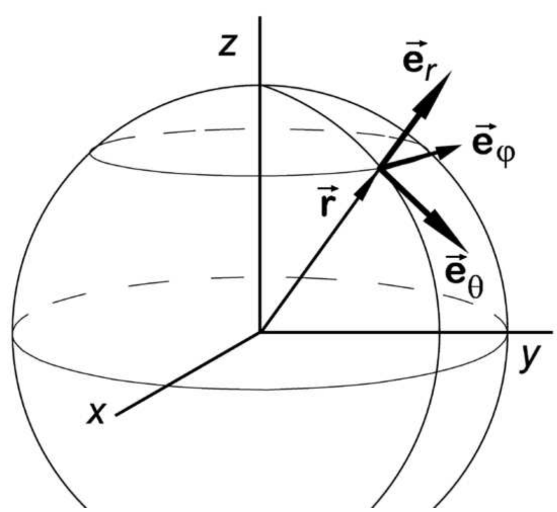
$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi$$

$$\mathbf{A} \times \mathbf{B} \;=\; (A_\theta B_\phi - A_\phi B_\theta)\hat{\mathbf{e}}_r \;+\; (A_\phi B_r - A_r B_\phi)\hat{\mathbf{e}}_\theta \;+\; (A_r B_\theta - A_\theta B_r)\hat{\mathbf{e}}_\phi$$

For the cross product, it's important to maintain the proper order:
(x, y, z) or (r, ϕ, z) or (r, θ, ϕ).

Note that some of the cylindrical and spherical unit vectors are **not fixed in space** like the Cartesian ones are. As the coordinates change, the unit vectors swivel around...



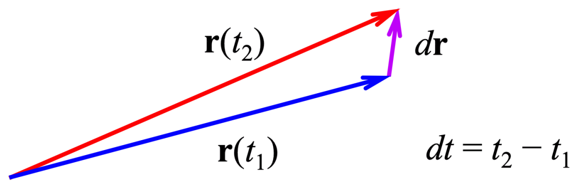


For any coordinate w, $\hat{\mathbf{e}}_w$ points in the direction of increasing w.

The fact that some unit vectors swivel will cause trouble later when we define new kinds of **vector partial derivatives.**

Describing Motion in Other Coordinate Systems

Recall the infinitesimal displacement vector?



Its components depend on the coordinate system chosen:

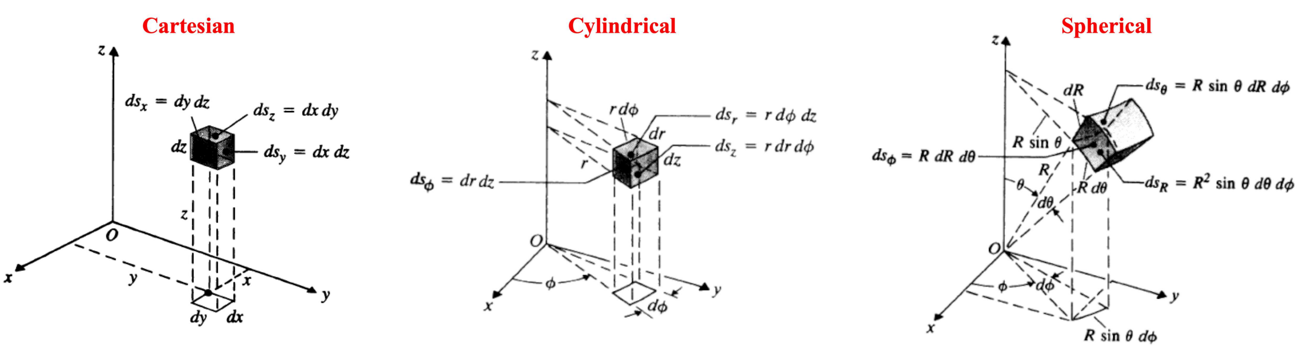
$$\text{Cartesian:} \qquad d\mathbf{r} \;=\; (dx)\hat{\mathbf{e}}_x + (dy)\hat{\mathbf{e}}_y + (dz)\hat{\mathbf{e}}_z$$

$$\text{Cylindrical:} \qquad d\mathbf{r} \;=\; (dr)\hat{\mathbf{e}}_r + (r\,d\phi)\hat{\mathbf{e}}_\phi + (dz)\hat{\mathbf{e}}_z$$

$$\text{Spherical:} \qquad d\mathbf{r} \;=\; (dr)\hat{\mathbf{e}}_r + (r\,d\theta)\hat{\mathbf{e}}_\theta + (r\,\sin\theta\,d\phi)\hat{\mathbf{e}}_\phi$$

Each component (in parentheses) must have units of length.

These definitions make more sense when one tries to construct little **"volume elements"** using only the ingredients of each system:



We can construct the **velocity vector** in the same way we did before...
by dividing $d\mathbf{r}$ by dt.

$$\mathbf{v} \;=\; (\dot{x})\hat{\mathbf{e}}_x + (\dot{y})\hat{\mathbf{e}}_y + (\dot{z})\hat{\mathbf{e}}_z$$

$$\mathbf{v} \;=\; (\dot{r})\hat{\mathbf{e}}_r + (r\,\dot{\varphi})\hat{\mathbf{e}}_\phi + (\dot{z})\hat{\mathbf{e}}_z$$

$$\mathbf{v} \;=\; (\dot{r})\hat{\mathbf{e}}_r + (r\,\dot{\theta})\hat{\mathbf{e}}_\theta + (r\,\sin\theta\,\dot{\varphi})\hat{\mathbf{e}}_\phi$$

using "dot notation" for time derivatives: $\dot{x} = dx/dt, \ddot{x} = d^2x/dt^2$, and so on.

One interesting quantity we'll use a lot later is the **angular velocity** ω around the z-axis, measured in radians per second. We've already defined it: $\boxed{\omega = \dot{\varphi}.}$

. . . .

How about **acceleration?**

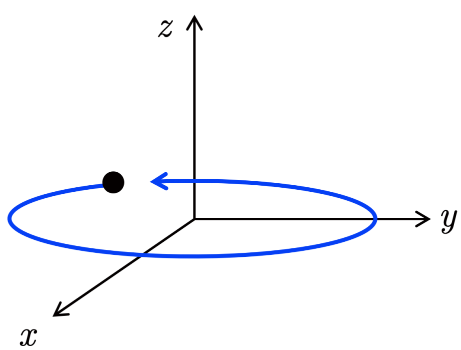
Unfortunately, in non-Cartesian coordinate systems, the expressions for **a** are even more complicated than the ones for **v**. We'll see them soon, but you don't need to memorize them.

Why are they so complicated?

Cartesian coordinates are special because:

- **IF** you're moving with a constant velocity, parallel to one of the coordinate axes,
- **THEN** your acceleration must be zero: $\mathbf{a} = 0$.

However, this isn't true for non-Cartesian coordinates. As an example, consider **circular motion** in the x-y plane, described in cylindrical coordinates:



- The r coordinate remains constant, so there's zero velocity along the r-axis.
- The z coordinate remains constant, so there's zero velocity along the z-axis.
- There's a constant velocity along the ϕ-axis, but...
- **a** is not zero!

Due to the change in direction, there's centripetal acceleration, which you've learned has a magnitude $a_{\rm cen} = v_\phi^2/r$.

We just learned that, in cylindrical coordinates,

$$v_\phi = r\,\dot{\varphi} \ , \qquad \text{so} \qquad a_{\rm cen} = \frac{r^2\,\dot{\varphi}^2}{r} = r\,\dot{\varphi}^2$$

and you also know it's pointed inwards to the origin, so the full version is

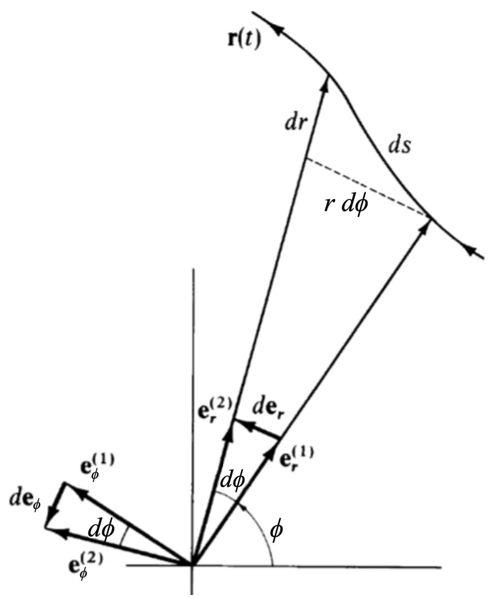
$$\mathbf{a}_{\rm cen} = (-r\,\dot{\varphi}^2)\,\hat{\mathbf{e}}_r \ .$$

When this shows up in nature, it's driven by a force (after all, $\mathbf{F} = m\mathbf{a}$), but its derivation is really just **coordinate book-keeping...** i.e., in this case it's "do whatever is needed to maintain the path being parallel to the ϕ-axis."

The above was just one possible way to move with constant speed along one coordinate and still have an acceleration.

To see the more general version, we can derive $\mathbf{v}$ and $\mathbf{a}$ by being more careful with the derivatives. Specifically, in 2D polar coordinates (i.e., the xy plane of cylindrical coordinates):

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\hat{\mathbf{e}}_r) = \dot{r}\hat{\mathbf{e}}_r + r\frac{d\hat{\mathbf{e}}_r}{dt}$$



Recall that, in non-Cartesian coordinates, the unit vectors aren't fixed, so they have derivatives too.

As $\mathbf{r}$ changes, the unit vectors pointing to $\mathbf{r}$ swivel around in ϕ but don't change in radial distance. Geometry provides:

$$d\hat{\mathbf{e}}_r = d\phi\,\hat{\mathbf{e}}_\phi \ , \quad d\hat{\mathbf{e}}_\phi = -d\phi\,\hat{\mathbf{e}}_r$$

and dividing each side by dt gives

$$\frac{d\hat{\mathbf{e}}_r}{dt} = \dot{\varphi}\,\hat{\mathbf{e}}_\phi \ , \quad \frac{d\hat{\mathbf{e}}_\phi}{dt} = -\dot{\varphi}\,\hat{\mathbf{e}}_r$$

This gives the 2D result: $\mathbf{v} = (\dot{r})\hat{\mathbf{e}}_r + (r\,\dot{\varphi})\hat{\mathbf{e}}_\phi$ (equivalent to cylindrical).

In 3D, one can take the next set of derivatives to give the acceleration $\mathbf{a} = d\mathbf{v}/dt$, in all three coordinate systems.

Cartesian:

$$\mathbf{a} = (\ddot{x})\hat{\mathbf{e}}_x + (\ddot{y})\hat{\mathbf{e}}_y + (\ddot{z})\hat{\mathbf{e}}_z$$

Cylindrical:

$$\mathbf{a} = \left(\ddot{r} - r\dot{\varphi}^2\right)\hat{\mathbf{e}}_r + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)\hat{\mathbf{e}}_\phi + (\ddot{z})\hat{\mathbf{e}}_z$$

Spherical:

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\hat{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\hat{\mathbf{e}}_\theta + \\ + \left(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\,\dot{\theta}\,\dot{\varphi}\,\cos\theta\right)\hat{\mathbf{e}}_\phi$$

DIFFERENTIAL EQUATIONS

As one goes further in physics & astrophysics, one will see more differential equations. Let's look at how to handle a few important types.

We will mostly see **first-order** ordinary differential equations (ODEs). These can be boiled down to

$$f\left(x,\, y,\, \frac{dy}{dx}\right) \;=\; 0 \ \ .$$

and the goal is to solve it for $y(x)$.

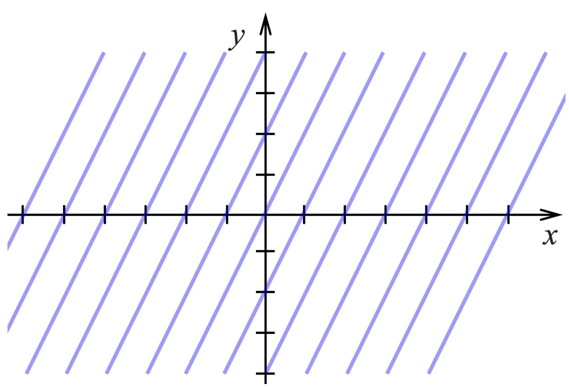
We'll occasionally see **second-order** differential equations, which contain d^2y/dx^2 as well as dy/dx, but: (1) you won't have to solve them, and (2) we'll only bring them up when needed.

There's one major difficulty with solving a differential equation for $y(x)$. Usually the equation itself **doesn't** contain all the information you need!

Let's look at a simple example:

$$f\left(x, y, \frac{dy}{dx}\right) \;=\; \frac{dy}{dx} - 2 \qquad\quad \text{so the ODE is:} \qquad\quad \frac{dy}{dx} \;=\; 2 \ .$$

You know that dy/dx is the slope of the curve $y(x)$, so all this equation says is that the slope must always be 2. This can satisifed by a straight line, but *which* straight line is the actual desired solution?



Thus, in order to solve a first-order ODE, we need both the equation and some **boundary condition;** i.e., one special value of x (call it x_0) where you specify what $y(x)$ must be (call it y_0). That pins down which solution to choose.

. . . .

Many of the first-order ODEs we will see are **separable.** This means that it's possible to write them in the form

$$f(x)\,dx \;=\; g(y)\,dy\;.$$

Let's look at an example:

$$\frac{dy}{dx} = x^5\,y \quad\longrightarrow\quad \int\frac{dy}{y} = \int dx\,x^5 \quad\longrightarrow\quad \ln y \;=\; \frac{x^6}{6} + C\;.$$

There was an integration constant on each side, but it's fair to just combine them together when we don't know their values. Thus,

$$y(x) \;=\; \exp\left(C + \frac{x^6}{6}\right) \;=\; e^C\,e^{x^6/6} \;=\; y_0\,e^{x^6/6}$$

and if we know the value of y at $x = 0$, that gives us y_0.

By the way, there's no guarantee that you'll always be able to **explicitly** solve the integrated version of an ODE for $y(x)$. BUT, if you're able to do the integration and apply the boundary condition, you've still got an **implicit** solution that can be solved by a computer.

We will look at a few more separable examples, but first there's one more trick to put in your back pockets (and it's also in the "useful formula" handout, so no need to memorize it) for one additional type of first-order ODE:

$$\frac{dy}{dx} + P(x)y(x) \;=\; Q(x)$$

It's not separable, but there's a nice recipe for solving the equation, using something called an **integrating factor:**

$$\mu(x) \;=\; \exp\left[\int^x P(x')\,dx'\right] \qquad \text{(doing the integral without } C\text{)}$$

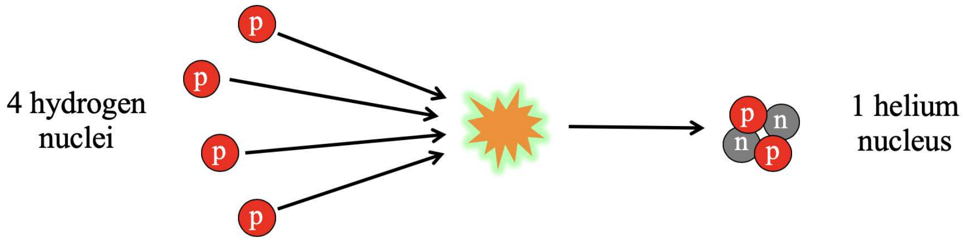
and this lets us find the solution

$$y(x) = \frac{1}{\mu(x)}\left[\int^x Q(x')\mu(x')\,dx' \;+\; C\right]\;.$$

. . . .

Example 1: Stellar Nucleosynthesis

We'll discuss nuclear physics (i.e., fusion & fission) much later in the semester, but you probably already know what's going on in, say, the center of our Sun:



The core of a star is hot & dense enough for hydrogen nuclei to undergo fusion into helium nuclei ("alpha particles"), and the small amount of leftover energy released as photons provides the stellar luminosity!

For a small bit of volume near the center of the star, we would like to follow the change in **number density** (i.e., particles per unit volume) of hydrogen and helium nuclei over time.

$$\text{Thus, we want to solve for} \quad n_p(t) \quad \text{and} \quad n_\alpha(t) \ .$$

In reality, the cartoon picture above is wrong. It's very unlikely for 4 protons to come together at the same time & place. Fusion actually proceeds in a chain of pairwise fusion reactions ($A + B \to AB$) in combination with a few "weak" reactions ($p \to n$).

We'll see that it's okay to assume that the **reaction rates** (i.e., the number of reactions that occur per unit time, per unit volume) are proportional to n_p^2.

The equations to solve are:

$$\frac{dn_p}{dt} = -4qn_p^2 \qquad\qquad \frac{dn_\alpha}{dt} = +qn_p^2$$

where q is just some number (appropriate for the star's current temperature & density) that puts the rate into the right units.

Why are the right-hand sides different? We can see that the reason is that for every 4 protons "destroyed," there's one helium created.

To prove that, let's first take the first equation and ADD it to:
{ 4 times the second equation }:

$$\frac{dn_p}{dt} + 4\frac{dn_\alpha}{dt} = 0$$

$$\frac{d}{dt}(n_p + 4n_\alpha) = 0 \qquad \Longrightarrow \qquad n_p + 4n_\alpha = \text{constant.}$$

Another way to write this is: $\qquad (n_p + 4n_\alpha)_{\text{initial}} = (n_p + 4n_\alpha)_{\text{final}}$.

This makes sense. Let's assume the *initial* state has 100 protons and no helium at all. After some time, there's one reaction. The *final* state has 96 protons and 1 helium. ✓

If we've convinced ourselves the equations make sense, we can now solve them. We're lucky they're not truly *coupled ODEs,* because the n_p equation can be solved on its own, without the other one:

$$\frac{dn_p}{n_p^2} = -4q\,dt$$

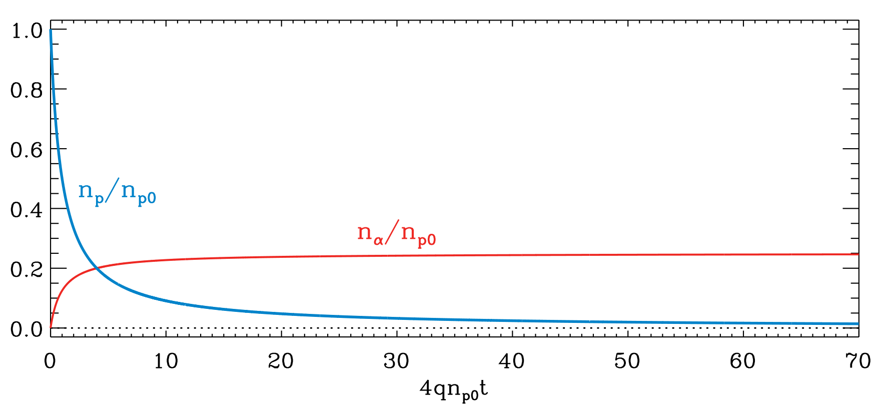
Let's integrate from $t' = 0$ to t, and from n_{p0} to $n_p(t)$.

$$\left(-\frac{1}{n_p}\right) - \left(-\frac{1}{n_{p0}}\right) = -4qt \qquad \Longrightarrow \qquad \boxed{n_p(t) = \frac{n_{p0}}{1 + 4qn_{p0}t}} \ .$$

To complete the solution, we need $n_\alpha(t)$. We don't have to integrate again, because we already know that $n_p + 4n_\alpha$ is a constant. If we assume something simple (i.e., at $t = 0$, there's no helium), then

$$n_{p0} = n_p + 4n_\alpha \qquad \Longrightarrow \qquad \boxed{n_\alpha(t) = \frac{n_{p0} - n_p(t)}{4}} \ .$$

and the two curves look like what we thought they would:



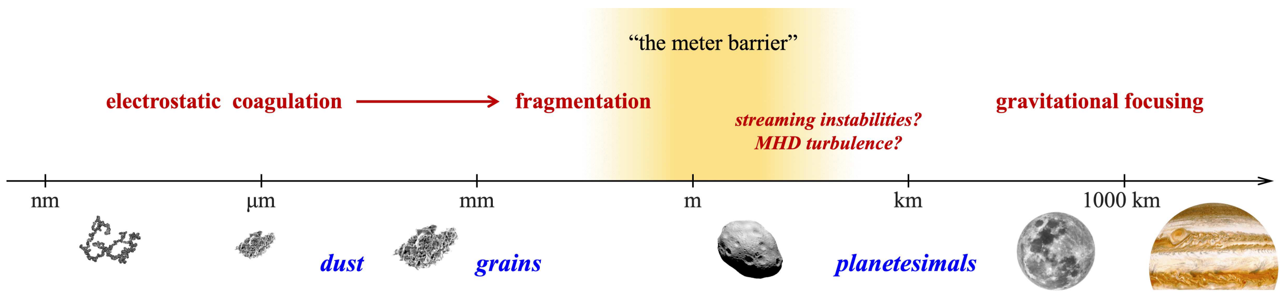
The proton "fuel" actually never gets *completely* used up; its number density just asymptotically approaches zero.

By the way, for the present-day Sun, $4q \approx 10^{-43}$ cm^3/s, and $n_{p0} \approx 10^{26}$ protons/cm^3. Thus, one "time unit" on this plot is about 10^{17} seconds, or $\sim$3 billion years. Right order of magnitude for stellar evolution!

. . . .

Example 2: Planet Formation

How are planets formed out of the gas/dust accretion disks that surround young stars?



When gas cools, molecules form. When they cool even more, they can clump up into larger dust grains. Lab experiments show that dust grains can undergo electrostatic coagulation (i.e., static cling) and grow to become rocks you could hold in your arms.

However, once they reach centimeter/meter size, they tend to *fragment* when they collide. Much of current planet formation research is about how to grow past this "meter barrier."

We'll assume they've solved the problem & become $\sim$1 km **planetesimals.** Our problem: how rapidly do they grow into true planets?

The ODE that we'll solve is sometimes called Safronov's equation. We follow the radius R of a spherical planetesimal, as it plows through a protoplanetary disk filled with similarly-sized planetesimals and occasionally collides with them...

$$\frac{dR}{dt} = f\,\sigma \left[1 + \left(\frac{R}{R_{\rm crit}} \right)^2 \right]$$

where

- f is the "filling factor" of the disk; i.e., what fraction of the volume is filled with planetesimals. (Typical value: 10^{-9} ?)
- σ gives the mean "thermal speed" that they're all moving around with respect to one another. (Typical value: $\sim$3 km/s)
- $R_{\rm crit}$ is a size above which the **self-gravity** of our planetesimal is strong enough to pull others towards it, and thus enhance the rate of collisions. (Typical value: $\sim$1000 km)

Note that when $R \ll R_{\rm crit}$, dR/dt is just a constant; i.e., R grows linearly with time, as the planetesimal just scoops up a neighbor every once in a while.

What about when R starts to exceed $R_{\rm crit}$? Let's solve the separable ODE:

$$\int \frac{dR}{1+(R/R_{\rm crit})^2} = f\sigma \int dt$$

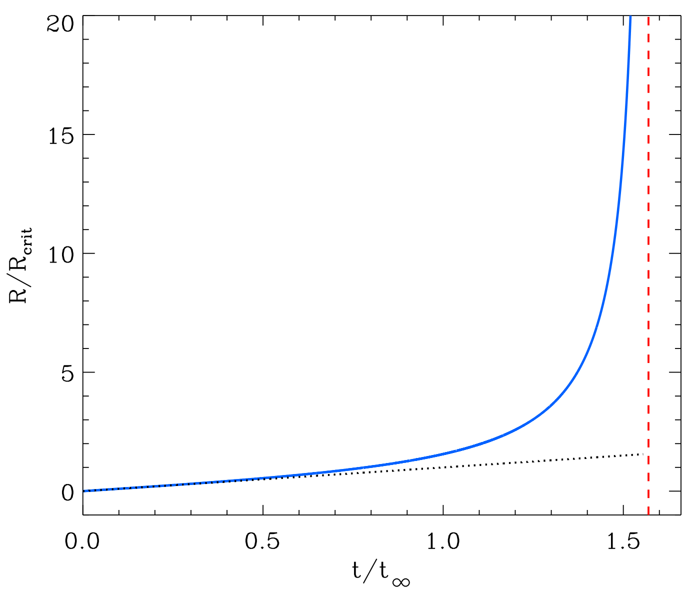
and we integrate from R_0 (at $t=0$) to $R(t)$, with

$$R_{\rm crit}\left[\tan^{-1}\left(\frac{R}{R_{\rm crit}}\right) - \tan^{-1}\left(\frac{R_0}{R_{\rm crit}}\right)\right] = f\sigma(t-0)$$

Thus,

$$\frac{R}{R_{\rm crit}} = \tan\left[\tan^{-1}\left(\frac{R_0}{R_{\rm crit}}\right) + \frac{t}{t_\infty}\right] \qquad \text{defining} \qquad t_\infty = \frac{R_{\rm crit}}{f\sigma} \; .$$

This can be simplified with some other identities, but let's just plot it:



The above plot used $R_0/R_{\text{crit}} = 0.001$ for a planetesimal that started at $\sim$1 km. The solution would've looked similar if we just chose $R_0 = 0$.

Note the linear growth at early times (when R is still $\ll R_{\text{crit}}$), but then it takes off!

There's a maximum time $\approx t_\infty$ at which the tangent function goes to infinity, and R increases without bound. This is **runaway growth!**

(Note: for $t > t_\infty$, there may still be mathematical solutions of the ODE, but they aren't *physically* realistic.)

Using the example numbers given above, $t_\infty \approx 10,000$ years, which is quick on the scale of the age of the solar system.

What "real physics" stops the asymptote from going all the way to infinity? Mostly, it's the fact that the new planet soon "clears its orbit" by sweeping up all available stuff. f eventually becomes zero, so $dR/dt = 0$, and $R =$ constant. The planet has formed.

The point of these examples was mainly to give examples of solving ODEs, but hopefully they also convey some "astrophysical insight" about physical parameters, keeping track of units, and making approximations.
