

A selection of hopefully useful formulae for ASTR–2100

Steven R. CranmerSpring 2020

The inspiration (and some of the L^AT_EX!) for this document comes from the [NRL Plasma Formulary](#), excellently curated by Joseph Huba. Generally, I add to this document when I find myself repeatedly looking up something that I probably should have memorized. C’est la vie!



“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.”

— Eugene Wigner (1960)



Other Sources of Useful Data

NRL Plasma Formulary <http://www.nrl.navy.mil/ppd/content/nrl-plasma-formulary>
Fundamental physical constants from NIST <http://physics.nist.gov/cuu/Constants/>
Zombeck’s *Handbook of Space Astronomy & Astrophysics* ... <http://ads.harvard.edu/books/hsaa/>
Wolfram Alpha (just type a question or equation) <http://www.wolframalpha.com/>

1 Mathematics

1.1 On the Art of Approximation

You may be used to solving problems that have exact solutions. At some point, however, we run out of those, and we must rely increasingly on approximation/assumption...

It's kind of an art to figure out $\left\{ \begin{array}{l} \text{what to simplify} \\ \text{what to neglect} \\ \text{what to flat-out ignore} \end{array} \right.$

Hopefully, by seeing it done in upper-level courses, you'll start to get a feel for doing it yourself. It takes a while...

- = the "exact equality" will often give way to
- \approx "is approximately equal to," or sometimes even
- \sim "very roughly equal to" (within an order of magnitude!?)
- \propto and sometimes we just care about which quantities are "proportional to" one another, ignoring normalizing constants.

[Mahajan \(2018\)](#) says a lot more about the mind-decluttering power of \sim ("twiddle").

1.2 Algebra and Trigonometry

Quadratic Formulae

$$ax^2 + bx + c = 0 \quad \implies \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

where the latter is useful when $a \rightarrow 0$. For the two roots x_1 and x_2 , Vieta's formulae are

$$x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad x_1 x_2 = \frac{c}{a} .$$

Logarithms

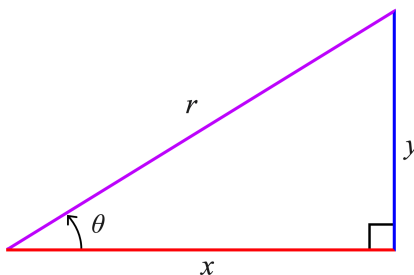
Definitions	$y = a^x \iff x = \log_a y$
Product	$\log_a(xy) = \log_a x + \log_a y$
Quotient	$\log_a(x/y) = \log_a x - \log_a y$
Power	$\log_a(y^n) = n \log_a y$
Change of base	$\log_a y = (\log_a b)(\log_b y)$

Trigonometric Identities

$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$



$$x^2 + y^2 = r^2$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}, \quad \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left(\frac{x \pm y}{1 \mp xy} \right)$$

$$\sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1, \quad \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Undoing the inverse...

$$\sin(\cos^{-1} x) = \cos(\sin^{-1} x) = \sqrt{1 - x^2}$$

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1 + x^2}}, \quad \cos(\tan^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}, \quad \tan(\cos^{-1} x) = \frac{\sqrt{1 - x^2}}{x}$$

Law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

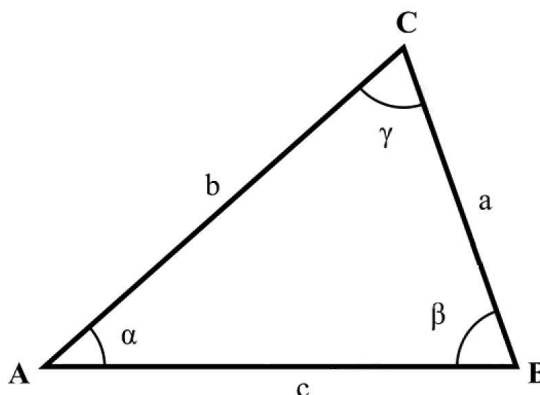
$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

Sum of angles

$$\alpha + \beta + \gamma = 180$$



Angles

$$1 \text{ circle} = 360^\circ = 2\pi \text{ radians (rad)}$$

$$1 \text{ radian} = 360^\circ / 2\pi = 180^\circ / \pi \approx 57.296^\circ$$

$$1 \text{ degree} = 60' \text{ (i.e., 60 arcminutes)} = 3600'' \text{ (i.e., 3600 arcseconds)}$$

$$1 \text{ arcminute} = 60'' \text{ (i.e., 60 arcseconds)}$$

$$\text{Thus, } 1 \text{ radian} \approx 206,265'' \text{ and } 1 \text{ circle} = 1,296,000''$$

$$1 \text{ sphere} = 4\pi \text{ steradians (sr)}$$

1.3 Vector Identities

Notation: f, g , are scalars; \mathbf{A}, \mathbf{B} , etc., are vectors.

- (1) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$
- (2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
- (3) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$
- (4) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$
- (5) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D}$
- (6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$
- (7) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
- (8) $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$
- (9) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$
- (10) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
- (11) $\mathbf{A} \times (\nabla \times \mathbf{B}) = (\nabla \mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
- (12) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
- (13) $\nabla^2 f = \nabla \cdot \nabla f$
- (14) $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$
- (15) $\nabla \times \nabla f = 0$ (i.e., if $\nabla \times \mathbf{F} = 0$, it implies one can write $\mathbf{F} = \nabla \phi$)
- (16) $\nabla \cdot \nabla \times \mathbf{A} = 0$
- (17) $\nabla^2(fg) = f\nabla^2 g + 2(\nabla f) \cdot (\nabla g) + g\nabla^2 f$

Also, if vectors \mathbf{A} & \mathbf{B} depend on time t , then

$$\frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} \quad \frac{\partial}{\partial t}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{B}$$

1.4 Coordinate Systems

Conversions between Cartesian and Cylindrical (sometimes ρ or ϖ or R are used for r)

$$\begin{aligned} x &= r \cos \phi & r &= \sqrt{x^2 + y^2} \\ y &= r \sin \phi & \phi &= \tan^{-1}(y/x) \\ z &= z & z &= z \end{aligned}$$

Conversions between Cartesian and Spherical

$$\begin{aligned} x &= r \cos \phi \sin \theta & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \phi \sin \theta & \theta &= \cos^{-1}(z/r) \\ z &= r \cos \theta & \phi &= \tan^{-1}(y/x) \end{aligned}$$

Vector & Differential Operators in CARTESIAN Coordinates

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{e}_x(A_y B_z - A_z B_y) + \mathbf{e}_y(A_z B_x - A_x B_z) + \mathbf{e}_z(A_x B_y - A_y B_x)$$

$$\nabla f = \mathbf{e}_x \frac{\partial f}{\partial x} + \mathbf{e}_y \frac{\partial f}{\partial y} + \mathbf{e}_z \frac{\partial f}{\partial z} \quad \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times \mathbf{F} = \mathbf{e}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$[(\mathbf{A} \cdot \nabla)\mathbf{B}]_i = A_x \frac{\partial B_i}{\partial x} + A_y \frac{\partial B_i}{\partial y} + A_z \frac{\partial B_i}{\partial z} \quad \text{for } i = x, y, z$$

$$\text{Volume element} \quad dV = dx dy dz$$

$$\text{Line element displacement} \quad d\mathbf{r} = (dx)\hat{\mathbf{e}}_x + (dy)\hat{\mathbf{e}}_y + (dz)\hat{\mathbf{e}}_z$$

Vector & Differential Operators in CYLINDRICAL Coordinates (r, ϕ, z)

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = (A_\phi B_z - A_z B_\phi)\hat{\mathbf{e}}_r + (A_z B_r - A_r B_z)\hat{\mathbf{e}}_\phi + (A_r B_\phi - A_\phi B_r)\hat{\mathbf{e}}_z$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\phi = \frac{1}{r} \frac{\partial f}{\partial \phi}; \quad (\nabla f)_z = \frac{\partial f}{\partial z}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r} \frac{\partial A_r}{\partial \phi}$$

Laplacian

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2}{r^2} \frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi + \frac{2}{r^2} \frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2}$$

$$(\nabla^2 \mathbf{A})_z = \nabla^2 A_z$$

Components of $(\mathbf{A} \cdot \nabla) \mathbf{B}$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_r = A_r \frac{\partial B_r}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_r}{\partial \phi} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_\phi = A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_\phi}{\partial \phi} + A_z \frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_z = A_r \frac{\partial B_z}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_z}{\partial \phi} + A_z \frac{\partial B_z}{\partial z}$$

Divergence of a tensor

$$(\nabla \cdot \mathbb{T})_r = \frac{1}{r} \frac{\partial}{\partial r} (r T_{rr}) + \frac{1}{r} \frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} - \frac{T_{\phi\phi}}{r}$$

$$(\nabla \cdot \mathbb{T})_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r T_{r\phi}) + \frac{1}{r} \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{\phi r}}{r}$$

$$(\nabla \cdot \mathbb{T})_z = \frac{1}{r} \frac{\partial}{\partial r} (r T_{rz}) + \frac{1}{r} \frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z}$$

$$\text{Volume element} \quad dV = r \, dr \, d\phi \, dz$$

$$\text{Line element displacement} \quad d\mathbf{r} = (dr)\hat{\mathbf{e}}_r + (r \, d\phi)\hat{\mathbf{e}}_\phi + (dz)\hat{\mathbf{e}}_z$$

Vector & Differential Operators in SPHERICAL Coordinates (r, θ, ϕ)

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi$$

$$\mathbf{A} \times \mathbf{B} = (A_\theta B_\phi - A_\phi B_\theta)\hat{\mathbf{e}}_r + (A_\phi B_r - A_r B_\phi)\hat{\mathbf{e}}_\theta + (A_r B_\theta - A_\theta B_r)\hat{\mathbf{e}}_\phi$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta}; \quad (\nabla f)_\phi = \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi}$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi)$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta}$$

Laplacian

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} - \frac{2 \cot \theta A_\theta}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$(\nabla^2 \mathbf{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \phi}$$

Components of $(\mathbf{A} \cdot \nabla) \mathbf{B}$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_r = A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_r}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_r}{\partial \phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_\theta = A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\theta}{\partial \phi} + \frac{A_\theta B_r}{r} - \frac{\cot \theta A_\phi B_\phi}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_\phi = A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\phi}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi B_r}{r} + \frac{\cot \theta A_\phi B_\theta}{r}$$

Divergence of a tensor

$$(\nabla \cdot \mathbb{T})_r = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta T_{\theta r}) + \frac{1}{r \sin \theta} \frac{\partial T_{\phi r}}{\partial \phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r}$$

$$(\nabla \cdot \mathbb{T})_\theta = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta T_{\theta\theta}) + \frac{1}{r \sin \theta} \frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{T_{\theta r}}{r} - \frac{\cot \theta T_{\phi\phi}}{r}$$

$$(\nabla \cdot \mathbb{T})_\phi = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta T_{\theta\phi}) + \frac{1}{r \sin \theta} \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{T_{\phi r}}{r} + \frac{\cot \theta T_{\phi\theta}}{r}$$

$$\text{Volume element} \quad dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$\text{Line element displacement} \quad d\mathbf{r} = (dr) \hat{\mathbf{e}}_r + (r \, d\theta) \hat{\mathbf{e}}_\theta + (r \sin \theta \, d\phi) \hat{\mathbf{e}}_\phi$$

1.5 Special Functions & Series Expansions

Binomial Series

$$\text{For } |x| \ll 1, \quad (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$$

Trigonometric Functions

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots \quad (x^2 < 1)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots \quad (x^2 < 1)$$

Exponential Functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad 10^x = \exp(2.30259x) \quad (\ln 10 = 2.30259)$$

Full-width at half-maximum (FWHM):

$$\text{For } y = \exp\left[-\frac{x^2}{2\sigma^2}\right] = \exp\left[-\left(\frac{x}{V_{1/e}}\right)^2\right], \quad \text{the FWHM} = V_{1/e}2\sqrt{\ln 2} \approx 1.66511V_{1/e}.$$

Euler's formulae:

$$e^{ix} = \cos x + i \sin x \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Logarithmic Functions

$$\text{For } |x| \ll 1, \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

1.6 Derivatives and Integrals

Definite Integrals of Gaussians

$$\begin{aligned}
 \int_0^\infty dx e^{-(x/\sigma)^2} &= \frac{\sigma\sqrt{\pi}}{2} & \int_0^\infty dx x e^{-(x/\sigma)^2} &= \frac{\sigma^2}{2} \\
 \int_0^\infty dx x^2 e^{-(x/\sigma)^2} &= \frac{\sigma^3\sqrt{\pi}}{4} & \int_0^\infty dx x^3 e^{-(x/\sigma)^2} &= \frac{\sigma^4}{2} \\
 \int_0^\infty dx x^4 e^{-(x/\sigma)^2} &= \frac{3\sigma^5\sqrt{\pi}}{8} & \int_0^\infty dx x^5 e^{-(x/\sigma)^2} &= \sigma^6 \\
 \int_0^\infty dx x^6 e^{-(x/\sigma)^2} &= \frac{15\sigma^7\sqrt{\pi}}{16} & \int_0^\infty dx x^n e^{-(x/\sigma)^2} &= \frac{\sigma^{n+1}}{2} \Gamma\left(\frac{n+1}{2}\right) \\
 \int_{-\infty}^\infty dx e^{-ax^2+bx} &= e^{b^2/4a} \sqrt{\frac{\pi}{a}}
 \end{aligned}$$

Definite Integrals relevant to the Planck function

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = \zeta(n+1) \Gamma(n+1) = \zeta(n+1) n! \quad (\text{for integer } n)$$

$\zeta(n)$ is the Riemann zeta function, which ≈ 1 for $n \gtrsim 3$. In that case, $\int_0^\infty dx \frac{x^n}{e^x - 1} \approx n!$

Integration by Parts

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du \quad \int u(x) \frac{dv}{dx} dx = u(x)v(x) - \int v(x) \frac{du}{dx} dx$$

and a vector version comes from Gauss' divergence theorem (Binney & Tremaine, eqn B.45):

$$\int d^3\mathbf{r} g \nabla \cdot \mathbf{F} = \oint g \mathbf{F} \cdot d^2\mathbf{S} - \int d^3\mathbf{r} (\mathbf{F} \cdot \nabla) g$$

1.7 Differential Equations

For a 1st order linear equation of the form

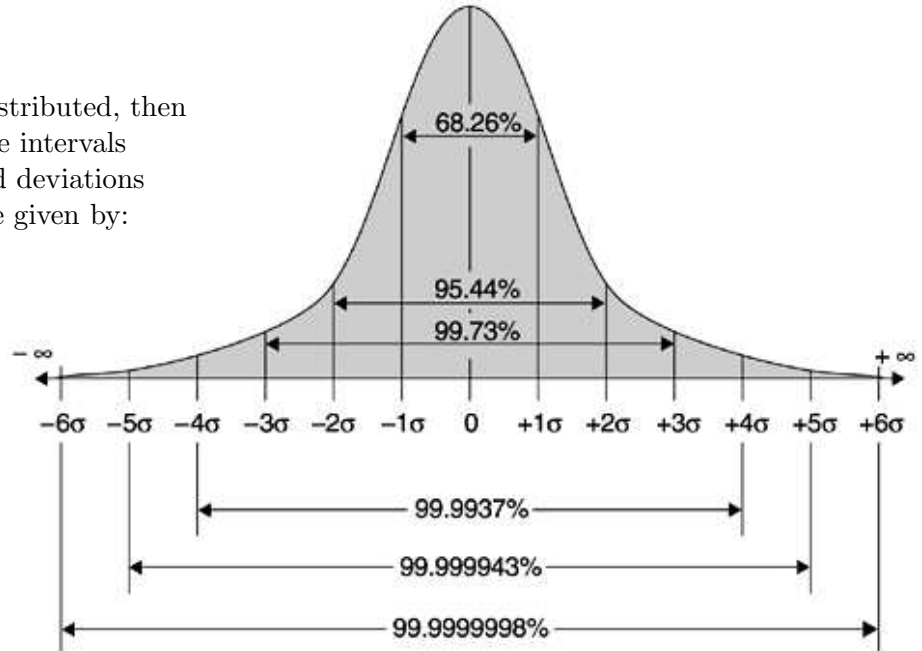
$$\frac{dy}{dx} + P(x)y(x) = Q(x)$$

use an *integrating factor*

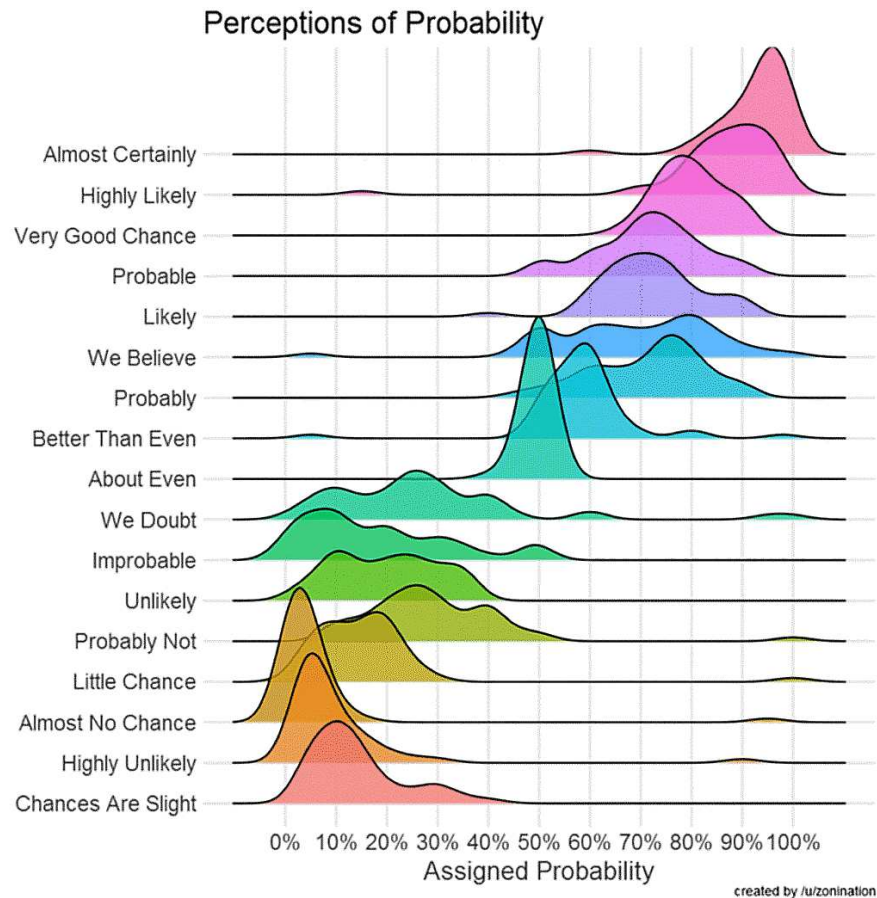
$$\mu(x) = \exp\left[\int^x P(x') dx'\right] \quad \text{and the solution is} \quad y(x) = \frac{1}{\mu(x)} \left[\int^x Q(x')\mu(x') dx' + C\right].$$

1.8 Probability and Statistics

If events are normally distributed, then the traditional confidence intervals (in units of $\pm N$ standard deviations away from the mean) are given by:



Perceptions of probability associated with common phrases ([github link](#)):



Permutations and Combinations

Consider a pile of n books that you want to read. How many different ways are there to order them? There are n ways of choosing the first one in your list. There are then $n - 1$ ways of choosing the second one (because there's now one fewer book to choose from), $n - 2$ ways of choosing the third one, and so on. The total number of uniquely ordered arrangements, or **permutations**, is thus given by

$$n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1 = n! \quad (\text{"}n \text{ factorial"}).$$

For $n \gg 1$, Stirling's approximation gives

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

What if you don't have enough time to read all n books? If you have enough time to read only r books (where $r \leq n$), how many ways can you order them? Like before, you start with n ways of choosing the first one, $n - 1$ ways of choosing the second one, and so on. But you stop when you reach the r th book, for which there are $n - r + 1$ options. This number of permutations is denoted

$${}_n P_r = n \times (n - 1) \times (n - 2) \times \cdots \times (n - r + 1) = \frac{n!}{(n - r)!}.$$

Another way of thinking about that last version of ${}_n P_r$ is the following: Because you're only reading r books, that means there are $(n - r)$ books that you *won't* be reading. Thus, you can divide out the $(n - r)!$ ways that those books could have been ordered from the overall total ($n!$).

Lastly, what if you wanted to compute the number of unique subsets of r books that can be extracted from the larger pool of n books, but *without regard to their ordering*? You first compute ${}_n P_r$, then divide it by the number of possible orderings of the r books that you will read (i.e., $r!$). This gives the number of **combinations**,

$${}_n C_r = \frac{n!}{(n - r)! r!} = \binom{n}{r}$$

These are also called binomial coefficients, since

$$(x + y)^n = \sum_{r=0}^n \binom{n}{r} x^{n-r} y^r.$$

The above explanations were derived from similar ones by [Spiegel \(1975\)](#) and [Arbuckle \(2008\)](#).

2 Physics

Metric Prefixes

Multiple	Prefix	Symbol	Multiple	Prefix	Symbol
10^{-1}	deci	d	10	deca	da
10^{-2}	centi	c	10^2	hecto	h
10^{-3}	milli	m	10^3	kilo	k
10^{-6}	micro	μ	10^6	mega	M
10^{-9}	nano	n	10^9	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E
10^{-21}	zepto	z	10^{21}	zetta	Z
10^{-24}	yocto	y	10^{24}	yotta	Y

Physical Constants (SI)

Speed of light in vacuum	$c = 2.99792458 \times 10^8 \text{ m s}^{-1}$
Newton's gravitation constant ..	$G = 6.67384 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
Boltzmann's ideal gas constant .	$k_B = 1.3806488 \times 10^{-23} \text{ J K}^{-1}$
Stefan-Boltzmann constant	$\sigma = 5.670373 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Radiation pressure constant	$a = 4\sigma/c = 7.5657314 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$
Planck's constant	$h = 2\pi\hbar = 6.62606957 \times 10^{-34} \text{ J s}$
Permittivity of free space	$\epsilon_0 = 8.8542 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7} \text{ N s}^2 \text{ C}^{-2}$
Energy associated with 1 eV ...	$E_{\text{eV}} = 1.602176565 \times 10^{-19} \text{ J}$

Atomic Constants (SI)

Electron charge	$e = 1.602176565 \times 10^{-19} \text{ C}$
Electron mass	$m_e = 9.10938291 \times 10^{-31} \text{ kg}$
Proton mass	$m_p = 1.672621777 \times 10^{-27} \text{ kg} \approx 1836 m_e$
Neutron mass	$m_n = 1.674927351 \times 10^{-27} \text{ kg} \approx 1839 m_e$
Atomic mass unit	$1 \text{ u} = m(^{12}\text{C})/12 = 1.660538921 \times 10^{-27} \text{ kg}$
Bohr radius	$a_0 = 4\pi\epsilon_0 \hbar^2 / (m_e e^2) = 5.2917721092 \times 10^{-11} \text{ m}$
Classical electron radius	$r_e = e^2 / (4\pi\epsilon_0 m_e c^2) = 2.8179403267 \times 10^{-15} \text{ m}$
Thomson cross section	$\sigma_T = (8\pi/3)r_e^2 = 6.652458734 \times 10^{-29} \text{ m}^2$

Astronomical Constants (SI)

Solar mass	$M_\odot = 1.989 \times 10^{30} \text{ kg}$
Solar radius	$R_\odot = 6.963 \times 10^8 \text{ m}$
Solar luminosity	$L_\odot = 3.83 \times 10^{26} \text{ J s}^{-1}$
Solar effective temperature	$T_{\text{eff}} = 5770 \text{ K}$
Solar surface gravity	$g_\odot = 273.79 \text{ m s}^{-2} \quad \log g_\odot (\text{cgs}) = 4.4374$
Earth's mass	$M_\oplus = 5.9736 \times 10^{24} \text{ kg} \approx 3 \times 10^{-6} M_\odot$
Earth's radius	$R_\oplus = 6371 \text{ km}$
Astronomical unit	$1 \text{ AU} = 1.495978707 \times 10^{11} \text{ m} \approx 215 R_\odot$
Parsec	$1 \text{ pc} = 3.085677 \times 10^{16} \text{ m} = 3.2616 \text{ light years}$

Unit Conversions

Changing from one set of units to another is enabled by thinking of them as ratios that get multiplied together in a chain. Example:

$$\frac{55 \text{ miles}}{\text{hour}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{12 \text{ inches}}{1 \text{ foot}} \times \frac{2.54 \text{ cm}}{1 \text{ inch}} \times \frac{1 \text{ m}}{100 \text{ cm}} \times \frac{1 \text{ hour}}{3600 \text{ s}} = 24.59 \text{ m/s}$$

Other Units

$$1 \text{ year} \approx 365.25 \times 24 \times 60 \times 60 = 3.15576 \times 10^7 \text{ seconds} \sim \pi \times 10^7 \text{ seconds.}$$

$$1 \text{ Angstrom} = 1 \text{ \AA} = 10^{-10} \text{ m} = 0.1 \text{ nm.}$$

Significant Figures

You should already know how to count up the number of significant digits in a quantity. In scientific notation, it's usually assumed that every digit given prior to the exponential is significant (e.g., 4.1800×10^7 has five significant digits).

- When combining two quantities (e.g., adding, multiplying), the answer should be given with the *least* number of significant digits from the initial quantities.
- However, when working through multi-step calculations, it's useful to keep at least one more significant digit in the intermediate results than will be needed in the final answer.
- If asked to guess or approximate, the final answer should only have (at most) two significant digits.